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Upper and Lower Weakly s-(τ_1, τ_2)-continuous Multifunctions

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Abstract. This article presents new classes of multifunctions called upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions are discussed.

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1. Introduction

In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of continuity for functions and multifunctions. As weak forms of continuity in topological spaces, weak continuity [44], quasicontinuity [47], semi-continuity [45] and almost continuity in the sense of Husain [36] are well-known. Lee [43] studied the concept of semiconnected functions. Kohli [40] introduced the notion of s-continuous functions and investigated some characterizations of semilocally connected spaces in terms of s-continuous functions. The notion of s-continuity as a generalization of continuity and semiconnectedness. Moreover, Kohli [41] introduced the concepts of s-regular spaces and completely s-regular spaces and proved that s-regularity and complete s-regularity are preserved under certain s-continuous functions. Viriyapong and Boonpok [68] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [12]. Dungthaisong et al. [33] introduced

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and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [32] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions and slightly $(\tau_1, \tau_2)s$-continuous functions were presented in [60], [63], [16], [54], [25], [11], [8], [10], [4], [1], [2], [26], [23], [18] and [59], respectively. Srisarakham et al. [61] introduced and studied the concept of faintly (τ_1, τ_2)-continuous functions. Thongmoon et al. [66] introduced and investigated the notion of rarely (τ_1, τ_2)-continuous functions. Kong-ied at al. [42] introduced and studied the concept of almost quasi (τ_1, τ_2)-continuous functions. Chiangpradit et al. [30] introduced and investigated the notion of weakly quasi (τ_1, τ_2)-continuous functions. Prachanpol et al. [53] introduced and studied the concept of weakly $\delta(\tau_1, \tau_2)$-continuous functions.

In 1989, Lipski [46] extended the concept of s-continuous functions to the setting of multifunctions. Popa [49] introduced the concept of precontinuous multifunctions and showed that H-almost continuity and precontinuity are equivalent for multifunctions. Ewert and Lipski [35] introduced and studied the concept of s-quasi-continuous multifunctions. Popa and Noiri [52] introduced and investigated the notion of s-precontinuous multifunctions as a generalization of s-continuous multifunctions and precontinuous multifunctions. In particular, Popa and Noiri [51] introduced and studied the notion of s-β-continuous multifunctions. Popa and Noiri [50] introduced and investigated the concept of s-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. Moreover, several characterizations and some properties concerning $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, almost quasi (τ_1, τ_2)-continuous multifunctions, c-(τ_1, τ_2)-continuous multifunctions, c-quasi (τ_1, τ_2)-continuous multifunctions, s-$(\tau_1, \tau_2)p$-continuous multifunctions, slightly (τ_1, τ_2)-continuous multifunctions and slightly $(\tau_1, \tau_2)p$-continuous multifunctions were established in [5], [28], [69], [3], [7], [17], [24], [6], [21], [20], [15], [9], [19], [22], [37], [13], [27], [62], [14], [57], [39], [65], [58], [56], [38], [55], [73], [70] and [71], respectively. Noiri and Popa [48] introduced and studied the notion of weakly s-m-continuous multifunctions as a generalization of both weakly m-continuous multifunctions and s-m-continuous multifunctions. The class of weakly s-m-continuous multifunctions contains weakly s-precontinuous multifunctions due to Ekici and Park [34]. In this paper, we introduce the concepts of upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [29] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [29] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [29] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [29] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is called $\alpha(\tau_1, \tau_2)$-*open* [72] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [67] (resp. $(\tau_1, \tau_2)s$-*open* [5], $(\tau_1, \tau_2)p$-*open* [5], $(\tau_1, \tau_2)\beta$-*open* [5]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [67] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [67] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [67] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [67] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2. [67] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower wekly s-(τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of upper weakly s-(τ_1, τ_2)-continuous multifunctions and lower weakly s-(τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly s-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly s-(τ_1, τ_2)-continuous if F is upper weakly s-(τ_1, τ_2)-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly s-(τ_1, τ_2)-continuous;

(2) $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B))))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-connected;

(6) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(7) $\tau_1\tau_2$-*Cl*$(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(8) $\tau_1\tau_2$-*Cl*$(F^-(\sigma_1\sigma_2$-*Int*$(K))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-connected* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having the $\sigma_1\sigma_2$-connected complement and $x \in F^+(V)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, we have $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\tau_1\tau_2$-open, we have $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y having $\sigma_1\sigma_2$-connected complement. By (2), we have

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \tau_1\tau_2\text{-Int}(F^+(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= \tau_1\tau_2\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(3) ⇒ (4): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-connected in Y and by (3), we have

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4) ⇒ (5): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-connected. Then by (4),

$$\begin{aligned} X - \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= \tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By (5), we have

$$x \in F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence F is upper weakly s-(τ_1, τ_2)-continuous.

