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Almost Nearly Quasi (τ_1, τ_2)-continuous Multifunctions

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Abstract. This paper deals with the concept of almost nearly quasi (τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations and some properties concerning almost nearly quasi (τ_1, τ_2)-continuous multifunctions are considered.

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1. Introduction

The concept of quasi continuous functions was introduced by Marcus [44]. Popa [48] introduced and investigated the notion of almost quasi continuous functions. Neubrunnovaá [45] showed that quasi continuity is equivalent to semi-continuity due to Levine [42]. Popa and Noiri [50] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Malghan and Hanchinamani [43] introduced the notion of N-continuous functions. Noiri and Ergun [46] investigated some characterizations of N-continuous functions. Viriyapong and Boonpok [68] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [12]. Dungthaisong et al. [35] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [34] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Srisarakham et al. [60] introduced and studied the concept of almost (Λ, p)-continuous functions. Furthermore, several characterizations of strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions,

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$\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, slightly $(\tau_1, \tau_2)s$-continuous functions and $\delta(\tau_1, \tau_2)$-continuous functions were presented in [63], [16], [52], [25], [11], [8], [10], [4], [1], [2], [26], [23], [18], [57] and [51], respectively. Srisarakham et al. [61] introduced and studied the concept of faintly (τ_1, τ_2)-continuous functions. Thongmoon et al. [66] introduced and investigated the notion of rarely (τ_1, τ_2)-continuous functions. Chiangpradit et al. [32] introduced and studied the concept of weakly quasi (τ_1, τ_2)-continuous functions. Kong-ied at al. [41] introduced and investigated the notion of almost quasi (τ_1, τ_2)-continuous functions.

In 2003, Ekici [36] introduced and studied the concept of nearly continuous multifunctions as a generalization of semi-continuous multifunctions and N-continuous functions. Ekici [37] introduced and investigated the notion of almost nearly continuous multifunctions as a generalization of nearly continuous multifunctions and almost continuous multifunctions [48]. Noiri and Popa [47] introduced and studied the notion of almost nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological spaces. Carpintero et al. [30] introduced and studied the notion of nearly ω-continuous multifunctions as a weaker form of nearly continuous multifunctions. Rosas et al. [58] introduced and studied upper and lower almost nearly continuous multifunctions using notions of topological ideals. Moreover, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^{\star}$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, almost quasi (τ_1, τ_2)-continuous multifunctions, c-(τ_1, τ_2)-continuous multifunctions, c-quasi (τ_1, τ_2)-continuous multifunctions, s-$(\tau_1, \tau_2)p$-continuous multifunctions, slightly (τ_1, τ_2)-continuous multifunctions and slightly $(\tau_1, \tau_2)p$-continuous multifunctions were established in [5], [28], [69], [3], [7], [17], [24], [6], [21], [20], [15], [9], [19], [22], [38], [13], [27], [62], [14], [55], [40], [65], [56], [54], [39], [53], [73], [70] and [71], respectively. Rychlewicz [59] introduced and studied the notion of nearly quasi-continuous multifunctions as a generalization of almost nearly continuous multifunctions and almost quasi continuous multifunctions [49]. In this paper, we introduce the concept of almost nearly quasi (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of almost quasi (τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [29] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [29] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [29] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [29] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [29] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [67] (resp. $(\tau_1, \tau_2)s$-*open* [5], $(\tau_1, \tau_2)p$-*open* [5], $(\tau_1, \tau_2)\beta$-*open* [5]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [72] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\mathscr{N}(\tau_1, \tau_2)$-*closed* [64] if every cover of A by $(\tau_1, \tau_2)r$-open sets of X has a finite subcover.

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cup A$ [5];

(2) $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$ [54].

Lemma 3. [33] *Let (X, τ_1, τ_2) be a bitopological space. If V is a $\tau_1\tau_2$-open set of X having $\mathscr{N}(\tau_1, \tau_2)$-closed complement, then $\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(V))$ is a $(\tau_1, \tau_2)r$-open set having $\mathscr{N}(\tau_1, \tau_2)$-closed complement.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A}F(x)$.

