EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5731
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

An Application of Legendre Polynomials to Bi-Bazilevic Functions associated with q-Ruscheweyh Operator

Waleed Al-Rawashdeh

Department of Mathematics, Zarqa University, 2000 Zarqa, 13110, Jordan

Abstract. In this paper, we make use of the concept of fractional q-calculus to introduce a novel class of bi-Bazilevic functions involving q-Ruscheweyh differential operator that are subordinate to Legendre Polynomials. This study explores the characteristics and behaviors of these functions, providing estimates for the modulus of the initial Taylor series coefficients a_2 and a_3 within this specific class and its various subclasses. Additionally, the research delves into the traditional Fekete-Szegö functional problem of functions f belong to the newly defined class and several of its subclasses.

2020 Mathematics Subject Classifications: 30C45, 30C50, 33C45, 33C05, 11B39

Key Words and Phrases: Bi-Univalent Functions; Bi-Bazilevic; q-Ruscheweyh Differential Operator; Jackson q-Derivative Operator; q-Gamma Function; Fractional q-Calculus Operator; Legendre Polynomials; Coefficient estimates; Fekete-Szegö functional problem; Convolution; Hadamard Product

1. Introduction

The q-calculus offers essential tools that are widely utilized to examine different categories of analytic functions. Various geometric properties, such as coefficient estimates, convexity, near-convexity, distortion bounds, and radii of starlikeness, have been investigated within these classes of functions. Moreover, q-analysis has garnered considerable attention in operator theory, as highlighted by the extensive research documented in [10]. The progress made in operator theory within this domain has inspired many researchers, leading to the publication of a variety of scholarly articles.

Recently, Srivastava [46] has released a comprehensive survey and expository review paper, which serves as a significant resource for researchers interested in the field of geometric function theory. This survey meticulously investigates the mathematical frameworks and applications of fractional q-derivative operators and fractional q-calculus, particularly in relation to geometric function theory. It addresses the complexities involved in utilizing

DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5731

Email address: walrawashdeh@zu.edu.jo (W. Al-Rawashdeh)

these fractional operators and calculus concepts to characterize mathematical functions and their geometric attributes. Furthermore, the review highlights the practical applications and ramifications of fractional q-derivative operators within the expansive scope of geometric function theory, thereby offering an in-depth analysis of both the theoretical underpinnings and practical implementations of these mathematical instruments in the relevant domain.

Many researchers have employed the concept of q-calculus to establish novel subclasses of analytic and univalent functions. This investigation seeks to enhance the comprehension of the properties and attributes of analytic and univalent functions, particularly in relation to the newly introduced q-derivative, thereby elucidating the criteria that determine membership within the specified subclasses, see, for example, the articles [9], [22], [28], [38], [40], [47], [51] and the related references included therein.

In this research paper, the central focus lies in the application of the concept of the q-derivative to derive specific differential operator. This operator is introduced with the aim of generalizing the class of $q$analogue of the Ruscheweyh operator within the set of univalent functions. By utilizing the newly defined operator, we define a novel class of bi-Bazilvic functions associated with the Legendre polynomials.

Now, consider the set $\mathcal{H}$, which consists of all functions $f(z)$ that are analytic within the open unit disk denoted as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. The exploration of such functions contributes to a deeper comprehension of complex analysis and its applications. Moreover, any function f belongs to the set $\mathcal{H}$ can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1}$$

The Hadamard product (or convolution) of two analytic functions $f(z)$ given by Equation (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as:

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The convolution facilitates deeper mathematical exploration and enhances better understanding of the geometric and symmetric properties of $f \in \mathcal{H}$. The significance of convolution, within operator theory and geometric function theory, is well-documented in the literature. For more information about convolution in the geometric function theory, we invite the interested reader to see the monograph [10], the articles [23], [39] [47], and the related references provided therein.

Let us consider two functions, f and g, which are analytic within the open unit disk $\mathbb{D}$. We say that f is subordinated to g in $\mathbb{D}$, denoted as $f(z) \prec g(z)$ for every z in $\mathbb{D}$, if there exists a Schwarz function w that meets the criteria of $w(0) = 0$ and $|w(z)| < 1$ for all z in $\mathbb{D}$. This means that for every z in $\mathbb{D}$, the relationship $f(z) = g(w(z))$ holds true. This concept of subordination is essential in complex analysis, as it allows us to analyze and compare the behaviors of two analytic functions within the unit disk. Importantly, when g is a univalent function in $\mathbb{D}$, the condition $f(z) \prec g(z)$ translates to the equivalence of $f(0) = g(0)$ and the inclusion $f(\mathbb{D}) \subset g(\mathbb{D})$. This equivalence underscores the importance of the subordination principle in elucidating the connections between analytic functions. For those seeking a deeper understanding and more detailed discussions on the Subordination Principle, it is recommended to consult the monographs [18], [17], [34], and [36], which offer thorough explanations and applications of this principle within the realms of complex analysis and geometric function theory.

In this context, $\mathcal{S}$ represents the set of functions that are univalent in the open unit disk $\mathbb{D}$ and belong to the set $\mathcal{H}$. As known univalent functions are injective functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem, the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{2}$$

Now, we introduce the class Σ in the following manner. A function $f \in \mathcal{H}$ is considered bi-univalent if both the function itself and its inverse, f^{-1}, are univalent within the unit disk $\mathbb{D}$. Consequently, we define Σ as the collection of all bi-univalent functions in $\mathcal{H}$ that are represented by equation (1). For more information about univalent and bi-univalent functions we refer the readers to the articles [30], [32], [37] the monograph [18], [21] and the references provided therein.

Research in geometric function theory illuminates the complex connections between coefficients and the geometric properties of functions. By analyzing the constraints on the modulus of a function's coefficients, we can better understand the behavior and interactions of these functions within the mathematical landscape. This analytical perspective not only deepens our grasp of the fundamental principles of geometric function theory but also opens avenues for further investigation and innovation in this vibrant area of study. For instance, within the class $\mathcal{S}$, it is shown that the modulus of the coefficient a_n is limited by the value of n. These constraints on the modulus of coefficients yield important insights into the geometric features of these functions. In particular, the bounds on the second coefficients of functions in the class $\mathcal{S}$ provide essential information about the growth and distortion bounds relevant to this class.

