EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5729
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fourth Order Functional Differential Equations of Neutral Type: Enhanced Oscillation Theorems

Waed Muhsin[1], Barakah Almarri[2], Mohammad S. Jazmati[3], Osama Moaaz[1,3,*], Elmetwally M. Elabbasy[1]

[1] *Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt*
[2] *Department of General Studies, Jubail Industrial College, 8244, Rd Number 6, Al Huwaylat, 35718, Al Jubail, Saudi Arabia*
[3] *Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah, 51452 Saudi Arabia*

Abstract. In this paper, we investigate the oscillatory properties of fourth-order neutral delay differential equation solutions in the canonical situation. To our knowledge, this equation has received minimal research. We prove new improved features and relationships for the solution and the accompanying function. Based on these relationships, oscillation theorems were developed that guarantee oscillation of all solutions to the considered equation. The comparison principle used in our results is one of the most significant methods for investigating the oscillatory behavior of delay differential equations. The findings of our study extend and develop a number of previous findings in the literature.

2020 Mathematics Subject Classifications: 34C10, 34K11

Key Words and Phrases: differential equation; oscillation criteria; fourth-order differential equations; neutral delay argument

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1. Introduction

The objective of this paper is to investigate the oscillation of the fourth-order neutral delay differential equations (NDDEs)

$$\left(a_3(u)\left(a_2(u)\left(a_1(u)\Omega'(u)\right)'\right)'\right)' + q(u)\,x(\rho(u)) = 0,\ \ u \geq u_0, \tag{1}$$

where $\Omega(u) = x(u) + h(u)\,x(\kappa(u))$. Based on the following :

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5729

Email addresses: waed.zarebah@gmail.com (W. Muhsin), marribj@rcjy.edu.sa (B. Almarri), Jzmaty@qu.edu.sa (M. S. Jazmati), o_moaaz@mans.edu.eg (O. Moaaz), emelabbasy@mans.edu.eg (E. M. Elabbasy)

(A$_1$) $a_i \in \mathbf{C}^{(4-i)}\left([u_0,\infty),(0,\infty)\right),\ i=1,2,3$ and satisfies

$$I_i\left(u\right) := \int_{u_0}^{u} \frac{1}{a_i\left(\xi\right)} \mathrm{d}\xi \to \infty \text{ as } u \to \infty; \tag{2}$$

(A$_2$) $\kappa,\ \rho \in \mathbf{C}\left([u_0,\infty),(0,\infty)\right)$ satisfy $\kappa\left(u\right) < u,\ \rho\left(u\right) < u,\ \rho'\left(u\right) > 0,\ \lim_{u\to\infty}\kappa\left(u\right) = \infty$ and $\lim_{u\to\infty}\rho\left(u\right) = \infty$;

(A$_3$) $h,\ q \in \mathbf{C}\left([u_0,\infty),(0,\infty)\right)$.

A studied solution to (1) is defined as a real-valued function x that is four times differentiable, that has the property $a_3\left(u\right)\left(a_2\left(u\right)\left(a_1\left(u\right)\Omega'\left(u\right)\right)'\right)' \in \mathbf{C}^1\left([u_0,\infty),(0,\infty)\right)$ and fulfills (1) for any suitably large u on $[u_0,\infty)$. We concentrate only on those solutions of (1) that satisfy

$$\sup\{|x\left(u\right)| : u \geq U\} > 0,$$

for all $U \geq u_0$, i.e., it is a nontrivial solution. A solution x to (1) is said to be oscillating or non-oscillating based on whether it is positive, negative, or neither positive nor negative eventually in the first place. That is, what interests us is the behavior of the solution in the neighborhood of the infinite points. If all solutions of the equation oscillate (if it has arbitrarily large zeros), then the equation is oscillatory; otherwise, it is said to be non-oscillatory [4].

Understanding, studying, and then analyzing functional differential equations is the basis of many diverse disciplines in mathematics (both pure and applied), engineering, and physics, all of which focus on the properties of differential equations (DEs) in their various forms.

As we know, many technological, physical, or biological processes are modeled using dynamic differentials. The existence and uniqueness of solutions of these equations is a key focus when studying the differential equations of these models, as a closed-form solution may not be found for nonlinear dynamic differentials obtained when modeling many phenomena. Numerical techniques can be used as an alternative method to approximate the results [12].

Due to the difficulty of finding these solutions, researchers have focused on knowing the properties of these solutions under the name of qualitative theory of equations, of which oscillation theory is one of its main subfields. Asymptotic and oscillatory characteristics of solutions are the focus of this theory; see [27]. Fite produced an essay [16] on differential equation oscillation theory, written in the early 20th century. Many inquiries into the oscillation theory of delay differential equations began with his publication.