3. Upper almost nearly quasi (τ_1, τ_2)-continuous multifunctions and lower almost nearly quasi (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper almost nearly quasi (τ_1, τ_2)-continuous multifunctions and lower almost nearly quasi (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of upper almost nearly quasi (τ_1, τ_2)-continuous multifunctions and lower almost nearly quasi (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost nearly quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^+(V)$ and for every $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and $W \subseteq F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost nearly quasi (τ_1, τ_2)-continuous if F is upper almost nearly quasi (τ_1, τ_2)-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost nearly quasi (τ_1, τ_2)-continuous;

(2) for each $x \in X$ and for each $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^+(V)$ and for every $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and $W \subseteq F^+(V)$;

(3) for each $x \in X$ and for every $\sigma_1\sigma_2$-closed and $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y such that $x \in F^+(Y-K)$ and for every $\tau_1\tau_2$-closed set H of X such that $x \in X-H$, there exists a $\tau_1\tau_2$-closed set M such that $H \subseteq M$, $M \neq X$ and $F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K))) \subseteq M$;

(4) for each $x \in X$ and for every $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^+(V)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))$;

(5) $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;

(6) $F^-(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $(\sigma_1, \sigma_2)r$-closed and $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y.

Proof. (1) ⇒ (2): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$ and let U be any $\tau_1\tau_2$-open set of X containing x. By (1), there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and

$$W \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = F^+(V).$$

(2) ⇒ (1): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$ and let U be any $\tau_1\tau_2$-open set of X containing x. By Lemma 3, we have $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open and $Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $\mathscr{N}(\sigma_1, \sigma_2)$-closed. Since $F(x) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, therefore there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and $W \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(1) ⇒ (3): Let $x \in X$ and K be any $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set of Y such that $x \in F^+(Y - K)$. It is clear that $Y - K$ is a $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. Let H be a $\tau_1\tau_2$-closed set of X such that $x \in X - H$. By (1), there there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq X - H$ and $W \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))$. Let us observe that

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)) &= \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Int}(K)) \\ &= Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)).\end{aligned}$$

It follows that $W \subseteq F^+(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$. Let $M = X - W$, then $X - M \subseteq X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$ since

$$F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq M.$$

It is evident that M is a $\tau_1\tau_2$-closed set and $M \neq X$.

(3) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$. Then, we have $K = Y - V$ is (σ_1, σ_2)-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set of Y and $x \in F^+(Y - K)$. Let U be a $\tau_1\tau_2$-open set of X containing x. Then, $H = X - U$ is a $\tau_1\tau_2$-closed set such that $x \in X - H$. By the hypothesis, there exists a $\tau_1\tau_2$-closed set M such that $H \subseteq M$, $M \neq X$ and $F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq M$. The last inclusion implies that $X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq M = X - W$, where $W = X - M$ is a nonempty $\tau_1\tau_2$-open set. It was shown that $W \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. It is easy to see that $W \subseteq U$.

(1) ⇒ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$. Then, for any $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W_U such that $W_U \subseteq U$ and $W_U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Let $G = \{x\} \cup [\cup\{W_U \mid U \text{ is a } \tau_1\tau_2\text{-open set containing } x\}]$. Then, we have

$$G \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(G))$$

and hence G is a $(\tau_1, \tau_2)s$-open set such that $x \in G$ and $G \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$. Let U be a $\tau_1\tau_2$-open set of X containing x. By the hypothesis, there exists a $(\tau_1, \tau_2)s$-open set G such that $x \in G$ and $G \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Let $W = \tau_1\tau_2\text{-Int}(G) \cap U$. Since $U \cap G \neq \emptyset$, we have $W \neq \emptyset$. It is easy to check that $W \subseteq U$ and $W \subseteq G$. Thus, $W \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement and $x \in F^+(V)$. Then $F(x) \subseteq V$. Under the assumption, there exists a $(\tau_1, \tau_2)s$-open set G_x such that $x \in G_x$ and $G_x \subseteq F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. It is easily seen that the set $G = \cup\{G_x \mid x \in F^+(V)\}$ is $(\tau_1, \tau_2)s$-open and equal to the set $F^+(V)$.

