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k-Hop Domination Defect in a Graph

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Abstract. In this paper, we introduce a new graph parameter called the hop domination defect and investigate it for some classes of graphs. The hop domination number $\gamma_h(G)$ of a graph G is the minimum number of vertices required to hop dominate all the vertices of G. The minimality of $\gamma_h(G)$ implies that if $W \subseteq V(G)$ and $|W| < \gamma_h(G)$, then there is at least one vertex in G that is not hop dominated by W. Given a positive integer $k < \gamma_h(G)$, where $\gamma_h(G) \geq 2$, the k-hop domination defect of G, denoted by $\zeta_k^h(G)$, is the minimum number of vertices of G that is not hop dominated by any subset of vertices of G with cardinality $\gamma_h(G) - k$. We give some bounds on the k-hop domination defect of a graph in terms of its order and maximum hop degree. Furthermore, we determine the k-hop domination defects of the join of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, k-hop domination defect, hop degree of a graph, join

1. Introduction

Natarajan and Ayyaswamy in [13] introduced and studied hop domination, which, in some sense, is related to the standard domination. According to the authors, the concept has its origin from the field of Inorganic Chemistry. This parameter has been widely studied since its appearance in the literature and a significant number of variants of the parameter have already been defined and investigated (see for example [1], [2], [4], [5], [6], [7], [8], [9], [14], [15]), and [16]).

Recently, Das et al. [3] introduced a domination parameter called the domination defect. As mentioned in their paper, the motivation of this study was mainly on dealing with problems associated with guarding facilities or placing monitoring devices in networks when there is only fewer than the minimum number of guards or devices required. In their

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study, the authors were able to establish various bounds on the domination defect of a graph in terms of, among others, the domination number, order, degree sequence, graph homomorphisms, and efficient dominating set. Other studies on the topic (see [10–12]) focused on characterizing the k-domination defect sets and determining the k-domination defect in the join, corona, edge corona, and composition of two graphs.

Since hop domination has similar applications (e.g. in facility location, protection strategy, management in social networks) as domination, it is also a bit interesting to study the effect of having fewer than the required minimum number of nodes in a hop dominating set. In this study, we define the parameter k-hop domination defect and study it for some known graphs. The study hopes to give bounds of the newly defined parameter in terms of the order, hop degree of the graph, and other parameters. In particular, the authors would like to find what specific conditions to impose so that these bounds and some results which seemingly run parallel to the ones found in [3], hold in the sense of hop domination. The k-hop domination defects of some join of graphs are also obtained.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The distance between two subsets A and B of $V(G)$ is given by $d_G(A, B) = \min\{d_G(a, b) : a \in A \text{ and } b \in B\}$. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v \in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A*. The degree of vertex v is $deg_G(v) = |N_G(v)|$. A vertex v of G is isolated if $|N_G(v)| = 0$. The *maximum degree* $\Delta(G)$ of G is given by $\Delta(G) = \max\{|N_G(v)| : v \in V(G)\}$ and the minimum degree $\delta(G)$ of G is given by $\delta(G) = \min\{|N_G(v)| : v \in V(G)\}$. A vertex is called an *endvertex* if its degree is 1. A vertex is called a *support vertex* if it is adjacent to an end-vertex. The $I(G)$ is the set containing all the isolated vertices of G.

The *open hop neighborhood* of a point u is the set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$. The *closed hop neighborhood* of u is $N^2_G[u] = N^2_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N^2_G(A) = \bigcup_{v \in A} N^2_G(v)$ is called the *open hop neighborhood of A* and $N^2_G[A] = N^2_G(A) \cup A$ is called the *closed hop neighborhood of A*. The *maximum hop degree* and *minimum hop degree* of G, denoted by $\Delta_h(G)$ and $\delta_h(G)$, respectively, is given by $\Delta_h(G) = \max\{|N^2_G(v)| : v \in V(G)\}$ and $\delta_h(G) = \min\{|N^2_G(v)| : v \in V(G)\}$.

A set $S \subseteq V(G)$ is a *hop dominating* set if $N^2_G[S] = V(G)$. The minimum cardinality of a hop dominating set of a graph G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A set $S \subseteq V(G)$ is a point-wise non-dominating set of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G is denoted by $pnd(G)$.

Let G be a non-trivial graph of order n and let $1 \leq k < \gamma_h(G)$. Let $S \subseteq V(G)$ with cardinality $|S| = \gamma_h(G) - k$. The set $V(G) \setminus N_G^2[S]$ is called the *k-hop defect set* of S and the *k-hop defect* of S is $\zeta_k^h(S) = |V(G) \setminus N_G^2[S]| = n - |N_G^2[S]|$. The minimum cardinality of a *k-hop defect set* in G, denoted by $\zeta_k^h(G)$, is called the *k-hop domination defect* of G, i.e.,

$$\zeta_k^h(G) = \min\{\zeta_k^h(S) : S \subseteq V(G) \text{ with } |S| = \gamma_h(G) - k\}.$$

A set $S \subseteq V(G)$ of cardinality $\gamma_h(G) - k$ for which $|V(G) \setminus N_G^2[S]| = \zeta_k^h(G)$ is called a ζ_k^h*-set* of G. Thus, $\langle N_G^2[S] \rangle$ is an induced subgraph of G with $n - \zeta_k^h(G)$ vertices and hop domination number $\gamma_h(G) - k$.