Delay and neutral delay differential equations have a very diverse history and are used in many applications in the natural sciences and engineering; for further details, see Hale [18]. In a differential equation, when the highest order derivative of the function, whether known or unknown, is present in both the delay-containing and delay-free portions of this differential equation, the equation is said to include a neutral delay (ND).

Neutral delay differential equations (NDDEs) are equations that depend on the present and delayed values of a function and its derivatives and are of significant importance due to their widespread applications in various fields, including engineering, physics, and biology. They are complex because they include delays in the derivatives and are fundamental to the study of dynamical systems that contain time delays. Unlike ordinary delay differential equations (ODDEs). Neutral differential equations are characterized by the presence of both the derivative of the dependent variable and its delayed term in the highest derivative. This distinguishes them from other functional differential equations, such as delay differential equations, which involve delays only in the dependent variable without its derivatives.

There has long been a lot of research into the oscillatory behavior and asymptotic features of the solutions to many different types of functional differential equations. Here, we recall the pioneering works of [14, 26, 46, 48].

In the last few years, there has been a major expansion and evolution of oscillation theory, which currently includes the study of oscillation for fractional and ordinary DEs solutions with delay, neutral, mixed, or damping components. For instance, [43–45] contains mixed equations; [5, 11, 23, 32, 33, 35, 39] contains neutral equations; and [7, 47] demonstrates the advancement in the investigation of higher-order equations.

Additionally, the oscillation of damping equations can be traced in [6, 8]. On the other hand, refs.[19] dealt with fractional DEs, while [21, 22] is concerned with dynamic equations.

Oscillation theory is interesting both theoretically and practically. It is known that homogeneous first-order ordinary differential equations do not have oscillatory solutions. However, the presence of deviant arguments can cause oscillation of solutions; see [15]. Therefore, the focus has been on studying and understanding the behavior of second- and higher-order differential equation solutions.

We should not fail to include the ancient papers that were and probably still are a source of inspiration for many studies that were interested in discovering oscillation criterion when studying second-order DE (linear, half-linear, superlinear, and sublinear); see [3, 20].

The extensive applications of fourth-order DDEs in civil, aeronautical engineering, and mechanical make them extremely valuable in daily life. Because this kind of equation is so important in so many different domains, research on it is still ongoing (see references [10, 17, 34]). It also has biological applications, such as analyzing oscillations in neuromuscular systems [38].

Most of the recently published results on fourth-order DEs in the canonical case deal with equation (1) when $a_1(u) = a_2(u) = 1$. Therefore, the goal of this work is to extend the results obtained by studying this form of fourth-order DEs involving the neutral delay. Additionally, some similar findings served as inspiration for our study in particular: Masood et al. [29] took into account the oscillatory characteristics of solutions of

$$\left(a_3(u)\left(x'''(u)\right)^{\alpha}\right)' + q(u)\,x^{\alpha}(\rho(u)) = 0,\ u \geq u_0,$$

where α is a ratio of two odd integers. They found some properties for a class of positive solutions of the latter quasi-linear DDE and then created an oscillation criterion.

By using some inequalities and the theory of comparison, Bazighifan et al. [9] studied the following equation

$$\left(a_3(u)(\Omega'''(u))^{\alpha}\right)' + q(u)x^{\beta}(\rho(u)) = 0,\ u \geq u_0,$$

where α and β are quotients of two odd positive integers. They presented some oscillation criteria for the above equation and improved some previously published results.

In 2013, Agarwal [1] investigated the oscillatory behavior of the n-order equation

$$\Omega^{(n)}(u) + q(u)x(\rho(u)) = 0, \tag{3}$$

and arrived at criteria that enhance findings reported in the literature.
In what follows, the oscillation for (3) was explored by Li and Rogovchenko [28]. They employed first-order delay equation comparison as their method.

In addition, Salah et al. [42] studied equation (3) when $n = 4$, i.e.

$$\Omega^4(u) + q(u)x(\rho(u)) = 0,\ u \geq u_0.$$

For the oscillation of every solution to their equation, aptly, they derived a new single-oscillation criterion. For some instances of positive solutions to the investigated equation, they established additional monotonic properties. Furthermore, they employed an iterative process to enhance these attributes. The evolution of these monotonic features helps to produce new and more effective standards for confirming the equation's oscillation. The results obtained, utilizing an example Euler-type equation an, extend and improve upon earlier findings in the literature.

On the other hand, Jadlovska [24] used an iteratively enhanced monotonicity characteristic of non-oscillatory solutions approach to offer a single-condition sharp criterion for the oscillation of the spacial case of the above equation when $h(u) = 0$, i.e

$$x^4(u) + q(u)x(\rho(u)) = 0,\ u \geq u_0.$$

Perhaps one of the oldest studies in the literature, [25], that presented some oscillation criteria for equation

$$\left[a_3(u)x''(u)\right]'' + x(u)F\left(x^2, u\right) = 0,\ u \geq 0, \tag{4}$$

the function F in this equation is given as follows: $xF\left(x^2, u\right)$ is continuous for $|x| < \infty$, $u \geq 0$, and $F(y, u)$ is positive for $y > 0$, $u \geq 0$.