The study of coefficient-related characteristics of functions within the bi-univalent class Σ began in the 1970s. A pivotal moment occurred in 1967 when Lewin [30] investigated the bi-univalent function class and set a limit for the coefficient $|a_2|$. In 1969, Netanyahu [37] furthered this research by establishing that the maximum value of $|a_2|$ for functions in Σ is $\frac{4}{3}$. Later, in 1979, Brannan and Clunie [11] proved that for functions belonging to this class, the inequality $|a_2| \leq \sqrt{2}$ is valid. This foundational research has led to a multitude of studies focused on the coefficient bounds for various subclasses of bi-univalent functions. However, despite the extensive investigations into coefficient bounds, there is still a considerable lack of understanding regarding the general coefficients $|a_2|$ when $n \geq 4$. The difficulty in estimating these coefficients, especially the general coefficient $|a_n|$, remains an open question in the field, underscoring the complexity and depth of the bi-univalent function class and indicating that further research is essential to grasp the behavior of these coefficients in higher dimensions.

In 1933, Fekete and Szegö [19] established the upper bound of the expression $|a_3 - \lambda a_2^2|$ for univalent functions f, where the $0 \leq \lambda \leq 1$. This pivotal finding gave rise to the Fekete-Szegö problem, which focuses on maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for functions f belonging to the class $\mathcal{H}$, with λ being any complex number. A significant body of research has since been dedicated to exploring the Fekete-Szegö functional and related coefficient estimation issues. Noteworthy contributions to this area can be found in various publications, including [3], [4], [6], [12], [15], [24], [26], [31], [32], [48] and the references provided therein. These investigations have significantly enhanced the comprehension of the Fekete-Szegö problem and its relevance within the domain of geometric function theory.

2. Preliminaries and Lemmas

The information provided in this section is crucial for comprehending the key findings of this study. In 1975, Ruscheweyh [43] introduced the operator $\mathcal{R}$, which is defined through the convolution (Hadamard product) of two power series. In particular, for a function $f \in \mathcal{H}$, a variable $z \in \mathbb{D}$, and a real number $\alpha > -1$, the Ruscheweyh operator is articulated as follows:

$$\mathcal{R}^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}.$$

For $\alpha = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get the Ruscheweyh derivative $\mathcal{R}^\alpha$ of the function f:

$$\mathcal{R}^\alpha f(z) = z\frac{\left(z^{\alpha-1} f(z)\right)^{(\alpha)}}{\Gamma(\alpha+1)}.$$

Moreover, the Taylor-Maclaurin series (see, for example [27]) of $\mathcal{R}^\alpha f$ is given by

$$\mathcal{R}^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha+1)} a_n z^n.$$

In this framework, we revisit the concept of q-difference operators, which play a crucial role in the fields of hypergeometric series, quantum mechanics, and the theory of geometric functions. The introduction of q-calculus can be traced back to Jackson [23]. Following this, Kanas and Răducanu [25] utilized fractional q-calculus operators to explore particular categories of analytic functions associated with conic domains.

The q-integer number, for $0 < q < 1$ and non-negative integer n, is defined as follows

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad \text{with} \quad [0]_q = 0.$$

In general, for any non-negative real number x, we have $[x]_q = \dfrac{1-q^x}{1-q}$. Moreover, the q-shifted factorial is defined by

$$[n]_q! = [n]_q[n-1]_q[n-2]_q \cdots [2]_q[1]_q, \quad \text{with} \quad [0]_q! = 1.$$

It is obvious that $\lim_{q\to 1^-} [n]_q = n$ and $\lim_{q\to 1^-} [n]_q! = n!$.

Let the function f belong to the set $\mathcal{H}$ and represented by Equation (1). The q-Jackson derivative operator (or q-difference operator) is defined by

$$\mathcal{D}_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & \text{if} \ \ z \neq 0 \\ f'(0), & \text{if} \ \ z = 0 \\ f'(z), & \text{as } q \to 1^-. \end{cases}$$

Therefore, for a function $f \in \mathcal{H}$ that is given by Equation (1), it is easy to see that

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

For example, if $n \in \mathbb{N} = \{1, 2, \cdots\}$ and $z \in \mathbb{D}$, then

$$\mathcal{D}_q (z^n) = \frac{(q^n-1)z^{n-1}}{(q-1)} = [n]_q z^{n-1}.$$

Also, $\lim_{q\to 1^-} \mathcal{D}_q (z^n) = \lim_{q\to 1^-} [n]_q z^{n-1} = nz^{n-1}$, which is the ordinary derivative of the function z_n.

Moreover, for $m \in \mathbb{N}$, we have the following

$$\mathcal{D}_q^0 f(z) = f(z), \quad \text{and} \quad \mathcal{D}_q^m f(z) = \mathcal{D}_q \left(\mathcal{D}_q^{m-1} f(z) \right).$$

It is known that, for $f, g \in \mathcal{H}$, we have the following rules for the q-difference operator

(i) $\mathcal{D}_q(mf(z) \pm ng(z)) = m\mathcal{D}_q f(z) \pm n\mathcal{D}_q g(z), \;\text{ for }\; m, n \in \mathbb{C}.$

(ii) $\mathcal{D}_q(fg)(z) = f(z)\mathcal{D}_q g(z) + g(z)\mathcal{D}_q f(z).$

(iii) $\mathcal{D}_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\mathcal{D}_q f(z) - f(z)\mathcal{D}_q g(z)}{g(z)g(qz)}$, where $g(z)g(qz) \neq 0$.

For any real number x and natural number n, the q-generalized Pochhammer symbol is defined as follows

$$[x; n]_q = [x]_q [x+1]_q [x+2]_q \cdots [x+n-1]_q.$$

Moreover, for $x > 0$, the q-Gamman function is defined as follows

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \text{with} \quad \Gamma_q(1) = 1.$$

Now, we present a q-analogue of the Ruscheweyh differential operator by employing the convolution alongside the q-difference operator $\mathcal{R}_q^\alpha : \mathcal{H} \to \mathcal{H}$. Thus, for any $f \in \mathcal{H}$ and $\alpha > -1$, this linear operator is defined as $\mathcal{R}_q^\alpha f(z) = \mathcal{F}_{q,\alpha+1}(z) * f(z)$, where

$$\mathcal{F}_{q,\alpha+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\alpha)}{[n-1]_q! \Gamma_q(\alpha+1)} z^n.$$