Let G and H be undirected graphs. The *join* $G + H$ of G and H is the graph with vertex-set $V(G + H) = V(G) \dot\cup V(H)$ and edge-set $E(G + H) = E(G) \dot\cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The graph $G - v$ is the graph $\langle V(G) \setminus \{v\} \rangle$ induced by $V(G) \setminus \{v\}$ and $G - \{u, v\} = \langle V(G) \setminus \{u, v\} \rangle$.

3. Results

Theorem 1 ([16]). *Let m and n be positive integers. Then each of the following holds.*

(i) *For a complete graph K_n, $\gamma_h(K_n) = n$.*

(ii) *For a complete bipartite graph $K_{m,n}$, $\gamma_h(K_{m,n}) = 2$.*

(iii) *For a path P_n on n vertices, we have*

$$\gamma_h(P_n) = \begin{cases} 2r & \text{if } n = 6r \\ 2r + 1 & \text{if } n = 6r + 1 \\ 2r + 2 & \text{if } n = 6r + s; \quad 2 \leq s \leq 5. \end{cases}$$

(iv) *For a cycle C_n on n vertices, we have*

$$\gamma_h(C_n) = \begin{cases} 2r & \text{if } n = 6r \\ 2r + 1 & \text{if } n = 6r + 1 \\ 2r + 2 & \text{if } n = 6r + s; \quad 2 \leq s \leq 5. \end{cases}$$

(v) *$\gamma_h(W_n) = 3$ where W_n is a wheel with n spokes.*

(vi) *$\gamma_h(P) = 2$ where P denotes the Petersen graph.*

Theorem 2. *[8] Let G and H be any two graphs of orders m and n, respectively. Then $\gamma_h(G + H) = pnd(G) + pnd(H)$. In particular,*

(i) *$\gamma_h(G + H) = m + n$ if G and H are complete;*

(ii) *$\gamma_h(G + H) = 2$ if G and H have isolated vertices;*

(iii) $\gamma_h(G + H) = 1 + pnd(H)$ *if* $G = K_1$;

(iv) $\gamma_h(G + H) = 4$ *if* $G = P_m$ *and* $H = P_n$ $(m, n \geq 2)$; *and*

(v) $\gamma_h(G + H) = 4$ *if* $G = C_m$ *and* $H = C_n$ $(mn \geq 4)$.

Theorem 3. *Let* G *be a graph with* $I(G) \neq \varnothing$ *and suppose* $|I(G)| = r$. *Then* $\zeta_j^h(G) = j$ *for every* $j \in [r] = \{1, 2, \cdots, r\}$ *and* $\zeta_k^h(G) = r + \zeta_{k-r}^h(G')$ *for every* $k \in \{r+1, \cdots, \gamma_h(G)-1\}$, *where* $G' = \langle V(G) \setminus I(G) \rangle$.

Proof. Let $I(G) = \{v_1, v_2, \cdots, v_r\}$ and let S be a γ_h-set in G. Then $I(G) \subseteq S$. Let $j \in [r]$. Then $D = S \setminus \{v_1, v_2, \cdots, v_j\}$ is a ζ_j^h-set in G and $|N_G^2[D]| = |N_G^2[S]| - |N_G^2[\{v_1, v_2, \cdots, v_j\}]| = |V(G)| - j$. Hence, $\zeta_j^h(G) = |V(G)| - (|V(G)| - j) = j$.

Next, let $k \in \{r+1, \cdots, \gamma_h(G)-1\}$. Then $S_0 = S \setminus I(G)$ is γ_h-set in $G' = \langle V(G) \setminus I(G) \rangle$. Hence, $\gamma_h(G') = \gamma_h(G) - r$. Since $k \leq \gamma_h(G) - 1$, $k - r \leq \gamma_h(G) - (r+1) < \gamma_h(G) - r$. Let S' be a ζ_{k-r}^h-set in G'. Then $|S'| = (\gamma_h(G) - r) - (k - r) = \gamma_h(G) - k$ and $\zeta_{k-r}^h(G') = |V(G')| - |N_{G'}^2[S']| = (|V(G)| - r) - |N_G^2[S']|$. This implies that $|V(G)| - |N_G^2[S']| = r + \zeta_{k-r}^h(G')$. Therefore, since S' is also a ζ_k^h-set in G, $\zeta_j^h(G) = r + \zeta_{k-r}^h(G')$. □

Remark 1. *A* ζ_k^h*-set of a graph* G *need not be contained in any* γ_h*-set of* G.

To see this, consider the graph in Figure 1 with $\gamma_h(G) = 2$. The set $\{1, 4\}$ is the only γ_h-set and $\{7\}$ is the only ζ_1^h-set in G and hence, $\zeta_1^h(G) = 5$. Observe that $\{7\} \nsubseteq \{1, 4\}$.

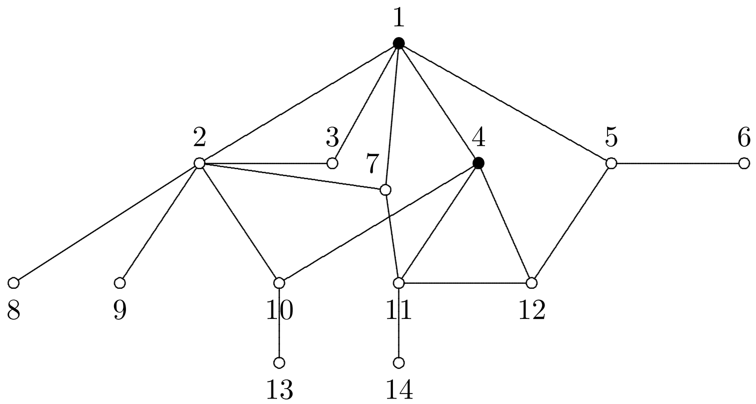


Figure 1: A graph G with $\gamma_h(G) = 2$.