Following Nehari [37] and Wong [13], where equations of the form (4) are classified based on the nonlinearity of $xF\left(x^2, u\right)$ with respect to x. They determined the effect that $a_3(u)$ exercises upon the oscillatory character of (4) in conjunction with its nonlinearity.

New monotonic properties of the second-order NDE

$$\left(a_3(u)\left(\Omega'(u)\right)^{\alpha}\right)' + q(u)x^{\alpha}(\rho(u)) = 0,$$

were derived by Moaaz et al. [30]. Subsequently, they employed these characteristics to derive optimal oscillation parameters, employing various approaches to achieve this objective. Additionally, they provided new standards that guarantee the oscillation of the fourth-order NDE

$$\left(a_3(u)\left(\Omega'''(u)\right)^{\alpha}\right)' + q(u)\,x^{\alpha}(\rho(u)) = 0. \tag{5}$$

Earlier, Agarwal et al.[2] evaluated the oscillatory behavior of the solutions of the fourth-order functional DE

$$\left[a_3^{-1}(u)\left(\left[a_2^{-1}(u)\left(\left[a_1^{-1}(u)\left(x'(u)\right)^{\alpha_1}\right]'\right)^{\alpha_2}\right]'\right)^{\alpha_3}\right]' + \lambda q(u)\,f(x(\rho(u))) = 0,$$

where $\int_{u_0}^{\infty} a_i^{-1/\alpha_i}(\xi)\,\mathrm{d}\xi \to \infty$ as $u \to \infty$, $i = 1,2,3$, $\lambda = \pm 1$, $f \in \mathbf{C}((0,\infty),(0,\infty))$, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$.

Very recently, Nabih et al. [36] concentrated on examining the oscillation of (5); they discovered novel characteristics that allow them to employ more efficient terms. Using the comparison approach and the generic form of Riccati, they were able to derive criteria that eliminated positive decreasing solutions.

Lemma 1. *[40] Let $\Psi \in \mathbf{C}^n([u_0,\infty),\mathbb{R})$. If $\Psi^{(n)}(u)$ is eventually of fixed sign for all large u, then there exist a $u_x \geq u_0$ and a μ, $0 \leq \mu \leq n$, with $n+\mu$ even for the derivative $\Psi^{(n)}(u) \geq 0$, or $n+\mu$ odd for the derivative $\Psi^{(n)}(u) \leq 0$ such that*

(i) $\mu > 0$ *implies* $\Psi^{(\kappa)}(u) > 0$ *for* $u \geq u_x$, $\kappa = 0,1,...,\mu-1$;
(ii) $\mu \leq n-1$ *implies* $(-1)^{\mu+\kappa}\,\Psi^{(\kappa)}(u) > 0$ *for* $u \geq u_x$, $\kappa = \mu, \mu+1, ..., n-1$.

Lemma 2. *[31, Lemma 2.1] Suppose that there is an eventual positive solution to (1) for $x(u)$. Then, eventually*

$$x(u) > \sum_{t=0}^{\gamma}\left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right)\right)\left[\frac{\Omega\left(\kappa^{(2t)}(u)\right)}{h\left(\kappa^{(2t)}(u)\right)} - \Omega\left(\kappa^{(2t+1)}(u)\right)\right], \tag{6}$$

for any integer $\gamma \geq 0$.

This search is intended to look into the oscillatory behavior of solutions to NDDEs of the fourth order. Any functional differential equation has three types of solutions: oscillatory, negative, and positive. As we know, negative solutions are thus disregarded due to the symmetry between the solutions (the positive and the negative) of the investigated equation. The novelty of the paper lies in the demonstrated improved relationships and features between the corresponding function and the solution; these relationships affect the oscillation criteria, and improving them leads to improving these criteria. The new criteria ensure that all solutions of the studied equation are oscillatory based on the comparison principle with first-order differential equations. It should be noted that if all the solutions of the equation are oscillatory, this means that the equation itself is an oscillatory equation. The findings discovered complement several well-known in the literature.

This is how our paper will be structured: In Section 2.1, by using the comparison principle with first-order equations, we introduce oscillation criteria, where the classical relationship that links the solution $x(u)$ to its corresponding function $\Omega(u)$ has been applied. In Section 2.2, we present the improved criteria, which apply the new improved relationship between the corresponding function and the solution.