More precisely, the q-Rucheweyh differential operator can be written as follows

$$\mathcal{R}_q^\alpha f(z) = z + \sum_{n=2}^{\infty} \psi_n(q, \alpha) a_n z^n,$$

where

$$\psi_n = \psi_n(q, \alpha) = \frac{\Gamma_q(\alpha+n)}{[n-1]_q! \Gamma_q(\alpha+1)}.$$

It is clear that,

$$\mathcal{R}_q^0 f(z) = f(z), \quad \mathcal{R}_q^1 f(z) = z\mathcal{D}_q f(z), \text{ and}$$

$$\mathcal{R}_q^n f(z) = \frac{z\mathcal{D}_q^n \left(z^{n-1} f(z)\right)}{[n]_q!}, \quad n \in \mathbb{N}$$

It is worth mention that,

$$\lim_{q \to 1^-} \mathcal{F}_{q,\alpha+1}(z) = \frac{z}{(1-z)^{\alpha+1}},$$

and

$$\lim_{q\to 1^-} \mathcal{R}_q^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}} = \mathcal{R}^\alpha f(z).$$

For more information about q-Rucheweyh differential operator and q-derivative operator, we refer the interested readers to consult the articles [8], [9], [13], [16], [23], [25], [27], [44], [46], [50], [51], [52] and the references provided therein.

Legendre polynomials belong to a well-established family of classical orthogonal polynomials. They are defined by their compliance with a second-order linear differential equation, which emerges naturally in the context of solving initial value problems in three-dimensional spaces exhibiting spherical symmetry. The equation associated with Legendre polynomials is classified as a Legendre second-order differential equation:

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \quad -1 < x < 1. \tag{3}$$

The process of identifying the parameters $\lambda \in \mathbb{R}$ that allows Equation (3) to possess a bounded solution is referred to as a singular Sturm-Liouville problem. The significance of these eigenvalues λ lies in their role in determining the nature of the solutions to the differential equation. In this context, the necessity for boundary conditions is eliminated, as the boundedness of the solution itself serves as a substitute for these conditions. It has been established that the only permissible values of λ that yield bounded solutions are of the form $\lambda = n(n+1)$, where n is a natural number. These values of λ are called the eigenvalues of the Sturm-Liouville problem.

The polynomial solutions of Legendre's differential equation can be explicitly expressed as follows. These solutions play a significant role in various applications, particularly in mathematical physics and engineering, where they are utilized to solve problems involving spherical symmetry.

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, \tag{4}$$

where $\lfloor z \rfloor$ is the floor of z, i.e. the greatest integer $m \leq z$. It is worth to mention that when n is even, the polynomial $P_n(x)$ exclusively comprises even powers of x, while for odd n it contains only odd powers. Consequently, $P_n(x)$ is classified as an even function for even n and as an odd function for odd n. Their unique properties, such as orthogonality and recurrence relations, further enhance their utility in both theoretical and applied mathematics. The first few of them are: $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x),\ P_4(x) = \frac{1}{8}(35x^4-30x^2+3),\ P_5(x) = \frac{1}{8}(63x^5-70x^3+15x).$

It is important to highlight that the Legendre polynomials can be represented in a more concise manner. Specifically, the n^{th} Legendre polynomial, denoted as P_n, can be formulated using Rodrigues' formula (5), which serves as a foundational tool in the study of these polynomials.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{5}$$

As a result of Rodrigue's formula, one can observe a specific connection that exists between three consecutive Legendre polynomials. This relationship plays a crucial role in understanding the properties and behaviors of these polynomials and their applications in mathematical physics.

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

It can be demonstrated that the Legendre polynomials are produced by the generating function

$$g(x,t) = \frac{1}{\sqrt{t^2-2xt+1}}.$$

This relationship highlights the significance of this function in generating these important polynomials, which have numerous applications in physics and engineering. Additionally, when the function $g(x,t)$ is expanded as a Taylor series in terms of t, the coefficient corresponding to t^n is the Legendre polynomial $P_n(x)$:

$$g(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n, \quad \text{where} \quad |x|<1 \quad \text{and} \quad t \in \mathbb{D}. \tag{6}$$

In this paper, the symbol $\mathcal{P}$ denotes the Caratheodory class, which is formally defined as

$$\mathcal{P} = \{\Omega \in \mathcal{H} : \Omega(0)=1,\ \mathcal{R}(\Omega(z))>0,\ z\in\mathbb{D}\}.$$

It is established in the literature (for example, see [21], page 102) that the function $\phi(z)$ is a member of the class $\mathcal{P}$ for any real number θ, with ϕ expressed as

$$\phi(z) = \frac{1-z}{\sqrt{1-(2\cos\theta)z+z^2}}.$$

Notably, the function $\phi(z)$ transforms the open unit disk $\mathbb{D}$ onto the right half-plane $\mathcal{R}(w)>0$, with the exception of the slit along the positive real axis extending from $|\cos(\alpha/2)|^{-1}$ to infinity. Consequently, ϕ exhibits starlikeness with respect to the point 1. By consulting Equation (6), it is straightforward to verify the following equation, for any z within the open unit disk $\mathbb{D}$.

$$\mathcal{L}(z) = 1 + \sum_{n=1}^{\infty} \left[P_n(\cos\theta) - P_{n-1}(\cos\theta)\right] z^n \tag{7}$$

$$= 1 + \sum_{n=1}^{\infty} \beta_n(\theta) z^n. \tag{8}$$

Using the Rodregue's formula (5), we easily obtain the following initial values of $\beta_n(\theta) = P_n(\cos\theta) - P_{n-1}(\cos\theta)$ which are listed below:

$$\beta_1(\theta) = \cos\theta - 1,\ \beta_2(\theta) = \frac{1}{2}(\cos\theta - 1)(1 + 3\cos\theta).$$

Additional information regarding the Legengre polynomials readers are encouraged to consult the articles referenced as [1], [2], [5], [7], [13], [35] and [41], as well as the monographs [18], [21], [42], [49], and the related sources.

Expanding on these foundational concepts, our objective is to introduce a novel class. This class is comprised of bi-Bazilevic functions characterized by the q-Ruscheweyh differential operator associated with Legendre polynomials. We denote this class as $\mathcal{B}^{\lambda}(\delta, R_q^{\alpha}, \phi)$, and we next provide a formal definition for this class.