In view of Theorem 3, all graphs considered henceforth, unless specified, do not have isolated vertices. Furthermore, given a graph G, the positive integer k, when not specified, always satisfies the condition $k \leq \gamma_h(G) - 1$.

Theorem 4. *If* G *is a graph of order* $n \geq 2$, *then* $1 \leq \zeta_k^h(G) \leq n - 1$.

Proof. Let S be a ζ_k^h-set in G. Then $|S| = \gamma_h(G) - 1$. Hence, $V(G) \setminus N_G^2[S] \neq \varnothing$. This implies that $\zeta_k^h(G) = |V(G)| - |N_G^2[S]| \geq 1$. Moreover, since $|N_G^2[S]| \geq 1$, $\zeta_k^h(G) = |V(G)| - |N_G^2[S]| \leq n - 1$. This proves the assertion. □

Theorem 5. *Let G be a non-trivial graph of order n. Then $\zeta_1^h(G) = 1$ if and only if there exists $v \in V(G)$ such that $\gamma_h(G - v) = \gamma_h(G) - 1$.*

Proof. Suppose $\zeta_1^h(G) = 1$ and let S be a ζ_1^h-set in G. Then $|S| = \gamma_h(G) - 1$ and $|V(G) \setminus N_G^2[S]| = 1$. Let $v \in V(G) \setminus N_G^2[S]$. Then $N_G^2[S] = V(G) \setminus \{v\}$. Therefore, $\gamma_h(G - v) = \gamma_h(\langle N_G^2[S]\rangle) = \gamma_h(G) - 1$.

Conversely, let $v \in V(G)$ such that $\gamma_h(G-v) = \gamma_h(G)-1$. Then, there exists $S' \subseteq V(G)$ with $|S'| = \gamma_h(G)-1$ and $N_G^2[S'] = V(G)\setminus\{v\}$. It follows that $\zeta_1^h(G) = |V(G)\setminus N_G^2[S']| = |\{v\}| = 1$. Therefore, $\zeta_1^h(G) = 1$. □

Theorem 6. *If G is a graph on n vertices, then $\zeta_k^h(G) \leq k(1 + \Delta_h(G))$.*

Proof. Let S be a γ_h-set in G. For each $v \in V(G)$, we have $|N_G^2[v]| \leq 1 + \Delta_h(G)$. Let $S' \subset S$ with $|S'| = k$ and set $S^* = S \setminus S'$. Then

$$\begin{aligned}
\zeta_k^h(S^*) &= |V(G) - N_G^2[S^*]| \\
&= |N_G^2[S]| - |N_G^2[S^*]| \\
&\leq |N_G^2[S']| + |N_G^2[S^*]| - |N_G^2[S^*]| \\
&= |N_G^2[S']| = \sum_{v\in S'} |N_G^2[v]| \\
&\leq k(1 + \Delta_h(G)).
\end{aligned}$$

Therefore, $\zeta_k^h(G) \leq k(1 + \Delta_h(G))$. □

Remark 2. *Let $G_1, G_2, \cdots, G_r$ be the components of a graph G. Then each of the following holds:*

(i) $\gamma_h(G) = \sum_{j=1}^{r} \gamma_h(G_j)$.

(ii) If $A_j \subseteq V(G_i)$ for each $j \in [r] = \{1, 2, \cdots, r\}$ and $A = \cup_{j=1}^{r} A_j$, then $N_G^2[A] = \cup_{j=1}^{r} N_G^2[A_j]$ (a disjoint union).

Theorem 7. *Let $G_1, G_2, \cdots, G_r$ be the components of graph G and let $\zeta_1^h(G_i)$ be the hop domination defect of G_i for each $i \in [r] = \{1, 2, \cdots, r\}$. Then*

$$\zeta_1^h(G) = \min\{\zeta_1^h(G_i) : i \in [r]\}.$$

Proof. Let $\gamma_h(G_i)$ and $\gamma_h(G)$ be the hop domination numbers of G_i and G, respectively. By Remark 2(i), $\gamma_h(G) = \sum_{j=1}^{r} \gamma_h(G_j)$. For each $i \in [r]$, let D_i be a ζ_1^h-set of G_i. Then $|D_i| = \gamma_h(G_i) - 1$ and $\zeta_1^h(G_i) = |V(G_i) - N_G^2[D_i]|$. Let $j \in [r]$ be such that $\zeta_1^h(G_j) = \min\{\zeta_1^h(G_i) : i \in [r]\}$. Let S_i be a γ_h-set in G_i for each $i \in [r]$ and let $S = (\cup_{i\in[r]\setminus\{j\}} S_i) \cup D_j$. Then

$$|S| = \sum_{i\in[r]\setminus\{j\}} |S_i| + |D_j| = \gamma_h(G) - 1$$

and, by Remark 2(ii),

$$\begin{aligned}|N_G^2[S]| &= |N_{G_j}^2[D_j]| + \sum_{i\in[r]\setminus\{j\}} |N_{G_i}^2[S_i]| \\ &= |V(G_j)| - \zeta_1^h(G_j) + \sum_{i\in[r]\setminus\{j\}} |V(G_i)| \\ &= \sum_{i=1}^{r} |V(G_i)| - \zeta_1^h(G_j).\end{aligned}$$