(5) ⇒ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V$. Then by Lemma 3, $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is a $(\sigma_1, \sigma_2)r$-open set having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. By (5), we have $F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $(\sigma_1, \sigma_2)s$-open in X. Of course, $x \in F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (6): Let K be any $(\sigma_1, \sigma_2)r$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set of Y. Then, $Y - K$ is a $(\sigma_1, \sigma_2)r$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. By (5), $F^+(Y - K) = X - F^-(K)$ is $(\tau_1, \tau_2)s$-open in X and hence $F^-(K)$ is $(\tau_1, \tau_2)s$-closed in X.

(6) ⇒ (5): The proof is similar to the above.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost nearly quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^-(V)$ and for every $\tau_1\tau_2$-open set of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and $W \subseteq F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost nearly quasi (τ_1, τ_2)-continuous if F is lower almost nearly quasi (τ_1, τ_2)-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost nearly quasi (τ_1, τ_2)-continuous;

(2) for each $x \in X$ and for each $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^-(V)$ and for every $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$ and $W \subseteq F^-(V)$;

(3) for each $x \in X$ and for every $\sigma_1\sigma_2$-closed and $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y such that $x \in F^-(Y - K)$ and for every $\tau_1\tau_2$-closed set H of X such that $x \in X - H$, there exists a $\tau_1\tau_2$-closed set M such that $H \subseteq M$, $M \neq X$ and $F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K))) \subseteq M$;

(4) for each $x \in X$ and for every $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^-(V)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $U \subseteq F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))$;

(5) $F^-(V)$ is $(\tau_1, \tau_2)s$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;

(6) $F^+(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $(\sigma_1, \sigma_2)r$-closed and $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y.

Proof. The proof is similar to that of Theorem 1.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, a multifunction

$$\mathrm{sCl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is defined in [31] as follows: $\mathrm{sCl}F_{\circledast}(x) = (\sigma_1, \sigma_2)\text{-sCl}(F(x))$ for each $x \in X$.

Theorem 3. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper almost nearly quasi (τ_1, τ_2)-continuous if and only if $sClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper almost nearly quasi (τ_1, τ_2)-continuous.*

Proof. Suppose that F is upper almost nearly quasi (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that

$$x \in \mathrm{sCl}F^{+}_{\circledast}(V).$$

Then, we have $\mathrm{sCl}F_{\circledast}(x) \subseteq V$ and hence $F(x) \subseteq V$. Since F is upper almost nearly quasi (τ_1, τ_2)-continuous, by Theorem 1 there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $U \subseteq F^{+}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Thus by Lemma 2, $U \subseteq F^{+}((\sigma_1, \sigma_2)\text{-sCl}(V))$. Therefore, $F(U) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$. For each $u \in U$, $(\sigma_1, \sigma_2)\text{-sCl}(F(u)) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$ and so $(\sigma_1, \sigma_2)\text{-sCl}(F(U)) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$. Thus, $\mathrm{sCl}F_{\circledast}(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and hence $x \in \mathrm{sCl}F^{+}_{\circledast}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 1, $\mathrm{sCl}F_{\circledast}$ is upper almost nearly quasi (τ_1, τ_2)-continuous.

Theorem 4. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost nearly quasi (τ_1, τ_2)-continuous if and only if $sClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost nearly quasi (τ_1, τ_2)-continuous.*

Proof. The proof is similar to that of Theorem 3.

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-clopen* [29] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed.

Definition 3. [29] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-connected if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets.*

Definition 4. [33] *A bitopological space (X, τ_1, τ_2) is said to be $\mathscr{N}(\tau_1, \tau_2)$-connected if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets having $\mathscr{N}(\tau_1, \tau_2)$-closed complements.*

Definition 5. *A bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)s$-connected if X cannot be written as the union of two disjoint nonempty $(\tau_1, \tau_2)s$-open sets.*