(4) ⇒ (6) and (6) ⇒ (7): The proofs are obvious.

(7) ⇒ (8): Let K be any $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)r$-closed set of Y. Then, we have $K = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $\sigma_1\sigma_2$-connected and by (7),

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).$$

(8) ⇒ (3): Let K be any $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set of Y. Since K is $\sigma_1\sigma_2$-connected, we have $\sigma_1\sigma_2\text{-Int}(K)$ is $\sigma_1\sigma_2$-connected and hence $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $\sigma_1\sigma_2$-connected. Let $H = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$. Then, H is a $(\sigma_1, \sigma_2)r$-closed $\sigma_1\sigma_2$-connected set of Y and $\sigma_1\sigma_2\text{-Int}(H) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = \sigma_1\sigma_2\text{-Int}(K)$. By (8), we have $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) = \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(H))) \subseteq F^-(H) \subseteq F^-(K)$.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly s-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for each $z \in U$. A multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is said to be lower weakly s-(τ_1, τ_2)-continuous if F is lower weakly s-(τ_1, τ_2)-continuous at each point x of X.

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly s-(τ_1, τ_2)-continuous;

(2) $F^-(V) \subseteq \tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ for every subset B of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-connected;

(6) $\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(7) $\tau_1\tau_2\text{-Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(8) $\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly s-(τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-$Cl(F^-(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^-(\sigma_1\sigma_2$-$Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(3) $\tau_1\tau_2$-$Cl(F^-(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^-(\sigma_1\sigma_2$-$Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure.

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 1(4).

(2) $\Rightarrow$ (3): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)s$-open, the proof follows from Theorem 1(7).

Theorem 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly s-(τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(3) $\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly s-(τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-$Cl(F^-(\sigma_1\sigma_2$-$Int((\sigma_1, \sigma_2)\theta$-$Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta$-$Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)\theta$-closure;

(3) $\tau_1\tau_2$-$Cl(F^-(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta$-$Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)\theta$-closure.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)\theta$-closure. Then, $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed and by Theorem1,

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(2) ⇒ (3): The proof is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq (\sigma_1, \sigma_2)\theta$-Cl$(B)$ for every subset B of Y.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed $\sigma_1\sigma_2$-connected set of Y. Then, we have $(\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = K$ and by (3),

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(K))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(K))) \\ &= F^-(K).\end{aligned}$$

Thus, $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ and by Theorem 1(8), F is upper weakly s-(τ_1, τ_2)-continuous.

Theorem 6. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly s-(τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)\theta$-closure;

(3) $\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)\theta$-closure.

Proof. The proof is similar to that of Theorem 5.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\text{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [29] we denote a multifunction defined as follows: $\text{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ for each $x \in X$.

Definition 3. [29] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$.

Lemma 3. [29] *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then $\text{Cl}F^+_{\circledast}(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Lemma 4. [29] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $\text{Cl}F^-_{\circledast}(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 7. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, F is upper weakly s-(τ_1, τ_2)-continuous if and only if $\text{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly s-(τ_1, τ_2)-continuous.*

Proof. Suppose that F is upper weakly s-(τ_1, τ_2)-continuous. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. By Theorem 1, Lemma 3 and Lemma 4, we have $\text{Cl}F^+_{\circledast}(V) = F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(\text{Cl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V)))$. Thus by Theorem 1, $\text{Cl}F_{\circledast}$ is upper weakly s-(τ_1, τ_2)-continuous.

Conversely, suppose that $\text{Cl}F_{\circledast}$ is upper weakly s-(τ_1, τ_2)-continuous. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. By Theorem 1, Lemma 3 and Lemma 4, we have

$$F^+(V) = \text{Cl}F^+_{\circledast}(V) \subseteq \tau_1\tau_2\text{-Int}(\text{Cl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

By Theorem 1, F is upper weakly s-(τ_1, τ_2)-continuous.