Thus, in G, $\zeta_1^h(S) = |V(G)| - |N_G^2[S]| = \zeta_1^h(G_j)$. It now remains to show that $\zeta_1^h(S)$ is the minimum among all subsets of $V(G)$ with cardinality $\gamma_h(G) - 1$. So assume there exists $S^{'} \subseteq V(G)$ such that $|S^{'}| = \gamma_h(G) - 1$ and $\zeta_1^h(S^{'}) < \zeta_1^h(S)$. Let $S^{'} = S_1^{'} \cup S_2^{'} \cup \cdots \cup S_r^{'}$ where $S_i^{'} \subseteq V(G_i)$ for each $i \in [r]$. Since $|S^{'}| = \gamma_h(G) - 1$, at least one $S_l^{'}$ is not a hop dominating set of G_l by Remark 2(i). Thus, $|S_l^{'}| = \gamma_h(G_l) - 1$ and $\zeta_1^h(S_l^{'}) \geq \zeta_1^h(G_l) \geq \zeta_1^h(G_j)$. Hence,

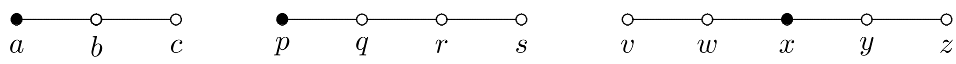
$$\begin{aligned}\zeta_1^h(S^{'}) = |V(G)| - |N_G^2[S^{'}]| &= \sum_{i=1}^{r}(|V(G_i)| - |N_{G_i}^2[S_i^{'}]|) \\ &\geq |V(G_l)| - |N_{G_l}^2[S_l^{'}]| \\ &= \zeta_1^h(S_l^{'}) \\ &\geq \zeta_1^h(G_j) \\ &= \zeta_1^h(S),\end{aligned}$$

contrary to the assumption that $\zeta_1^h(S^{'}) < \zeta_1^h(S)$. Therefore, $\zeta_1^h(G) = \zeta_1^h(S) = \zeta_1^h(G_j)$. □

Theorem 8. *$\zeta_k^h(K_n) = k$ for all $n \geq 2$ and $1 \leq k < n$.*

Proof. By Theorem 1(i), $\gamma_h(K_n) = n$. Let k be a positive integer such that $1 \leq k \leq n-1$ and let S be any subset of $V(K_n)$ with $|S| = \gamma_h(K_n) - k = n - k$. Since $\langle S \rangle$ is a complete graph, $N_{K_n}^2[S] = S$. This implies that $\zeta_k^h(S) = |V(K_n) \setminus N_{K_n}^2[S]| = n - |S| = n - (n-k) = k$. Since S was arbitrarily chosen, it follows that $\zeta_k^h(K_n) = k$. □

Note that $\gamma_h(P_3) = \gamma_h(P_4) = \gamma_h(P_5) = 2$. Therefore, $k = 1$. Consider $P_3 = [a, b, c]$, $P_4 = [p, q, r, s]$, and $P_5 = [v, w, x, y, z]$ below. Then $S_1 = \{a\}$, $S_2 = \{p\}$, and $S_3 = \{x\}$ are ζ_1^h-sets in P_3, P_4, and P_5, respectively. Since $|N_{P_3}^2[S_1]| = 2$, $|N_{P_4}^2[S_2]| = 2$, and $|N_{P_5}^2[S_3]| = 3$, it follows that $\zeta_1^h(P_3) = 1$, and $\zeta_1^h(P_4) = \zeta_1^h(P_5) = 2$.



Theorem 9. *If P_n is a path on $n \geq 6$ vertices, then*

$$\zeta_k^h(P_n) = \begin{cases} 3k & \text{if } n = 6r \\ 3k-2 & \text{if } n = 6r+1 \\ 3k+s-6 & \text{if } n = 6r+s;\ 2 \leq s \leq 5 \text{ and } 2 \leq k \leq \gamma_h(P_n)-1 \end{cases}$$

Proof. We denote the vertices of P_n as $\{1, 2, \cdots, n\}$. Now, consider the following cases:

Case 1: Suppose $n = 6r$. By Theorem 1 (iii), $\gamma_h(P_n) = 2r$. Choose a $(2r-k)$-element set $S = \{3, 6, \cdots, 6r-3k\}$. This implies that

$$\zeta_k^h(S) = n - |N^2_{P_n}[S]| = n - [(2r-k) + 2(2r-k)] = 6r - [6r-3k] = 3k.$$

Thus, it is the minimum value for any set S with $2r-k$ vertices. Hence, $\zeta_k^h(P_n) = 3k$.

Case 2: Suppose $n = 6r+1$. By Theorem 1 (iii), $\gamma_h(P_n) = 2r+1$. Choose a $(2r-k+1)$-element set $S = \{3, 6, \cdots, 6r-3k+3\}$. This implies that $\zeta_k^h(S) = n - |N^2_{P_n}[S]| = n - [(2r-k+1) + 2(2r-k+1)] = 6r - [6r-3k+3] + 1 = 3k-2$. Thus, it is the minimum value for any set S with $2r-k+1$ vertices. Hence, $\zeta_k^h(P_n) = 3k-2$.