2. Comparison with first order equations

2.1. Oscillation criteria

For the purpose of clarity, let

$$I_i(u) := \int_{u_0}^{u} \frac{1}{a_i(\xi)} \mathrm{d}\xi, \qquad I_{ij}(u) := \int_{u_0}^{u} \frac{1}{a_i(\xi)} I_j(\xi)\,\mathrm{d}\xi,$$

$$I_{ijk}(u) := \int_{u_0}^{u} \frac{1}{a_i(\xi)} I_{jk}(\xi)\,\mathrm{d}\xi \text{ where } i,\ j,\ k \in \{1,2,3\},$$

$$Q(u) := (1 - h(\rho(u)))\, q(u),$$

$$R_1(u) := \left(\frac{1}{a_2(u)} \int_{u}^{\infty} \frac{1}{a_3(\xi)} \int_{v}^{\infty} Q(v)\,\mathrm{d}v\mathrm{d}\xi \right) \int_{u_1}^{\rho(u)} \frac{1}{a_1(v)} \mathrm{d}v,$$

and

$$R_2(u) := Q(u) \int_{u_1}^{u} \frac{1}{a_1(v_1)} \int_{u_1}^{v_1} \frac{1}{a_2(\xi)} \int_{u_1}^{v} \frac{1}{a_3(v)} \mathrm{d}v\mathrm{d}\xi\mathrm{d}v_1,$$

for any integer $u \geq u_1$, where $u_1 \geq u_0$.

Lemma 3. *Let condition (2) hold and $x(u)$ be an eventually positive to (1). Then, $(a_3(a_2(a_1\Omega')')')'(u) \leq 0$ and either*

$$\Omega(u) \in L^- \iff \Omega(u) > 0 \quad \Omega'(u) > 0 \quad (a_1\Omega')'(u) < 0 \quad (a_2(a_1\Omega')')'(u) > 0;$$
$$\Omega(u) \in L^+ \iff \Omega(u) > 0 \quad \Omega'(u) > 0 \quad (a_1\Omega')'(u) > 0 \quad (a_2(a_1\Omega')')'(u) > 0.$$

Proof. Suppose (1) has an eventually positive solution at $x(u)$. From (1) we get $(a_3(a_2(a_1\Omega')')')'(u) \leq 0$. Lemma 1 must be used to deduce the cases L^- and L^+ for the corresponding function $\Omega(u)$ to the solution x (u) and its derivatives.

Remark 1. *The decomposition of (1) is found in the set L of all positive solutions*

$$L = L^- \cup L^+.$$

Now, we can derive the next theorem using the comparison principle:

Theorem 1. *Assume that both first-order DDEs*

$$g'(u) + R_1(u)\, g(\rho(u)) = 0 \tag{7}$$

and

$$g'(u) + R_2(u)\, g(\rho(u)) = 0 \tag{8}$$

are oscillatory. Then equation (1) is oscillatory.

Proof. Assume that x is an eventually positive solution of (1), say for $u \geq u_1$. Through Lemma 3, we can deduce that $\Omega(u) \in L^+$ or $\Omega(u) \in L^-$. It follows from the monotonicity of $a_1(u)\,\Omega'(u)$ that

$$\begin{aligned} \Omega(u) &\geq \int_{u_1}^{u} \frac{1}{a_1(v)} a_1(v)\,\Omega'(v)\,\mathrm{d}v \\ &\geq a_1(u)\,\Omega'(u) \int_{u_1}^{u} \frac{1}{a_1(v)} \mathrm{d}v, \end{aligned} \tag{9}$$

or

$$\Omega(u) \geq a_1(u)\,\Omega'(u)\, I_1(u).$$

By the definition of $\Omega(u)$, we obtain

$$\begin{aligned} x(u) &\geq \Omega(u) - h(u)\, x(\rho(u)) \geq \Omega(u) - h(u)\,\Omega(\rho(u)) \\ &\geq (1 - h(u))\,\Omega(u), \end{aligned}$$

which together with (1), implies

$$\begin{aligned} \left(a_3(a_2(a_1\Omega')')'\right)'(u) &\leq -q(u)\,(1 - h(\rho(u)))\,\Omega(\rho(u)) \\ &\leq -Q(u)\,\Omega(\rho(u)). \end{aligned} \tag{10}$$

Assume first that $\Omega(u) \in L^-$. Integrating (10) from u to ∞, we get

$$a_3(u)\,(a_2(a_1\Omega')')'(u) \geq \int_u^\infty Q(v)\,\Omega(\rho(v))\,\mathrm{d}v.$$

Using this fact $\Omega(\rho(u))$ is increasing, the last inequality becomes

$$(a_2(a_1\Omega')')'(u) \geq \Omega(\rho(u)) \frac{1}{a_3(u)} \int_u^\infty Q(v)\,\mathrm{d}v.$$

Integrating once more, we are led to

$$-(a_1\Omega')'(u) \geq \Omega(\rho(u)) \frac{1}{a_2(u)} \int_u^\infty \frac{1}{a_3(v)} \int_v^\infty Q(\xi)\,\mathrm{d}\xi \mathrm{d}v.$$