Theorem 8. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, F is lower weakly s-(τ_1, τ_2)-continuous if and only if $\text{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly s-(τ_1, τ_2)-continuous.*

Proof. The proof is similar to that of Theorem 7.

4. Some results on weak s-(τ_1, τ_2)-continuity

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [29] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed.

Definition 4. [29] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-connected if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets.*

Definition 5. [64] *A bitopological space (X, τ_1, τ_2) is said to be s-$\tau_1\tau_2$-connected if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets having $\tau_1\tau_2$-connected complements.*

Theorem 9. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an upper or lower weakly s-(τ_1, τ_2)-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$ and (X, τ_1, τ_2) is $\tau_1\tau_2$-connected, then (Y, σ_1, σ_2) is s-$\sigma_1\sigma_2$-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. There exist nonempty $\sigma_1\sigma_2$-open sets U and V of Y having $\sigma_1\sigma_2$-connected complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we shall show that $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-open in X. (i) Let F be upper weakly s-(τ_1, τ_2)-continuous. By Theorem 1,

$$F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(F^+(V))$$

since V is $\sigma_1\sigma_2$-clopen. Thus, $F^+(V) = \tau_1\tau_2\text{-Int}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-open in X. Similarly, we obtain $F^+(U)$ is $\tau_1\tau_2$-open in X. This shows that (X, τ_1, τ_2) is not $\tau_1\tau_2$-connected. (ii) Let F be lower weakly s-(τ_1, τ_2)-continuous. Since V is a $\sigma_1\sigma_2$-clopen set with $\sigma_1\sigma_2$-connected complement, by Theorem 2

$$\tau_1\tau_2\text{-Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V)) = F^+(V).$$

Thus, $F^+(V) = \tau_1\tau_2\text{-Cl}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-closed in X. Therefore, $F^+(U)$ is $\tau_1\tau_2$-open in X. Similarly, we obtain $F^+(V)$ is $\tau_1\tau_2$-open in X. Consequently, this shows that (X, τ_1, τ_2) is not $\tau_1\tau_2$-connected. This completes the proof.

The $\tau_1\tau_2$-*frontier* [26] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2\text{-fr}(A)$, is defined by $\tau_1\tau_2\text{-fr}(A) = \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A) = \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A)$.

Theorem 10. *The set of all points $x \in X$ at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper weakly s-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the upper inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. Let x be a point of X at which F is not upper weakly s-(τ_1, τ_2)-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $U \cap (X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Then, we have $x \in \tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence

$$x \in \tau_1\tau_2\text{-fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$$

since $x \in F^+(V) \subseteq \tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $x \in \tau_1\tau_2\text{-fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. If F is upper weakly s-(τ_1, τ_2)-continuous at $x \in X$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus,

$$x \in U \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

This contradicts that $x \in \tau_1\tau_2\text{-fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

Theorem 11. *The set of all points $x \in X$ at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not lower weakly s-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the lower inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets meeting $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. The proof is similar to that of Theorem 10.

A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Definition 6. [31] *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_1, τ_2)*-*T_2 *if for any pair of distinct points* x, y *in* X*, there exist disjoint* $\tau_1\tau_2$*-open sets* U *and* V *of* X *containing* x *and* y*, respectively.*

Definition 7. *A bitopological space* (X, τ_1, τ_2) *is said to be strongly s-*(τ_1, τ_2)*-normal if for every disjoint* $\tau_1\tau_2$*-closed sets* F *and* K *of* X*, there exist* $\tau_1\tau_2$*-open sets* U *and* V *having* $\tau_1\tau_2$*-connected complements such that* $F \subseteq U$*,* $K \subseteq V$ *and* $\tau_1\tau_2\text{-}Cl(U) \cap \tau_1\tau_2\text{-}Cl(V) = \emptyset$*.*

Theorem 12. *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is an injective upper weakly s-*(τ_1, τ_2)*-continuous multifunction into a strongly s-*(τ_1, τ_2)*-normal space* (Y, σ_1, σ_2) *and* $F(x)$ *is* $\sigma_1\sigma_2$*-closed for each* $x \in X$*, then* (X, τ_1, τ_2) *is* (τ_1, τ_2)*-*T_2*.*