Case 3: Suppose $n = 6r+s$ where $2 \leq s \leq 5$. By Theorem 1 (iii), $\gamma_h(P_n) = 2r+2$. Choose a $(2r-k+2)$-element set $S = \{3, 6, \cdots, 6r-3k+6\}$ for $2 \leq k \leq \gamma_h(P_n)-1$. This implies that $\zeta_k^h(S) = n - |N^2_{P_n}[S]| = n - [(2r-k+2) + 2(2r-k+2)] = 6r - [6r-3k+6] + s = 3k+s-6$. Thus, it is the minimum value for any set S with $2r-k+2$ vertices. Hence, $\zeta_k^h(P_n) = 3k+s-6$. □

From Theorem 8, we have $\zeta_k^h(C_3) = k$ for $k \in \{1, 2\}$. Now, since $\gamma_h(C_4) = \gamma_h(C_5) = 2$, $k = 1$. It can easily be verified that $|N^2_{C_4}[S]| = 2$ and $|N^2_{C_5}[S']| = 3$ for any singleton subsets S and S' of $V(C_4)$ and $V(C_5)$, respectively. Hence, $\zeta_k^h(C_4) = \zeta_k^h(C_5) = 2$.

The proof of the next result uses Theorem 1(iv) and follows along the same lines as that of Theorem 9.

Theorem 10. *If C_n is a cycle on $n \geq 6$ vertices, then*

$$\zeta_k^h(C_n) = \begin{cases} 3k & \text{if } n = 6r \\ 3k-2 & \text{if } n = 6r+1 \\ 3k+s-6 & \text{if } n = 6r+s;\ 2 \leq s \leq 5 \text{ and } 2 \leq k \leq \gamma_h(C_n)-1. \end{cases}$$

Lemma 1. *Let G be a nontrivial connected graph with $\gamma_h(G) \geq 2$ and let $k = \gamma_h(G)-1$. Then $S \subseteq V(G)$ is a ζ_k^h-set of G if and only if $S = \{x\}$ for some $x \in V(G)$ with $N^2_G(x) = \triangle_h(G)$.*

Proof. Let $k = \gamma_h(G)-1$ and let $S \subseteq V(G)$ be a ζ_k^h-set of G. Then $|S| = 1$, say,

$S = \{x\}$ and $\zeta_k^h(S) = n - |N_G^2[x]| = \zeta_k^h(G)$. This implies that $|N_G^2[x]|$ is maximum in G. Thus, $N_G^2(x) = \triangle_h(G)$.

Conversely, suppose $S = \{x\}$ with $N_G^2(x) = \triangle_h(G)$. Since $n - |N_G^2[S]| = n - |N_G^2[x]| = n - \triangle_h(G)$ is the minimum possible value for any singleton subset of $V(G)$, it follows that S is a ζ_k^h-set of G. □

Theorem 11. *If $G = K_{m_1,m_2,\cdots,m_t}$ is a complete multipartite graph with $1 \leq m_1 \leq m_2 \leq \cdots \leq m_t$, then $\gamma_h(G) = t$.*

Proof. Let $Q_1, Q_2, \cdots, Q_t$ be the partite sets of G and let S be a γ_h-set of G. Suppose there exists $j \in [t] = \{1, 2, \cdots, t\}$ such that $S \cap Q_j = \varnothing$. Then $Q_j \subseteq N_G(S)$. Hence, the vertices in Q_j are not hop dominated by any element of S. This implies that S is not a hop dominating set in G, a contradiction. Therefore, $S \cap Q_j \neq \varnothing$ for every $j \in [t]$. Since S is a γ_h-set of G, $|S \cap Q_j| = 1$ for every $j \in [t]$. Accordingly, $\gamma_h(G) = |S| = t$. □

Theorem 12. *For a complete multipartite graph $G = K_{m_1,m_2,\cdots,m_t}$ where $m_1 \leq m_2 \leq \cdots \leq m_t$, $\zeta_k^h(G) = \sum_{j=1}^k m_j$.*

Proof. Let $Q_1, Q_2, \cdots, Q_t$ be the partite sets of G with cardinalities $m_1 \leq m_2 \leq \cdots \leq m_t$. By Theorem 11, $\gamma_h(G) = t$. Choose a $(t-k)$-element set $S = \{q_{k+1}, q_{k+2}, \cdots, q_t\}$ where $q_j \in Q_j$ for each $j \in \{k+1, k+2, \cdots, t\}$. Then $N_G^2[S] = \cup_{j=k+1}^t N_G^2[q_j] = \cup_{j=k+1}^t Q_j$. It follows that $|N_G^2[S]| = \sum_{j=k+1}^t |Q_j| = \sum_{j=k+1}^t m_j$. This is the maximum value that can be obtained for any set S with $t-k$ vertices because $m_1 \leq m_2 \leq \cdots \leq m_t$. Thus,

$$\begin{aligned} \zeta_k^h(G) &= \zeta_k^h(S) \\ &= |V(G)| - |N_G^2[S]| \\ &= \sum_{j=1}^t m_j - \sum_{j=k+1}^t m_j \\ &= \sum_{j=1}^k m_j. \end{aligned}$$

The next result is a consequence of Theorem 12.