Definition 1. *A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{B}^{\lambda}(\delta, R_q^{\alpha}, \phi)$ if it obeys the following subordination conditions:*

$$\frac{e^{i\delta} z^{1-\lambda} \left(R_q^{\alpha} f(z)\right)'}{\left(R_q^{\alpha} f(z)\right)^{1-\lambda}} \prec \phi(z)\cos\delta + i\sin\delta,$$

and

$$\frac{e^{i\delta} w^{1-\lambda} \left(R_q^{\alpha} g(w)\right)'}{\left(R_q^{\alpha} g(w)\right)^{1-\lambda}} \prec \phi(w)\cos\delta + i\sin\delta,$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $\lambda \geq 0$, $0 < q < 1$, $\alpha > -1$, and $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

The following lemma, extensively elaborated upon in existing literature, represents well-established principles that hold significant importance for the research we are currently presenting.

Lemma 1. *[26] if Ω belongs to the Caratheodory class, then for $z \in \mathbb{D}$ the function Ω can be written as*

$$\Omega(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

Moreover, $|c_n| \leq 2$ for each natural number n.

The lemma presented in the following discussion is extensively referenced in existing literature and is regarded as a foundational principle that significantly influences the research we are conducting.

Lemma 2. *[26] Let K and L be real numbers. Let p and q be complex numbers. If $|p| < r$ and $|q| < r$,*

$$|(K+L)p + (K-L)q| \leq \begin{cases} 2r|K|, & \text{if } |K| \geq |L| \\ 2r|L|, & \text{if } |K| \leq |L|. \end{cases}$$

This paper seeks to explore two novel categories of bi-Bazilevic functions that are defined through the q-Ruscheweyh operator within the open unit disk $\mathbb{D}$, with a particular connection to Legendre polynomials. The central objective is to establish estimates for the magnitudes of the initial coefficients $|a_2|$ and $|a_3|$ that are linked to the Taylor-Maclaurin series representation of functions belonging to this class. Additionally, the research delves into the Fekete-Szegö functional problem pertinent to these functions, thereby enhancing the comprehension of their inherent characteristics. Moreover, some known corollaries are presented based on the choices of the parameters involved in defining our specific class.

3. Coefficient bounds of the function class

This section of the paper focuses on investigating the bounds pertaining to the modulus of the initial coefficients of functions belonging to the class $\mathcal{B}^{\lambda}(\delta, R_q^{\alpha}, \phi)$, along with several of its distinct subclasses, as delineated in Equation (1).

Theorem 1. *Let a function f be in the family Σ. If the function f belongs to the class $\mathcal{B}^{\lambda}(\delta, R_q^{\alpha}, \phi)$ and is represented by the equation (1), then the following inequalities hold:*

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\theta|\cos\delta}{\sqrt{|A\cos\delta(\cos\theta-1)+(1-3\cos\theta)(\lambda+1)^2\psi_2^2e^{i\delta}|}}, \tag{9}$$

and

$$|a_3| \leq \frac{|1-\cos\theta|\cos\delta}{(\lambda+2)\psi_3} + \frac{(1-\cos\theta)^2\cos^2\delta}{(\lambda+1)^2\psi_2^2}, \tag{10}$$

where

$$A = 2(\lambda+2)\psi_3 + (\lambda-1)(\lambda+2)\psi_2^2.$$

Proof. Suppose a function f belongs to the class $\mathcal{B}(\lambda, \delta, R_\eta^{\alpha}, \beta(t))$. Consulting the Definition 1 and Subordination Principle, we can find two Schwarz functions $k(z)$ and $h(w)$ defined on the open unit disk $\mathbb{D}$ such that

$$\frac{e^{i\delta}z^{1-\lambda}\left(R_q^{\alpha}f(z)\right)'}{\left(R_q^{\alpha}f(z)\right)^{1-\lambda}} = \phi(k(z))\cos\delta + i\sin\delta, \tag{11}$$

and

$$\frac{e^{i\delta}w^{1-\lambda}\left(R_q^{\alpha}g(w)\right)'}{\left(R_q^{\alpha}g(w)\right)^{1-\lambda}} = \phi(h(w))\cos\delta + i\sin\delta. \tag{12}$$

Now, using those Schwarz functions, we define two new analytic functions $\eta(z)$ and $\zeta(w)$ as follow:

$$\eta(z) = \frac{1+k(z)}{1-k(z)} \quad \text{and} \quad \zeta(w) = \frac{1+h(w)}{1-h(w)}.$$

It is clear that, these functions $\eta(z)$ and $\zeta(w)$ are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. Therefore, they can be written as follows

$$\eta(z) = \frac{1+k(z)}{1-k(z)} = 1 + \eta_1 z + \eta_2 z^2 + \cdots$$

and

$$\zeta(w) = \frac{1+h(w)}{1-h(w)} = 1 + \zeta_1 w + \zeta_2 w^2 + \cdots$$

Moreover, $\eta(0) = 1 = \zeta(0)$, $\Re(\eta) > 0$, $\Re(\zeta) > 0$, $|\eta_j| \leq 2$ and $|\zeta_j| \leq 2$ for all natural numbers j.

Equivalently, we get the following representations of $k(z)$ and $h(w)$

$$k(z) = \frac{\eta(z)-1}{\eta(z)+1} = \frac{\eta_1}{2} z + \left(\frac{\eta_2}{2} - \frac{\eta_1^2}{4}\right) z^2 + \cdots, \tag{13}$$

and

$$h(w) = \frac{\zeta(w)-1}{\zeta(w)+1} = \frac{\zeta_1}{2} w + \left(\frac{\zeta_2}{2} - \frac{\zeta_1^2}{4}\right) w^2 + \cdots. \tag{14}$$

Therefore, by consulting Equation (7) and Equation (13) the right-hand side of Equation (11) can be written as:

$$\begin{aligned} &\phi(k(z)) \cos\delta + i \sin\delta \\ &= \left(1 + \frac{\beta_1 \eta_1}{2} z + \left[\beta_1 \left(\frac{\eta_2}{2} - \frac{\eta_1^2}{4}\right) + \frac{\beta_2 \eta_1^2}{4}\right] z^2 + \cdots\right) \cos\delta + i \sin\delta. \end{aligned} \tag{15}$$

Moreover, the left-hand side of Equation (11) can be written as:

$$\begin{aligned} &\frac{e^{i\delta} z^{1-\lambda} \left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\lambda}} \\ &= e^{i\delta}(\lambda+1)\psi_2 a_2 z + e^{i\delta} \left[\frac{(\lambda-1)(\lambda+2)}{2} \psi_2^2 a_2^2 + (\lambda+2)\psi_3 a_3\right] z^2 + \cdots \end{aligned} \tag{16}$$