Theorem 5. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an upper or lower almost nearly quasi (τ_1, τ_2)-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-connected for every $x \in X$ and (X, τ_1, τ_2) is $(\tau_1, \tau_2)s$-connected, then (Y, σ_1, σ_2) is $\mathscr{N}(\sigma_1, \sigma_2)$-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\mathscr{N}(\sigma_1, \sigma_2)$-connected. There exist nonempty $\sigma_1\sigma_2$-open sets U and V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are $(\tau_1, \tau_2)s$-open in X. (i) Let F be upper almost nearly quasi (τ_1, τ_2)-continuous. Since U and V are $\sigma_1\sigma_2$-clopen in Y,

$$\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(U)) = U$$

and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = V$. Thus, U and V are $(\sigma_1, \sigma_2)r$-open sets having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements. Since F is upper almost nearly quasi (τ_1, τ_2)-continuous, by Theorem 1, $F^+(U)$ and $F^+(V)$ are $(\tau_1, \tau_2)s$-open sets. (ii) Let F be lower almost nearly quasi (τ_1, τ_2)-continuous. By Theorem 2, $F^+(U)$ is $(\tau_1, \tau_2)s$-closed in X because U is $\sigma_1\sigma_2$-clopen in Y. Thus, $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X. Similarly, we have $F^+(U)$ is (τ_1, τ_2)-open in X. Therefore, (X, τ_1, τ_2) is not $(\tau_1, \tau_2)s$-connected.

4. Almost nearly quasi (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the concept of almost nearly quasi (τ_1, τ_2)-continuous multifunctions. We also investigate some characterizations of almost nearly quasi (τ_1, τ_2)-continuous multifunctions.

Definition 6. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$ and for every $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set W such that $W \subseteq U$, $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in W$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly quasi (τ_1, τ_2)-continuous if F is almost nearly quasi (τ_1, τ_2)-continuous at each point x of X.*

Theorem 6. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is almost nearly quasi (τ_1, τ_2)-continuous at a point $x \in X$;

(2) for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$;

(3) $x \in (\tau_1, \tau_2)$*-sInt*$(F^+((\sigma_1, \sigma_2)$*-sCl*$(V_1)) \cap F^-((\sigma_1, \sigma_2)$*-sCl*$(V_2)))$ *for every* $\sigma_1\sigma_2$*-open sets* V_1, V_2 *of* Y *having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complements such that* $x \in F^+(V_1) \cap F^-(V_2)$*;*

(4) $x \in \tau_1\tau_2$*-Cl*$(\tau_1\tau_2$*-Int*$(F^+((\sigma_1, \sigma_2)$*-sCl*$(V_1)) \cap F^-((\sigma_1, \sigma_2)$*-sCl*$(V_2))))$ *for every* $\sigma_1\sigma_2$*-open sets* V_1, V_2 *of* Y *having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complements such that*

$$x \in F^+(V_1) \cap F^-(V_2).$$

Proof. (1) ⇒ (2): Let $\mathcal{U}(x)$ be the family of all $\tau_1\tau_2$-open sets of X containing x. Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. For each $H \in \mathcal{U}(x)$, there exists a nonempty $\tau_1\tau_2$-open set G_H of X such that $G_H \subseteq H$, $F(G_H) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in G_H$. Let $W = \cup\{G_H \mid H \in \mathcal{U}(x)\}$. Then, W is $\tau_1\tau_2$-open in X, $x \in \tau_1\tau_2$-Cl(W), $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(w) \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$. Then, $W \subseteq U \subseteq \tau_1\tau_2$-Cl$(W)$. Thus, U is a $(\tau_1, \tau_2)s$-open set of X containing x such that $F(U) \subseteq ((\sigma_1, \sigma_2)$-sCl$(V_1))$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$.

(2) ⇒ (3): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. Then, there exists a $(\tau_1, \tau_2)s$-open set of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2))$. Since U is $(\tau_1, \tau_2)s$-open, we have $x \in (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2)))$.

(3) ⇒ (4): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. Now put

$$U = (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2))).$$

Then, U is $(\tau_1, \tau_2)s$-open in X and

$$x \in \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))).$$

(4) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X containing x and V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have $x \in \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2))))$. Put

$$W = \tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2))) \cap U.$$

Then, W is a nonempty $\tau_1\tau_2$-open set of X such that $W \subseteq U$, $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(w) \neq \emptyset$ for every $w \in W$. This shows that F is almost nearly quasi (τ_1, τ_2)-continuous at x.