Corollary 1. *For a complete bipartite graph $K_{m,n}$ where $2 \leq m \leq n$, $\zeta_k^h(K_{m,n}) = m$.*

Theorem 13. *For a Petersen graph P, $\zeta_k^h(P) = 3$.*

Proof. By Theorem 1(vi), $\gamma_h(P) = 2$. It follows that $k = 1$. Let S be a ζ_k^h-set of P. Then $|S| = 1$, say $S = \{v\}$. Since $|N_P^2[x]| = 7$ for every $x \in V(P)$, it follows that $|N_P^2[S]| = 7$. Therefore, $\zeta_k^h(P) = \zeta_k^h(S) = |V(P)| - |N_P^2[S]| = 10 - 7 = 3$. □

Theorem 14. *Let G be a graph with $diam(G) \geq 3$ and let G' be a graph obtained by adding any number of edges xy to $E(G)$ with $d_G(x, y) \geq 3$ such that $\gamma_h(G) = \gamma_h(G')$. Then $\zeta_k^h(G') \leq \zeta_k^h(G)$ where $1 \leq k \leq \gamma_h(G) - 1$.*

Proof. Let $1 \leq k \leq \gamma_h(G) - 1$ and let S be a ζ_k^h-set of G. Since $\gamma_h(G) = \gamma_h(G')$, $|S| = \gamma_h(G) - k = \gamma_h(G') - k$. Let $x \in N_G^2[S]$. If $x \in S$, then $x \in N_{G'}^2[S]$. If $x \in N_G^2(S) \setminus S$, then there exists $z \in S$ such that $d_G(x, z) = 2$. It follows that $xz \notin E(G')$. Hence, $d_{G'}(x, z) = 2$. This implies that $x \in N_{G'}^2(S)$. Thus, $N_G^2[S] \subseteq N_{G'}^2[S]$. Therefore,

$$\begin{aligned}\zeta_k^h(G') &\leq n - |N_{G'}^2[S]| \\ &\leq n - |N_G^2[S]| \\ &= \zeta_k^h(G). \quad \square\end{aligned}$$

Theorem 15. *Let G be a graph of order n and let v be a vertex with the property that for every pair of vertices x and y with $d_G(x, y) = 2$ and $v \in N_G(x) \cap N_G(y)$, it holds that $|N_G(x) \cap N_G(y)| \geq 2$. If $G' = \langle V(G) \setminus \{v\} \rangle$ and $\gamma_h(G) > \gamma_h(G') \geq 2$, then $\zeta_{k+1}^h(G) \leq \zeta_k^h(G') + 1$ where $1 \leq k \leq \gamma_h(G) - 2$.*

Proof. Let S be a γ_h-set in G'. Then $v \notin S$. Let $S' = S \cup \{v\}$ and let $x \in V(G) \setminus S'$. Since $x \notin S'$, $x \notin S$ and $x \neq v$. Hence, $x \in V(G') \setminus S$. Since S is a hop dominating set in G', there exists $z \in S$ such that $d_{G'}(z, x) = 2$. This implies that there exists $z \in S'$ such that $d_G(z, x) = 2$. Therefore, S' is a hop dominating set in G and $\gamma_h(G) \leq |S'| = \gamma_h(G') + 1$. Since $\gamma_h(G') < \gamma_h(G)$, $\gamma_h(G') + 1 \leq \gamma_h(G)$. Thus, $\gamma_h(G) = \gamma_h(G') + 1$.

Now let D be a ζ_k^h-set of G' where $1 \leq k \leq \gamma_h(G) - 2$. Then $|D| = \gamma_h(G') - k$ and $\zeta_k^h(G') = (n-1) - |N_{G'}^2[D]|$. This implies that $|N_{G'}^2[D]| = (n-1) - \zeta_k^h(G')$. Since $|D| = \gamma_h(G') - k = \gamma_h(G) - (k+1)$, it follows that $\zeta_{k+1}^h(G) \leq n - |N_G^2[D]|$. Consider the following cases:

Case 1: $|N_G^2[D]| = |N_{G'}^2[D]|$.
Then $(n-1) - \zeta_k^h(G') = |N_{G'}^2[D]| = |N_G^2[D]| \leq n - \zeta_{k+1}^h(G)$. Thus, we have $\zeta_{k+1}^h(G) \leq \zeta_k^h(G') + 1$.

Case 2: $|N_G^2[D]| \neq |N_{G'}^2[D]|$.
Since $N_{G'}^2[D] \subseteq N_G^2[D]$, the assumption implies that there exists $w \in N_G^2(D) \setminus N_{G'}^2(D)$. Suppose $w \neq v$. Since $w \in N_G^2(D)$, there exists $u \in D$ such that $d_G(w, u) = 2$. Since $u \in D$, $u \neq v$. The assumption that $w \notin N_{G'}^2(D)$ would imply that $v \in N_G(w) \cap N_G(u)$ and the path $[w, v, u]$ is the only w-u geodesic in G, contradicting the property of v. Hence, $w = v$. Therefore, $|N_G^2[D]| = |N_{G'}^2[D]| + 1$ and

$$(n-1) - \zeta_k^h(G') = |N_{G'}^2[D]| = |N_G^2[D]| - 1 \leq n - \zeta_{k+1}^h(G) - 1.$$

Thus, $\zeta_{k+1}^h(G) \leq \zeta_k^h(G') \leq \zeta_k^h(G') + 1$.