Combining the last inequality with (9), gives

$$-(a_1\Omega')'(u) \geq \left(\Omega'(\rho(u)) \frac{a_1(\rho(u))}{a_2(u)} \int_u^\infty \frac{1}{a_3(v)} \int_v^\infty Q(\xi)\,\mathrm{d}\xi \mathrm{d}v \right) \int_{u_1}^{\rho(u)} \frac{1}{a_1(v)} \mathrm{d}v$$

$$= \quad a_1(\rho(u))\,\Omega'(\rho(u))\,R_1(u).$$

Thus, the function $g(u) := a_1(u)\,\Omega'(u)$ is a positive solution of the first-order DD inequality

$$g'(u) + R_1(u)\,g(\rho(u)) \le 0.$$

Therefore, we conclude that the related DE (7) as well has a positive solution by applying the Philos theorem [41]; this contradicts the assumptions made at the beginning of the theorem.

Now, we shall assume that $\Omega(u) \in L^+$. Since $a_3(u)\,(a_2(a_1\Omega')')'(u)$ is decreasing, one gets

$$\begin{aligned} a_2(u)\,(a_1\Omega')'(u) &\ge \int_{u_1}^{u} \frac{1}{a_3(v)} a_3(v)\left(a_2(a_1\Omega')'\right)'(v)\,\mathrm{d}v \\ &\ge a_3(u)\left(a_2(a_1\Omega')'\right)'(u) \int_{u_1}^{u} \frac{1}{a_3(v)}\mathrm{d}v, \end{aligned}$$

or

$$a_2(u)\,(a_1\Omega')'(u) \ge a_3(u)\left(a_2(a_1\Omega')'\right)'(u)\,I_3(u).$$

Integrating from u_1 to u, we get

$$\begin{aligned} \Omega'(u) &\ge \frac{1}{a_1(u)} \int_{u_1}^{u} \frac{1}{a_2(\xi)} \int_{u_1}^{v} \frac{1}{a_3(v)} a_3(v)\left(a_2(a_1\Omega')'\right)'(v)\,\mathrm{d}v\mathrm{d}\xi \\ &\ge a_3(u)\left(a_2(a_1\Omega')'\right)'(u) \frac{1}{a_1(u)} \int_{u_1}^{u} \frac{1}{a_2(\xi)} \int_{u_1}^{v} \frac{1}{a_3(v)}\mathrm{d}v\mathrm{d}\xi. \end{aligned}$$

Integrating once more, we arrive at

$$\Omega(u) \ge a_3(u)\left(a_2(a_1\Omega')'\right)'(u) \int_{u_1}^{u} \frac{1}{a_1(v_1)} \int_{u_1}^{v_1} \frac{1}{a_2(\xi)} \int_{u_1}^{v} \frac{1}{a_3(v)}\mathrm{d}v\mathrm{d}\xi\mathrm{d}v_1$$

We see that $g(u) = a_3(u)\,(a_2(a_1\Omega')')'(u)$ satisfies

$$\Omega(u) \ge g(u) \int_{u_1}^{u} \frac{1}{a_1(v_1)} \int_{u_1}^{v_1} \frac{1}{a_2(\xi)} \int_{u_1}^{v} \frac{1}{a_3(v)}\mathrm{d}v\mathrm{d}\xi\mathrm{d}v_1.$$

Setting the last estimate into

$$\left(a_3(a_2(a_1\Omega')')'\right)'(u) + Q(u)\,\Omega(\rho(u)) \le 0.$$

We note that $g(u)$ is a positive solution of the first-order DD inequality

$$g'(u) + R_2(u)\,g(\rho(u)) \le 0.$$

Therefore, we conclude that the related DE (7) as well has a positive solution by applying the Philos theorem [41]; this contradicts the assumptions made at the beginning of the theorem. This ends the proof.

Corollary 1. *Let (2) hold. If*

$$\liminf_{u\to\infty} \int_{\kappa(u)}^{u} R_i(\xi)\,\mathrm{d}\xi > \frac{1}{e}, \tag{11}$$

for $i = 1, 2$, *then (1) is oscillatory.*

2.2. Improved criteria

To be clear, let

$$U_1(u) := \sum_{t=0}^{\gamma}\left(\prod_{m=0}^{2t} h\left(\kappa^{[m]}(u)\right)\right)\left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1\right]\frac{I_{123}\left(\kappa^{[2t]}(u)\right)}{I_{123}(u)} q(u),$$

$$U_2(u) := \sum_{t=0}^{\gamma}\left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right)\right)\left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1\right]\frac{I_1\left(\kappa^{(2t)}(u)\right)}{I_1(u)} q(u),$$

$$S_1(u) := \left(\frac{1}{a_2(u)}\int_u^\infty \frac{1}{a_3(\xi)}\int_v^\infty U_1(v)\,\mathrm{d}v\mathrm{d}\xi\right)\int_{u_1}^{\rho(u)} \frac{1}{a_1(v)}\mathrm{d}v,$$

and

$$S_2(u) := U_2(u)\int_{u_1}^{u} \frac{1}{a_1(v_1)}\int_{u_1}^{v_1} \frac{1}{a_2(\xi)}\int_{u_1}^{v} \frac{1}{a_3(v)}\mathrm{d}v\mathrm{d}\xi\mathrm{d}v_1,$$

for any integer $\gamma \geq 0$.