Theorem 7. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is almost nearly quasi* (τ_1, τ_2)*-continuous;*

(2) for each $x \in X$ and for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$;

(3) $F^+(V_1) \cap F^-(V_2)$ is $(\tau_1, \tau_2)s$-open in X for every $(\sigma_1, \sigma_2)r$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements;

(4) $F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2)))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements;

(5)

$$(\tau_1, \tau_2)\text{-}sCl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_2)))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(B_2))$$

for every subsets B_1, B_2 of Y having the $\mathscr{N}(\sigma_1, \sigma_2)$-closed $\sigma_1\sigma_2$-closure;

(6) $F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$((F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2))))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements.

Proof. (1) $\Rightarrow$ (2): The proof follows immediately from Theorem 6, since F is almost nearly quasi (τ_1, τ_2)-continuous at each point of X.

(2) $\Rightarrow$ (3): Let V_1, V_2 be any $(\sigma_1, \sigma_2)r$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements and $x \in F^+(V_1) \cap F^-(V_2)$. Then, there exists a a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in (\tau_1, \tau_2)$-sInt$(F^+(V_1) \cap F^-(V_2))$. Therefore, $F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1, \tau_2)$-sInt$(F^+(V_1) \cap F^-(V_2))$. This shows that $F^+(V_1) \cap F^-(V_2)$ is $(\tau_1, \tau_2)s$-open in X.

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have $F(x) \subseteq V_1 \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and $\emptyset \neq V_2 \cap F(x) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_2) \cap F(x)$. Thus, $x \in F^+((\sigma_1, \sigma_2)$-sCl$(V_1))$ and $x \in F^-((\sigma_1, \sigma_2)$-sCl$(V_2))$. By (3), we have $F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2))$ is $(\tau_1, \tau_2)s$-open in X and $x \in (\tau_1, \tau_2)s$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2)))$. Therefore, $F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2)))$.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets of Y having the $\mathscr{N}(\sigma_1, \sigma_2)$-closed $\sigma_1\sigma_2$-closure. Then, $Y - \sigma_1\sigma_2$-Cl(B_1) and $Y - \sigma_1\sigma_2$-Cl(B_2) are $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complements. Thus by (4), we have

$$
\begin{aligned}
&X - (F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2))) \\
&= (X - F^-(\sigma_1\sigma_2\text{-Cl}(B_1))) \cap (X - F^+(\sigma_1\sigma_2\text{-Cl}(B_2))) \\
&= F^+(Y - \sigma_1\sigma_2\text{-Cl}(B_1)) \cap F^-(Y - \sigma_1\sigma_2\text{-Cl}(B_2)) \\
&\subseteq (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B_1))) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B_2)))) \\
&= X - (\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))))
\end{aligned}
$$

and hence

$$(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)).$$

(5) $\Rightarrow$ (6): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complements. Then, $Y - V_1$ and $Y - V_2$ are $\mathscr{N}(\sigma_1,\sigma_2)$-closed and $\sigma_1\sigma_2$-closed sets of Y. By (5) and Lemma 2,

$$\begin{aligned}
&\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_1))) \cup F^+(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V_2)))))\\
&\subseteq F^-(Y - V_1) \cup F^+(Y - V_2)\\
&= (X - F^+(V_1)) \cup (X - F^-(V_2))\\
&= X - (F^+(V_1) \cap F^-(V_2)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_1))) \cup F^+(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}((X - F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1))) \cup (X - F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(X - (F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))))\\
&= X - \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))).
\end{aligned}$$

Thus, $F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}((F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complements such that $x \in F^+(V_1) \cap F^-(V_2)$. By (6) and Lemma 2, we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))).$$

Put $U = (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))$. Then, U is a $(\tau_1,\tau_2)s$-open set of X containing x, $F(U) \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V_1)$ and $(\sigma_1,\sigma_2)\text{-sCl}(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. This shows that F is almost nearly quasi (τ_1,τ_2)-continuous.

Theorem 8. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is almost nearly quasi (τ_1,τ_2)-continuous;

(2) $(\tau_1,\tau_2)\text{-}sCl(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$ for every $(\sigma_1,\sigma_2)\beta$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complements;

(3) $(\tau_1,\tau_2)\text{-}sCl(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$ for every $(\sigma_1,\sigma_2)s$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complements;

(4) $F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1,\tau_2)\text{-}sInt(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$ for every $(\sigma_1,\sigma_2)p$-open sets V_1, V_2 of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complements.

Proof. The proof follows from Theorem 7 and is thus omitted.

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