Therefore, the assertion holds. □

Remark 3. *Equality of the two expressions (defects) given in Theorem 15 is attainable.*

Consider $G = W_4 = \langle\{v\}\rangle + [a, b, c, d, a]$. Let $G' = G - v = C_4 = [a, b, c, d, a]$. Observe that v satisfies the property given in Theorem 15, and $\gamma_h(G) = 3 > 2 = \gamma_h(G')$. Then $k = \gamma_h(G) - 2 = 1$. If D is ζ^h_k-set in G', then $|D| = \gamma_h(G') - 1 = 1$. In this case, we may take any of the four vertices of C_4 as the element of D, say $D = \{a\}$. Then $N^2_G[a] = \{a, c\}$ and $N^2_{G'}[a] = \{a, c\}$. It follows that

$$\zeta^h_{k+1}(G) = |V(G)| - |N^2_G[a]| = 3 = (|V(G')| - |N^2_{G'}[a]|) + 1 = \zeta^h_k(G') + 1.$$

Theorem 16. *Let G be a graph of order n such that $\gamma_h(G - v) = \gamma_h(G)$ for every $v \in V(G)$. If there exist $u, v \in V(G)$ such that $uv \notin E(G)$ and $\gamma_h(G - \{u, v\}) < \gamma_h(G)$, then $\zeta^h_1(G) = 2$.*

Proof. Let $u, v \in V(G)$ be such that $uv \notin E(G)$ and $\gamma_h(G - \{u, v\}) < \gamma_h(G)$. For convenience, let $H = G - \{u, v\}$. Then $\gamma_h(H) + 1 \leq \gamma_h(G)$. Let S be a γ_h-set of H. Then $u \notin S$. Set $S' = S \cup \{u\}$ and let $H' = G - v$. Let $x \in H' \setminus S'$. Then $x \notin \{u, v\}$. Hence, $x \in H \setminus S$. Since S is a hop dominating set of H, there exists $y \in S$ such that $d_H(x, y) = 2$. It follows that $y \in S'$ and $d_{H'}(x, y) = 2$. This shows that S' is a hop dominating set in H'. Since $\gamma_h(H)+1 \leq \gamma_h(G) = \gamma_h(H') \leq |S'| = \gamma_h(H)+1$, $\gamma_h(G) = \gamma_h(H') = |S'| = \gamma_h(H)+1$, that is, $\gamma_h(H) = \gamma_h(G) - 1$. Hence, $|S| = \gamma_h(G) - 1$. Clearly, $V(H) \subseteq N^2_G[S]$. Suppose $N^2_G[S] \neq V(H)$. Then $u \in N^2_G(S)$ or $v \in N^2_G(S)$, say $v \in N^2_G(S)$. Then there exists $w \in S$ such that $d_G(v, w) = 2$. Let $[v, p, w]$ be a v-w geodesic in G. Since $uv \notin E(G)$, $p \neq u$. This implies that $[v, p, w]$ is a v-w geodesic in $H^* = G \setminus u$. Hence, $d_{H^*}(v, w) = 2$. It follows that S is a hop dominating set H^*. This, however, is not possible because $\gamma_h(H) = |S| < \gamma_h(G) = \gamma_h(H^*)$ by assumption. Therefore, $N^2_G[S] = V(H)$. It follows that $\zeta^h_1(S) = n - |V(H)| = n - (n - 2) = 2$. Since $\gamma_h(G - \{v\}) = \gamma_h(G)$ for every $v \in V(G)$, $\zeta^h_1(G) \neq 1$ by Theorem 5. Therefore, $\zeta^h_1(G) = \zeta^h_1(S) = 2$. □

Example 1. *Let G be a graph obtained from $C_7 = [v_1, v_2, \cdots, v_7, v_1]$ by adding the pendant edge pv_1. Then $\gamma_h(G) = 7$ ($S = \{v_1, v_2, v_5\}$ is a γ_h-set in G). One can easily verify that $\gamma_h(G \setminus v) = \gamma_h(G)$ for every $v \in V(G)$. Consider the non-adjacent vertices p and v_2 of G. Then $G \setminus \{p, v_2\} = P_6$. Hence, $\gamma_h(G \setminus \{p, v_2\}) = 2 < \gamma_h(G)$. By Theorem 16, $\zeta^h_1(G) = 2$.*

It should be noted that the converse of Theorem 16 is not true. To see this, consider $G = C_4$. Then $\gamma_h(G) = 2$ and $\gamma_h(G \setminus v) = \gamma_h(P_3) = 2 = \gamma_h(G)$ for every $v \in V(C_4)$. Moreover, $\zeta^h_1(G) = 2$. However, one cannot find non-adjacent vertices $p, q \in V(G)$ such that $\gamma_h(G \setminus \{p, q\} = 1$.