We prove the following results, which provide details about the behavior of the positive solutions L^+.

Lemma 4. *Assume that $x \in L^+$. Then, eventually,*

$$\left(a_3(u)\left(a_2(u)\left(a_1(u)\Omega'(u)\right)'\right)'\right)' + U_1(\rho(u))\,\Omega(\rho(u)) \leq 0. \tag{12}$$

Proof. Assume that $x \in L^+$. We obtain

$$\begin{aligned} a_2(u)(a_1\Omega')'(u) &\geq \int_{u_1}^{u} \frac{1}{a_3(v)} a_3(v)\left(a_2(a_1\Omega')'\right)'(v)\,\mathrm{d}v \\ &\geq a_3(u)\left(a_2(a_1\Omega')'\right)'(u)\int_{u_1}^{u}\frac{1}{a_3(v)}\mathrm{d}v \\ &\geq a_3(u)\left(a_2(a_1\Omega')'\right)'(u)\, I_3(u), \end{aligned} \tag{13}$$

hence,

$$a_3(u)\left(a_2(a_1\Omega')'\right)'(u)\, I_3(u) - a_2(u)(a_1\Omega')'(u) \leq 0,$$

this implies

$$\begin{aligned} \left(\frac{a_2(u)(a_1\Omega')'(u)}{I_3(u)}\right)' &= \frac{I_3(u)\, a_3(u)\,(a_2(a_1\Omega')')'(u) - a_2(u)(a_1\Omega')'(u)}{a_3(u)\, I_3^2(u)} \\ &= \frac{1}{a_3(u)\, I_3^2(u)}\left[I_3(u)\, a_3(u)\left(a_2(a_1\Omega')'\right)'(u) - a_2(u)(a_1\Omega')'(u)\right] \\ &\leq 0. \end{aligned} \tag{14}$$

Applying this information, we determine that

$$a_1(u)\Omega'(u) \geq \int_{u_1}^{u} I_3(v)\frac{a_2(v)(a_1\Omega')'(v)}{a_2(v)\, I_3(v)}\mathrm{dv} \geq \frac{a_2(u)(a_1\Omega')'(u)}{I_3(u)} I_{23}(u),$$

or

$$\Omega'(u) \geq \frac{1}{a_1(u)} a_3(u) \left(a_2(a_1\Omega')'\right)'(u) I_{23}(u). \tag{15}$$

Yields,

$$\begin{aligned}\left(\frac{a_1(u)\Omega'(u)}{I_{23}(u)}\right)' &= \frac{I_{23}(u) a_2(u)(a_1\Omega')'(u) - I_3(u)(a_1\Omega')(u)}{a_2(u) I_{23}^2(u)} \\ &= \frac{1}{a_2(u) I_{23}^2(u)} \left[I_{23}(u) a_2(u)(a_1\Omega')'(u) - I_3(u)(a_1\Omega')(u)\right] \leq 0.\end{aligned}$$

Hence,

$$\begin{aligned}\Omega(t) &\geq \int_{u_1}^{u} I_{23}(v) \frac{a_1(v)\Omega'(v)}{a_1(v) I_{23}(v)} \mathrm{d}v \geq \frac{a_1(u)\Omega'(u)}{I_{23}(u)} \int_{u_1}^{u} I_{23}(v) \frac{1}{a_1(v)} \mathrm{d}v \\ &\geq \frac{a_2(u)(a_1\Omega')'(u)}{I_3(u)} I_{123}(u),\end{aligned} \tag{16}$$

we get,

$$\begin{aligned}\left(\frac{\Omega(u)}{I_{123}(u)}\right)' &= \frac{I_{123}(u) a_1(u)\Omega'(u) - I_{23}(u)\Omega(u)}{a_1(u) I_{123}^2(u)} \\ &= \left[I_{123}(u) a_1(u)\Omega'(u) - I_{23}(u)\Omega(u)\right] \frac{1}{a_1(u) I_{123}^2(u)} \leq 0.\end{aligned} \tag{17}$$