Now, consulting Equation (11), we get the right-hand sides of Equation (15) and Equation (16) are equal. Therefore comparing these equations coefficients we get the following two equations:

$$2e^{i\delta}(\lambda+1)\psi_2 a_2 = \beta_1 \eta_1 \cos\delta, \tag{17}$$

and

$$\begin{aligned} &e^{i\delta} \left[2(\lambda-1)(\lambda+2)\psi_2^2 a_2^2 + 4(\lambda+2)\psi_3 a_3\right] \\ &= \left[2\beta_1 \eta_2 + (\beta_2 - \beta_1)\eta_1^2\right] \cos\delta. \end{aligned} \tag{18}$$

On the other hand, by consulting Equation (7) and Equation (14) the right-hand side of Equation (12) can be written as:

$$\phi(h(w)) = 1 + \frac{\beta_1 \zeta_1}{2} w + \left[\beta_1 \left(\frac{\zeta_2}{2} - \frac{\zeta_1^2}{4}\right) + \frac{\beta_2 \zeta_1^2}{4}\right] w^2 + \cdots \tag{19}$$

Moreover, the left-hand side of Equation (12) can be written as:

$$\begin{aligned}&\frac{e^{i\delta}w^{1-\lambda}\left(R_q^{\alpha}g(w)\right)'}{\left(R_q^{\alpha}g(w)\right)^{1-\lambda}}\\&=-e^{i\delta}(\lambda+1)\psi_2a_2w+e^{i\delta}\left[\begin{pmatrix}2(\lambda+2)\psi_3\\+\dfrac{(\lambda-1)(\lambda+2)}{2}\psi_2^2\end{pmatrix}a_2^2-(\lambda+2)\psi_3a_3\right]w^2+\cdots\end{aligned} \tag{20}$$

Now, considering Equation (12) and comparing coefficients on bothsides of Equation (19) and Equation(20) we get the following two equations:

$$-2e^{i\delta}(\lambda+1)\psi_2a_2=\beta_1\zeta_1\cos\delta, \tag{21}$$

and

$$\begin{aligned}&e^{i\delta}\left(\left[8(\lambda+2)\psi_3+2(\lambda-1)(\lambda+2)\psi_2^2\right]a_2^2-4(\lambda+2)\psi_3a_3\right)\\&=\left[2\beta_1\zeta_2+(\beta_2-\beta_1)\zeta_1^2\right]\cos\delta.\end{aligned} \tag{22}$$

Therefore, using Equation (17) and Equation (21), we easily derive the following equation

$$e^{i\delta}a_2=\frac{\beta_1\eta_1\cos\delta}{2(\lambda+1)\psi_2}=\frac{-\beta_1\zeta_1\cos\delta}{2(\lambda+1)\psi_2}. \tag{23}$$

On one hand, adding Equation (18) to Equation (22), we obtain the following equation

$$\begin{aligned}&e^{i\delta}\left[4(\lambda-1)(\lambda+2)\psi_2^2+8(\lambda+2)\psi_3\right]a_2^2\\&=\left[2\beta_1(\eta_2+\zeta_2)+(\beta_2-\beta_1)(\eta_1^2+\zeta_1^2)\right]\cos\delta.\end{aligned} \tag{24}$$

on the other hand, consulting Equation (23), we obtain the following equation:

$$\eta_1^2+\zeta_1^2=\frac{8(\lambda+1)^2\psi_2^2e^{i(2\delta)}}{\beta_1^2\cos^2\delta}a_2^2. \tag{25}$$

Now, using Equation (24) and Equation (25), we easily derive the following equations

$$\begin{aligned}&\beta_1^2\cos\delta e^{i\delta}\left[4(\lambda-1)(\lambda+2)\psi_2^2+8(\lambda+2)\psi_3\right]a_2^2\\&=2\beta_1^3\cos^2\delta(\eta_2+\zeta_2)+8(\beta_2-\beta_1)(\lambda+1)^2\psi_2^2e^{i(2\delta)}a_2^2.\end{aligned}$$

Therefore, considering the initial values $\beta_1=\cos\theta-1$ and $2(\delta_2-\delta_1)=(3\cos\theta-1)(\cos\theta-1)$, we easily get the following equation

$$a_2^2=\frac{\beta_1^2\cos^2\delta(\eta_2+\zeta_2)e^{-i\delta}}{\beta_1\cos\delta[2(\lambda-1)(\lambda+2)\psi_2^2+4(\lambda+2)\psi_3]+2(1-3\cos\theta)(\lambda+1)^2\psi_2^2e^{i\delta}}. \tag{26}$$

Thus, using the constraints $|\eta_2| \leq 2$ and $|\zeta_2| \leq 2$, then simple calculations give the desired estimation of $|a_2|$ presented in Equation (9).

In the next step, we seek to determine the coefficient estimate for $|a_3|$. By substituting Equation (22) from Equation (18), we can derive the following equation:

$$8e^{i\delta}(\lambda+2)\psi_3(a_3 - a_2^2) = [2\beta_1(\eta_2 - \zeta_2) + (\beta_2 - \beta_1)(\eta_1^2 - \zeta_1^2)]\cos\delta.$$

Now, consulting Equation (23), we get $\eta_1 = -\zeta_1$. Hence, the last equation can be written as

$$a_3 = \frac{\beta_1(\eta_2 - \zeta_2)\cos\delta}{4e^{i\delta}(\lambda+2)\psi_3} + a_2^2. \tag{27}$$

Moreover, using Equation (25), the last equation can be written as:

$$a_3 = \frac{\beta_1(\eta_2 - \zeta_2)\cos\delta}{4e^{i\delta}(\lambda+2)\psi_3} + \frac{\beta_1^2(\eta_1^2 + \zeta_1^2)\cos^2\delta}{8\delta e^{i(2\delta)}(\lambda+1)^2\psi_2^2}. \tag{28}$$

Finally, considering the value $\beta_1 = \cos\theta - 1$, then using the constraints $|\eta_j| \leq 2$ and $|\zeta_j| \leq 2$ for all $j \in \mathbb{N}$, the last equation gives the required estimation of $|a_3|$ that is represented by the Inequality (10). Consequently, the proof of Theorem 1 is now concluded.

By selecting particular values of λ in Definition 1, it is possible to obtain the subsequent subclasses.