Lemma 2. *Let G and H be any two graphs of orders m and n, respectively. If $x \in V(G + H)$ and $|N^2_{G+H}(x)| = \Delta_h(G + H)$, then*

$$|N^2_{G+H}[x]| = \max\{m - \delta(G), n - \delta(H)\}.$$

Proof. Let $x \in V(G)$. Since $V(H) \subseteq N_G(x)$, it follows that $N^2_{G+H}[x] \cap V(H) = \varnothing$. Hence, $N^2_{G+H}[x] \subseteq V(G)$. Now $p \in N^2_{G+H}[x]$ if and only if $p = x$ or $d_{G+H}(x, p) = 2$. This implies that $p \in N^2_{G+H}[x]$ if and only if $p \in V(G) \backslash N_G(x)$. Thus, $N^2_{G+H}[x] = V(G) \backslash N_G(x)$. Consequently, $|N^2_{G+H}[x]| = m - |N_G(x)| = m - deg_G(x)$. Clearly,

$$\max\{|N^2_{G+H}[x]| : x \in V(G)\} = \max\{m - deg_G(x) : x \in V(G)\} = m - \delta(G).$$

Similarly,

$$\max\{|N^2_{G+H}[x]| : x \in V(H)\} = \max\{n - deg_H(x) : x \in V(H)\} = n - \delta(H).$$

Therefore, if $x \in V(G+H)$ such that $\Delta_h(G+H) = |N^2_{G+H}(x)|$, then

$$|N^2_{G+H}[x]| = \max\{m - \delta(G), n - \delta(H)\}.$$

This proves the assertion. □

Theorem 17. *Let G and H be graphs of orders m and n, respectively. Then each of the following holds:*

(i) $\zeta^h_k(G+H) = \min\{m + \delta(H), n + \delta(G)\}$ *where* $k = \gamma_h(G+H) - 1$.

(ii) $\zeta^h_1(G+H) = 1$ *if and only if there exists* $v \in V(G+H)$ *such that* $v \in V(G)$ *and* $pnd(G-v) = pnd(G) - 1$ *or* $v \in V(H)$ *and* $pnd(H-v) = pnd(H) - 1$.

(iii) *If* $I(G) \neq \varnothing$ *and* $I(H) \neq \varnothing$, *then* $\zeta^h_1(G+H) = \min\{m, n\}$.

(iv) *If* $I(G) \neq \varnothing$ *and* $H = K_n$, *then* $\zeta^h_k(G+H) = k$, *where* $1 \leq k \leq n$.

Proof. *(i)* Let S be a ζ^h_k-set in $G+H$. By Lemma 1, $S = \{x\}$ where $|N^2_{G+H}(x)| = \Delta_h(G+H)$. By Lemma 2, we may assume without loss of generality that $|N^2_{G+H}[x] = m - \delta(G) \geq n - \delta(H)$ for $x \in V(G)$. Then

$$\zeta^h_k(G+H) = (m+n) - (m - \delta(G)) = n + \delta(G) \leq m + \delta(H) = (m+n) - (n - \delta(H)).$$

(ii) By Theorem 5 and Theorem 2, $\zeta^h_1(G+H) = 1$ if and only if there exists a vertex $v \in V(G+H)$ such that $\gamma_h((G+H)-v) = \gamma_h((G+H)) - 1 = pnd(G) + pnd(H) - 1$. If $v \in V(G)$, then $(G+H)-v = (G-v)+H$. Otherwise, $(G+H)-v = G+(H-v)$. By Theorem 2, $\gamma_h((G+H)-v) = pnd(G-v) + pnd(H)$ or $\gamma_h((G+H)-v) = pnd(G) + pnd(H-v)$. Therefore, $\zeta^h_1(G+H) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $pnd(G-v) = pnd(G) - 1$ or $v \in V(H)$ with $pnd(H-v) = pnd(H) - 1$.

(iii) If $I(G) \neq \varnothing$ and $I(H) \neq \varnothing$, then $\gamma_h(G+H) = 2$, by Theorem 2. Thus, $k = 1$. Let $p \in I(G)$ and $q \in I(H)$. Since $N^2_G(p) = V(G) \setminus \{p\}$ and $N^2_H(q) = V(H) \setminus \{q\}$, it follows that $\Delta_h(G+H) = \max\{|N^2_G(p)|, |N^2_H(q)|\}$. Therefore, since $\delta(G) = |N_G(p)| = 0$ and $\delta(H) = |N_H(q)| = 0$, it follows from *(i)* that $\zeta^h_1(G+H) = \min\{m, n\}$.

(iv) Since $pnd(G) = 1$ and $pnd(K_n) = n$, $\gamma_h(G+K_n) = n+1$ by Theorem 2. Let k be such that $1 \leq k \leq n$ and let $S = S_G \cup S_n$ be a ζ^h_k-set in $G+K_n$, where $S_G \subseteq V(G)$ and

$S_n \subseteq V(K_n)$. Then $|S| = |S_G| + |S_n| = (n+1) - k$ and

$$|N^2_{G+K_n}[S]| = |N^2_{G+K_n}[S_G]| + |N^2_{G+K_n}[S_n]| = |N^2_{G+K_n}[S_G]| + |S_n|.$$

Since $N^2_{G+K_n}[t] = V(G)$ for any $t \in I(G)$ and the value $|N^2_{G+K_n}[S]|$ is maximum, it follows that $S_G = \{w\}$ for some $w \in I(G)$. This implies that $|S_n| = (n+1) - k - 1 = n - k$. Therefore,

$$\zeta^h_k(G+H) = (m+n) - |N^2_{G+K_n}[S]| = (m+n) - (m+n-k) = k. \quad \square$$

4. Conclusion

In this paper, we introduced and studied a new graph invariant called the k-hop domination defect of a graph. We obtained the k-hop domination defects of some known graphs including the join of some graphs. Also, we provided some bounds on the k-hop domination defect of a graph G in terms of its order and maximum hop degree and characterized the graphs that yield a k-domination defect equal to 1. It is recommended that the newly defined parameter be studied further for other classes of graphs.

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