Now, we determine that

$$\Omega\left(\kappa^{(2t)}(u)\right) \geq \Omega\left(\kappa^{(2t+1)}(u)\right),$$

based on the information that $\kappa^{(2t+1)}(u) \leq \kappa^{(2t)}(u) \leq u$ and $\Omega'(u) > 0$. Which, along with Lemma 2, implies that

$$\begin{aligned}x(u) &> \sum_{t=0}^{\gamma} \left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right)\right) \left[\frac{\Omega\left(\kappa^{(2t)}(u)\right)}{h\left(\kappa^{(2t)}(u)\right)} - \Omega\left(\kappa^{(2t+1)}(u)\right)\right], \\ &\geq \sum_{t=0}^{\gamma} \left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right)\right) \left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1\right] \Omega\left(\kappa^{(2t)}(u)\right).\end{aligned} \tag{18}$$

Moreover, as $\left(\Omega(u)/I_{123}(u)\right)' \leq 0$ and $\kappa^{(2t)}(u) \leq u$, we have

$$\frac{\Omega\left(\kappa^{(2t)}(u)\right)}{I_{123}\left(\kappa^{(2t)}(u)\right)} \geq \frac{\Omega(u)}{I_{123}(u)},$$

and

$$\Omega\left(\kappa^{(2t)}(u)\right) \geq \frac{I_{123}\left(\kappa^{(2t)}(u)\right)}{I_{123}(u)} \Omega(u).$$

Thus, by applying the subsequent inequality and putting into (18), yields

$$x(u) > \Omega(u) \sum_{t=0}^{\gamma} \left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right) \right) \left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1 \right] \frac{I_{123}\left(\kappa^{(2t)}(u)\right)}{I_{123}(u)},$$

this produces (12), when combined with (1). This ends the proof.

The results that we present below shed light on the behavior of the positive solutions L^-.

Lemma 5. *Assume that* $x \in L^-$. *Then, eventually,*

$$\left(a_3(u)\left(a_2(u)\left(a_1(u)\Omega'(u)\right)'\right)'\right)' + U_2(\rho(u))\,\Omega(\rho(u)) \leq 0. \tag{19}$$

Proof. Assume that $x \in L^-$. We derive, for $u \geq u_1$,

$$\Omega(u) \geq \int_{u_1}^{u} \frac{1}{a_1(\xi)} a_1(\xi)\,\Omega'(\xi)\,\mathrm{d}\xi \geq a_1(u)\,\Omega'(u)\,I_1(u).$$

Then,

$$\left(\frac{\Omega(u)}{I_1(u)}\right)' = \frac{a_1^{-1}(u)}{I_1^2(u)} \left[a_1(u)\,\Omega'(u)\,I_1(u) - \Omega(u)\right] \leq 0,$$

which, with the fact that $\kappa^{(2t)}(u) \leq u$, gives

$$\Omega\left(\kappa^{(2t)}(u)\right) \geq \frac{I_1\left(\kappa^{(2t)}(u)\right)}{I_1(u)} \Omega(u). \tag{20}$$

Using the facts $\Omega'(u) > 0$ and the inequality (20), relation (6) reduces to

$$\begin{aligned} x(u) &> \sum_{t=0}^{\gamma} \left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right) \right) \left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1 \right] \Omega\left(\kappa^{(2t)}(u)\right) \\ &> \Omega(u) \sum_{t=0}^{\gamma} \left(\prod_{m=0}^{2t} h\left(\kappa^{(m)}(u)\right) \right) \left[\frac{1}{h\left(\kappa^{(2t)}(u)\right)} - 1 \right] \frac{I_1\left(\kappa^{(2t)}(u)\right)}{I_1(u)}. \end{aligned}$$

Combining this inequality with (1), we get (19). This ends the proof.

Theorem 2. *Suppose that the two first-order DDEs*

$$g'(u) + S_1(u)\,g(\rho(u)) = 0 \tag{21}$$

and

$$g'(u) + S_2(u)\,g(\rho(u)) = 0 \tag{22}$$

are oscillatory, then equation (1) is oscillatory.

Proof. Assume that (1) has an eventually positive solution at x. One solution to Lemma 3 meets any of the the possibilities of L^{+} or L^{-}. By replacing the relationship (10) by (12) in the case L^{+} and by (19) in the case L^{-}. We get (21) and (22). This ends the proof.