Example 1. *A bi-univalent function f that represented as (1) belongs to the subclass $\mathcal{B}^0(\delta, R_q^\alpha, \phi)$ if the following subordinations hold:*

$$\frac{e^{i\delta}z\left(R_q^\alpha f(z)\right)'}{R_q^\alpha f(z)} \prec \phi(z)\cos\delta + i\sin\delta, \tag{29}$$

and

$$\frac{e^{i\delta}w\left(R_q^\alpha g(w)\right)'}{R_q^\alpha g(w)} \prec \phi(w)\cos\delta + i\sin\delta, \tag{30}$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, and $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Example 2. *A bi-univalent function f that represented as (1) belongs to the subclass $\mathcal{B}^1(\delta, R_q^\alpha, \phi)$ if the following subordinations hold:*

$$e^{i\delta}\left(R_q^\alpha f(z)\right)' \prec \phi(z)\cos\delta + i\sin\delta, \tag{31}$$

and

$$e^{i\delta}\left(R_q^\alpha g(w)\right)' \prec \phi(w)\cos\delta + i\sin\delta, \tag{32}$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, and $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Moreover, as $q \to 1^-$ and taking $\alpha = 0$, we get $R_q^0 f(z) = f(z)$. Therefore, we get the following close-to-starlike subclasses.

Example 3. *A bi-univalent function f that represented as (1) belongs to the subclass $\mathcal{S}^*(\delta, \phi)$ if the following subordinations hold:*

$$e^{i\delta}\left(\frac{zf'(z)}{f(z)}\right) \prec \phi(z)\cos\delta + i\sin\delta, \tag{33}$$

and

$$e^{i\delta}\left(\frac{wg'(w)}{g(w)}\right) \prec \phi(w)\cos\delta + i\sin\delta \tag{34}$$

where $g(w) = f^{-1}(w)$ is given by the Equation (2) and $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Example 4. *A bi-univalent function f that represented as (1) belongs to the subclass $\mathcal{G}^*(\delta, \phi)$ if the following subordinations hold:*

$$e^{i\delta}\left(f'(z)\right) \prec \phi(z)\cos\delta + i\sin\delta, \tag{35}$$

and

$$e^{i\delta}\left(g'(w)\right) \prec \phi(w)\cos\delta + i\sin\delta, \tag{36}$$

where $g(w) = f^{-1}(w)$ is given by the Equation (2) and $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

The subsequent corollaries are directly obtained from Theorem 1, contingent upon the conditions specified in the earlier examples. The techniques employed in deriving these corollaries closely mirror those applied in the proof of Theorem 1, which is the rationale behind our decision to exclude the detailed proofs.

Corollary 1. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{B}^0(\delta, R_q^\alpha, \phi)$, then it can be concluded that*

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\theta|\cos\delta}{\sqrt{|\cos\delta(\cos\theta - 1)(4\psi_3 - 2\psi_2^2) + (1 - 3\cos\theta)\psi_2^2 e^{i\delta}|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|\cos\delta}{2\psi_3} + \frac{(1-\cos\theta)^2\cos^2\delta}{\psi_2^2}.$$

Corollary 2. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{B}^1(\delta, R_q^\alpha, \phi)$, then it can be concluded that*

$$|a_2| \leq \frac{|1-\cos\theta|\cos\delta}{\sqrt{|2\cos\delta(\cos\theta - 1)\psi_3 + 2(1 - 3\cos\theta)\psi_2^2 e^{i\delta}|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|\cos\delta}{3\psi_3} + \frac{(1-\cos\theta)^2\cos^2\delta}{4\psi_2^2}.$$

Corollary 3. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{S}^*(\delta, \phi)$, then it can be concluded that*

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\theta|\cos\delta}{\sqrt{|2\cos\delta(\cos\theta-1)+(1-3\cos\theta)e^{i\delta}|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|\cos\delta}{2} + (1-\cos\theta)^2\cos^2\delta.$$

Corollary 4. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{G}^*(\delta, \phi)$, then it can be concluded that*

$$|a_2| \leq \frac{|1-\cos\theta|\cos\delta}{\sqrt{|2\cos\delta(\cos\theta-1)+2(1-3\cos\theta)e^{i\delta}|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|\cos\delta}{3} + \frac{(1-\cos\theta)^2\cos^2\delta}{4}.$$

4. Fekete-Szegö problem of the function class $\mathcal{B}^\lambda(\alpha, R_q^\alpha, \phi)$

In this section, we will derive the Fekete-Szegö inequalities for functions belonging to the class $\mathcal{B}^\lambda(\alpha, R_q^\alpha, \phi)$, which encompasses bi-Bazilevic functions defined through the q-Ruscheweyh differential operator and associated with Legendre Polynomials. Additionally, we aim to establish Fekete-Szegö inequalities for several subclasses within our defined class.

Theorem 2. *If a function f is a member of the class $\mathcal{B}^\lambda(\alpha, R_q^\alpha, \phi)$ and is represented by equation (1), then for a real number γ the following inequality holds*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|\cos\delta}{2(\lambda+2)\psi_3}, & \text{if } \ |1-\zeta| \leq |\Delta| \\ \frac{|1-\cos\theta||1-\gamma|\cos\delta}{|A\beta_1\cos\delta+B^2(1-3\cos\theta)e^{i\delta}|}, & \text{if } \ |1-\zeta| \geq |\Delta|, \end{cases} \tag{37}$$

where

$$A = 2(\lambda+2)\psi_3 + (\lambda-1)(\lambda+2)\psi_2^2, \quad B = (\lambda+1)\psi_2, \ \text{and}$$

$$\Delta = \frac{|A(\cos\theta-1)\cos\delta + B^2(1-3\cos\theta)e^{i\delta}|}{2|1-\cos\theta|\cos\delta(\lambda+2)\psi_3}.$$

Proof. For any real number γ, using Equation (26) and Equation (27), we easily derive the following equations

$$a_3 - \gamma a_2^2 = \frac{\beta_1\cos\delta e^{-i\delta}(\eta_2-\zeta_2)}{4(\lambda+2)\psi_3} + \frac{\beta_1^2\cos^2\delta e^{-i\delta}(\eta_2+\zeta_2)(1-\gamma)}{\beta_1\cos\delta[4(\lambda+2)\psi_3+2(\lambda-1)(\lambda+2)\psi_2^2]+2(1-\cos\theta)(\lambda+1)^2\psi_2^2e^{i\delta}}$$

$$= \left(\beta_1 \cos\delta e^{-i\delta}\right)\left\{\left(\mu + \frac{1}{4(\lambda+2)\psi_3}\right)\eta_2 + \left(\mu - \frac{1}{4(\lambda+2)\psi_3}\right)\zeta_2\right\},$$

where

$$\mu = \frac{(1-\gamma)\beta_1 \cos\delta e^{-i\delta}}{2A\beta_1\cos\delta + 2B^2(1-\cos\theta)e^{i\delta}}.$$