Proof. For any distinct points x, y of X, we have $F(x) \cap F(y) = \emptyset$ since F is injective. Since $F(x)$ is $\sigma_1\sigma_2$-closed for each $x \in X$ and (Y, σ_1, σ_2) is strongly s-(τ_1, τ_2)-normal, there exist $\sigma_1\sigma_2$-open sets V and W of Y having $\sigma_1\sigma_2$-connected complements such that $F(x) \subseteq V$, $F(y) \subseteq W$ and $\sigma_1\sigma_2\text{-Cl}(V) \cap \sigma_1\sigma_2\text{-Cl}(W) = \emptyset$. Since F is upper weakly s-(τ_1, τ_2)-continuous, there exist $\tau_1\tau_2$-open sets G, U of X containing x, y, respectively, such that $F(G) \subseteq V$ and $F(U) \subseteq W$. Thus, $G \cap U = \emptyset$ and hence (X, τ_1, τ_2) is (τ_1, τ_2)-T_2.

Definition 8. [64] *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be upper s-*(τ_1, τ_2)*-continuous at* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $F(x)$ *and having* $\sigma_1\sigma_2$*-connected complement, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V$*. A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be upper s-*(τ_1, τ_2)*-continuous if* F *has this property at each point* x *of* X*.*

Lemma 5. [64] *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper s-*(τ_1, τ_2)*-continuous;*

(2) $F^+(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *having* $\sigma_1\sigma_2$*-connected complement;*

(3) $F^-(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^+(B))$ *for every subset* B *of* Y *such that* $Y - \sigma_1\sigma_2\text{-}Int(B)$ *is* $\sigma_1\sigma_2$*-connected.*

Theorem 13. *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is upper weakly s-*(τ_1, τ_2)*-continuous and satisfies* $F^+(\sigma_1\sigma_2\text{-}Cl(V)) \subseteq F^+(V)$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *having* $\sigma_1\sigma_2$*-connected complement, then* F *is upper s-*(τ_1, τ_2)*-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. Since F is upper weakly s-(τ_1, τ_2)-continuous, by Theorem 1 we have

$$F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \tau_1\tau_2\text{-Int}(F^+(V))$$

and hence $F^+(V)$ is $\tau_1\tau_2$-open in X. By Lemma 5, F is upper s-(τ_1, τ_2)-continuous.

Definition 9. *A bitopological space (X, τ_1, τ_2) is said to be s-(τ_1, τ_2)-normal if for each disjoint $\tau_1\tau_2$-closed sets F and K of X, there exist $\tau_1\tau_2$-open sets U and V having $\tau_1\tau_2$-connected complements such that $F \subseteq U$, $K \subseteq V$ and $U \cap V = \emptyset$.*

Theorem 14. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-closed in Y for each $x \in X$ and (Y, σ_1, σ_2) is s-(σ_1, σ_2)-normal. Then, F is upper weakly s-(τ_1, τ_2)-continuous if and only if F is upper s-(τ_1, τ_2)-continuous.*

Proof. Suppose that F is upper weakly s-(τ_1, τ_2)-continuous. Let $x \in X$ and G be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. Since $F(x)$ is $\sigma_1\sigma_2$-closed in Y, by the s-(σ_1, σ_2)-normality of (Y, σ_1, σ_2) there exist $\sigma_1\sigma_2$-open sets V and W having $\sigma_1\sigma_2$-connected complements such that $F(x) \subseteq V$, $Y - G \subseteq W$ and $V \cap W = \emptyset$. Thus, $F(x) \subseteq V \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(Y - W) = Y - W$. Since F is upper weakly s-(τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq G$. This shows that F is upper s-(τ_1, τ_2)-continuous. The converse is obvious.

Definition 10. [64] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower s-(τ_1, τ_2)-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower s-(τ_1, τ_2)-continuous if F has this property at each point x of X.*

Theorem 15. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-open in Y for each $x \in X$. Then, F is lower weakly s-(τ_1, τ_2)-continuous if and only if F is lower s-(τ_1, τ_2)-continuous.*

Proof. Suppose that F is lower weakly s-(τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-connected complement. Since F is lower weakly s-(τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\sigma_1\sigma_2$-open, $F(z) \cap V \neq \emptyset$ for each $z \in U$ and hence F is lower s-(τ_1, τ_2)-continuous. The converse is obvious.

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