Example 1. *Consider*

$$\left(x\left(u\right)+h_0 x\left(\kappa_0 u\right)\right)^{(4)}+\frac{q_0}{u^4}x\left(\rho_0 u\right)=0, \tag{23}$$

where $u>0,\ h_0\in[0,1)\,,\ q_0>0,\ \rho,\ \kappa\in(0,1),$ *and* $a_1\left(u\right)=a_2\left(u\right)=a_3\left(u\right)=1.$
For our equation it is easy to verify that

$$Q\left(u\right):=\left(1-h\left(\rho\left(u\right)\right)\right)q\left(u\right)=q_0\left(1-h_0\right)\frac{1}{u^4},$$

and

$$\begin{aligned} R_1\left(u\right) &= \left(\frac{1}{a_2\left(u\right)}\int_u^{\infty}\frac{1}{a_3\left(\xi\right)}\int_{\xi}^{\infty}Q\left(v\right)\mathrm{d}v\mathrm{d}\xi\right)\int_{u_1}^{\rho(u)}\frac{1}{a_1\left(v\right)}\mathrm{d}v \\ &= \left(\int_u^{\infty}\int_{\xi}^{\infty}q_0\left(1-h_0\right)\frac{1}{v^4}\mathrm{d}v\mathrm{d}\xi\right)\int_{u_1}^{\rho_0 u}\mathrm{d}v \\ &= q_0\left(1-h_0\right)\left(\int_u^{\infty}\int_{\xi}^{\infty}\frac{1}{v^4}\mathrm{d}v\mathrm{d}\xi\right)\rho_0 u \\ &= q_0\left(1-h_0\right)\left(\int_u^{\infty}\frac{1}{3\xi^3}\mathrm{d}\xi\right)\rho_0 u \\ &= q_0\left(1-h_0\right)\left(\frac{1}{6u^2}\right)\rho_0 u \\ &= \frac{\rho_0 q_0\left(1-h_0\right)}{6}\frac{1}{u}. \end{aligned}$$

Thus, condition (11) for $i=1$ becomes

$$\begin{aligned} &\liminf_{u\to\infty}\int_{\kappa(u)}^{u}R_1\left(\xi\right)\mathrm{d}\xi \\ &=\liminf_{u\to\infty}\int_{\kappa_0 u}^{u}\frac{\rho_0 q_0\left(1-h_0\right)}{6}\frac{1}{\xi}\mathrm{d}\xi \\ &= \frac{\rho_0 q_0\left(1-h_0\right)}{6}\ln\frac{1}{\kappa_0}, \end{aligned}$$

which is satisfied when

$$\frac{\rho_0 q_0\left(1-h_0\right)}{6}\ln\frac{1}{\kappa_0}>\frac{1}{\mathrm{e}}. \tag{24}$$

Similarly, we have

$$R_2\left(u\right) = Q\left(u\right)\int_{u_1}^{u}\frac{1}{a_1\left(v_1\right)}\int_{u_1}^{v_1}\frac{1}{a_2\left(\xi\right)}\int_{u_1}^{\xi}\frac{1}{a_3\left(v\right)}\mathrm{d}v\mathrm{d}\xi\mathrm{d}v_1$$

$$
\begin{aligned}
&= q_0 (1-h_0) \frac{1}{u^4} \int_{u_1}^{u} \int_{u_1}^{v_1} \int_{u_1}^{\xi} \mathrm{d}v \mathrm{d}\xi \mathrm{d}v_1 \\
&= q_0 (1-h_0) \frac{1}{u^4} \frac{u^3}{6} \\
&= \frac{q_0 (1-h_0)}{6} \frac{1}{u}.
\end{aligned}
$$

Thus, condition (11) for $i = 2$ *becomes*

$$
\begin{aligned}
& \liminf_{u \to \infty} \int_{\kappa(u)}^{u} R_2 (\xi) \, \mathrm{d}\xi \\
= \; & \liminf_{u \to \infty} \int_{\kappa_0 u}^{u} \frac{q_0 (1-h_0)}{6} \frac{1}{\xi} \mathrm{d}\xi \\
= \; & \frac{q_0 (1-h_0)}{6} \ln \frac{1}{\kappa_0},
\end{aligned}
$$

which is satisfied when

$$
\frac{q_0 (1-h_0)}{6} \ln \frac{1}{\kappa_0} > \frac{1}{\mathrm{e}}. \tag{25}
$$

Using Corollary 1, we conclude that equation (23) is oscillatory if both conditions (24) and (25) are satisfied.

3. conclusion

In the canonical case, this work focuses on a fundamental matter, namely understanding the relationships between the corresponding function $\Omega(u)$ *and the solution* $x(u)$ for the fourth-order NDDE. *We have presented oscillation theories through which we have established oscillation criteria that guarantee the oscillation of the solutions to the equation under study., first after using the classical relation that links the solution to its corresponding function in the two cases (*L^- *and* L^+*) for positive solutions of our equation. Second, we have improved this classical relation in the two cases (*L^- *and* L^+*).* Notably, the main distinction between L^+ and L^- is the difference in the second derivative of the *corresponding function* $\Omega(u)$; however, this change affects several of the monotonic and asymptotic characteristics.

The results of this study complement many of previously published findings in the literature. To our knowledge, this equation has not been studied by many researchers, so it would be a good idea to apply these results to non-linear higher-order NDEs in the future.

Competing interests

There are no competing interests

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