Therefore, with the assistance of Lemma 2, we are able to achieve the following inequality

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{2\left|\beta_1 \cos\delta e^{-i\delta}\right|}{4(\lambda+2)\psi_3}, & \text{if } |\mu| \leq \frac{1}{4(\lambda+2)\psi_3} \\ 2\left|\beta_1 \cos\delta e^{-i\delta}\right| |\mu|, & \text{if } |\mu| \geq \frac{1}{4(\lambda+2)\psi_3}. \end{cases}$$

Finally, by streamlining the right-hand side of the final inequality, we arrive at the expected result as presented in inequality (37). This signifies the completion of the proof.

The subsequent corollaries emerge as logical extensions of Theorem 2, given the conditions outlined in the preceding examples. The methodology employed to derive this corollary closely resembles that utilized in the earlier theorem; therefore, we have opted to forgo a detailed proof for this corollary.

Corollary 5. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (29) and (30), then for a real number γ the following holds*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|\cos\delta}{4\psi_3}, & \text{if } |1-\zeta| \leq |\Delta_1| \\ \frac{|1-\cos\theta||1-\gamma|\cos\delta}{|\beta_1\cos\delta(4\psi_3-2\psi_2^2)+(1-3\cos\theta)\psi_2^2 e^{i\delta}|}, & \text{if } |1-\zeta| \geq |\Delta_1|, \end{cases}$$

where

$$\Delta_1 = \frac{|\beta_1\cos\delta(4\psi_3 - 2\psi_2^2) + (1-3\cos\theta)\psi_2^2 e^{i\delta}|}{4|1-\cos\theta|\cos\delta\psi_3}.$$

Corollary 6. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (31) and (32), then for a real number ζ the following holds*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|\cos\delta}{6\psi_3}, & \text{if } |1-\zeta| \leq |\Delta_2| \\ \frac{|1-\cos\theta||1-\gamma|\cos\delta}{|6\beta_1\cos\delta\psi_3+4(1-3\cos\theta)\psi_2^2 e^{i\delta}|}, & \text{if } |1-\zeta| \geq |\Delta_2|, \end{cases}$$

where

$$\Delta_2 = \frac{|3(\cos\theta - 1)\cos\delta\psi_3 + 2(1-3\cos\theta)\psi_2^2 e^{i\delta}|}{3|1-\cos\theta|\cos\delta\psi_3}.$$

Corollary 7. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (33) and (34), then for a real number ζ the following holds*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|\cos\delta}{4}, & \text{if } |1-\zeta| \leq |\Delta_3| \\ \frac{|1-\cos\theta||1-\gamma|\cos\delta}{|2\beta_1\cos\delta+(1-3\cos\theta)2e^{i\delta}|}, & \text{if } |1-\zeta| \geq |\Delta_3|, \end{cases}$$

where

$$\Delta_3 = \frac{|2\beta_1\cos\delta + (1-3\cos\theta)e^{i\delta}|}{4|1-\cos\theta|\cos\delta}.$$

Corollary 8. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (35) and (36), then for a real number ζ the following holds*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|\cos\delta}{6}, & \text{if } |1-\zeta| \leq |\Delta_4| \\ \frac{|1-\cos\theta||1-\gamma|\cos\delta}{|6\beta_1\cos\delta + 4(1-3\cos\theta)e^{i\delta}|}, & \text{if } |1-\zeta| \geq |\Delta_4|, \end{cases}$$

where

$$\Delta_4 = \frac{|3(\cos\theta - 1)\cos\delta + 2(1-3\cos\theta)e^{i\delta}|}{3|1-\cos\theta|\cos\delta}.$$

Remark 1. *Assuming $\delta = 0$, the results presented in this paper would give various new and known results. Moreover, taking $\delta = 0$ in Example 3, would lead to the known classes of starlike bi-univalent functions that studied by many researchers see, for example, [7], [14], [20], [29], [33], and [45]. More precisely, the results presented in those papers are just special case of the class mentioned in Example 3.*

5. Conclusion

This research paper investigates a new category of bi-Bazilevic functions that are defined through the q-Ruscheweyh differential operator and are linked to Legendre polynomials. The author has derived estimates for the initial coefficients and examined the Fekete-Szegö functional problem concerning functions within these specific classes. In conclusion, potential avenues for future research are suggested, particularly the exploration of substituting Legendre polynomials with other types of orthogonal polynomials, such as Gegenbauer polynomials. Furthermore, the findings presented in this study are anticipated to motivate researchers to expand the scope of this investigation to include meromorphic bi-univalent functions.

Acknowledgements

This research is partially funded by Zarqa University. The author would like to express his sincerest thanks to Zarqa University for the financial support. The author expresses sincere gratitude to the anonymous referees for their insightful comments and recommendations, which significantly enhanced the quality of this paper's presentation.

6. Conflicts of interest

The author confirms that there are no relevant conflicts of interest that are pertinent to the content of this article.

References

[1] V. Aboites. Easy route to tchebycheff polynomials. *Revista Mexicana de Fisica*, 65:12–14, 2019.

[2] M. Ahsan, M. Ahmad, W. Khan, E.E. Mahmoud, and A.H. Abdel-Aty. Meshless analysis of nonlocal boundary value problems in anisotropic and inhomogeneous media. *Mathemaics*, 11(8):2045, 2020.

[3] W. Al-Rawashdeh. Applications of gegenbauer polynomials to a certain subclass of p-valent functions. *WSEAS Transactions on Mathematics*, 22:1025–1030, 2023.

[4] W. Al-Rawashdeh. Horadam polynomials and a class of binivalent functions defined by ruscheweyh operator. *International Journal of Mathematics and Mathematical Sciences*, Article ID 2573044:7 pages, 2023.

[5] W. Al-Rawashdeh. A class of non-bazilevic functions subordinate to gegenbauer polynomials. *Int. J. of Analysis and Applications*, 22(29):1–9, 2024.

[6] W. Al-Rawashdeh. Fekete-szegö functional of a subclass of bi-univalent functions associated with gegenbauer polynomials. *European Journal of Pure and Applied Mathematics*, 17(1):105–115, 2024.

[7] W. Al-Rawashdeh. On the study of bi-univalent functions defined by the generalized sălăgean differential operator. *European Journal of Pure and Applied Mathematics*, 17(4):3899–3914, 2024.

[8] H. Aldweby and M. Darus. Some subordination results on q-analogue of ruscheweyh differential operator. *Abst. Appl. Anal.*, Article ID 985563:1–6, 2014.

[9] L. Andrei and V.A. Caus. Subordinations results on a q-derivative differential operator. *Mathematics*, 12:208, 2024.

[10] A. Aral, V. Gupta, and R.P. Agarwal. *Applications of q-Calculus in Operator Theory.* Springer, New York, US, 2013.

[11] D.A. Brannan and J.G. Clunie. *Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1–20, 1979).* Academic Press, New York and London, 1979.

[12] M. Cağlar, H. Orhan, and M. Kamali. Fekete-szegö problem for a subclass of analytic functions associated with chebyshev polynomials. *Boletim da Sociedade Paranaense de Matemática*, 40:1–6, 2022.

[13] Y. Cheng, R. Srivastava, and J.L. Liu. Applications of the $q-$derivative operator to new families of bi-univalent functions related to the legendre polynomials. *Axioms*, 11(595):1–13, 2022.

[14] N. E. Cho, V. Kumar, S. Kumar, and V. Ravichandran. Radius problems for starlike functions associated with the sine function. *Bull. Iranian Math. Society*, 45:213–232, 2019.

[15] J.H. Choi, Y.C. Kim, and T. Sugawa. A general approach to the fekete-szegö problem. *Journal of the Mathematical Society of Japan*, 59:707–727, 2007.

[16] L.I. Cotîrlă and G. Murugusundaramoorthy. Starlike functions based on ruscheweyh $q-$differential operator defined in janowski domain. *Fractal Fractional*, 7(148), 2023.

[17] P. Duren. Subordination in complex analysis, lecture notes in mathematics. *Springer, Berlin, Germany*, 599:22–29, 1977.

[18] P. Duren. *Univalent functions.* Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[19] M. Fekete and G. Szegö. Eine bemerkung Über ungerade schlichte funktionen. *Journal*

of London Mathematical Society, s1-8:85–89, 1933.

[20] P. Goel and S. Kumar. Certain class of starlike functions associated with modified sigmoid function. *Bull. Malaysian Math. Sci. Society*, 43:957–991, 2020.

[21] A. W. Goodman. *Univalent functions.* Mariner Publishing Co. Inc., Boston, 1983.

[22] R. Ibrahim, J. Suzan, and M.D. Obaiys. Studies on generalized differential-difference operator of normalized analytic functions. *Southeast Asian Bull. Math.*, 45:43–55, 2021.

[23] F.H. Jackson. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.*, 46:253–281, 1908.

[24] M. Kamali, M. Cağlar, E. Deniz, and M. Turabaev. Fekete szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with chebyshev polynomials. *Turkish J. Math.*, 45:1195–1208, 2012.

[25] S. Kanas and D. Răducanu. Some subclass of analytic functions related to conic domains. *Math. Slovaca*, 64:1183–1196, 2014.

[26] F.R. Keogh and E.P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society*, 20:8–12, 1969.

[27] B. Khan, H.M. Srivastava, and S. Arjika. A certain q-ruscheweyh type derivative operator and its applications involving multivalent functions. *Adv. Differ. Equ.*, 2012(279), 2021.

[28] M.F. Khan, I. Al-shbeil, S. Khan, N. Khan, W.U. Haq, and J. Gong. Applications of a q-differential operator to a class of harmonic mappings defined by q-mittag–leffler functions. *Symmetry*, 14:1905, 2022.

[29] S. Kumar and S. Banga. On a special type of ma-minda function. *arxiv preprint arXiv:2006.02111*, 2020.

[30] M. Lewin. On a coefficient problem for bi-univalent functions. *Proceedings of the American Mathematical Society*, 18(1):63–68, 1967.

[31] W. Ma and D. Minda. A unified treatment of some special classes of univalent functions. In I Conf. Proc. Lecture Notes Anal., editor, *Proceedings of the Conference on Complex Analysis*, pages 157–169. Int. Press, Cambridge, MA, 1992.

[32] N. Magesh and S. Bulut. Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions. *Afrika Matematika*, 29(1-2):203–209, 2018.

[33] R. Mendiratta, S. Nagpal, and V. Ravichandran. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Society*, 38:365–386, 2015.

[34] S. Miller and P. Mocabu. *Differential Subordination: Theory and Applications.* CRC Press, New York, 2000.

[35] E. Muthaiyan and A. Wanas. On some coefficient inequalities involving legendre polynomials in the class of bi-univalent functions. *Turkish Journal of Inequalities*, 7(2):39–46, 2023.

[36] Z. Nehari. *Conformal Mappings.* McGraw-Hill, New York, 1952.

[37] E. Netanyahu. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Archive for Rational Mechanics*

and Analysis, 32(2):100–112, 1969.

[38] V. Nezir and N. Mustafa. Analytic functions expressed with q-poisson distribution series. *Turk. J. Sci.*, 6:24–30, 2021.

[39] S.D. Purohit and R.K. Raina. Fractional q-calculus and certain subclasses of univalent analytic functions. *Mathematica*, 55:6274, 2013.

[40] C. Ramachandran and D. Kavitha. Coefficient estimates for a subclass of bi-univalent functions defined by sălăgean operator using quasi-subordination. *Applied Mathematical Sciences*, 11:1725–1732, 2017.

[41] A. Raposo, H. Weber, D. Alvarez-Castillo, and M. Kirchbach. Romanovski polynomials in selected physics problems. *Central European Journal of Physics*, 5(3):253–284, 2007.

[42] K. Riley, M. Hobson, and S. Bence. *Mathematical Methods for Physics and Engineering, Third Edition.* Cambridge University Press, Cambridge, UK, 2006.

[43] S. Ruscheweyh. New criteria for univalent functions. *Proceedings of the American Mathematical Society*, 49:109–115, 1975.

[44] T.M. Seoudy and M.K. Aouf. Coefficient estimates of new classes $q-$starlike and $q-$convex functions of complex order. *Journal of Mathematical Inequalities*, 10(1):130–145, 2016.

[45] J. Sokól and J. Stankiewicz. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Nauk. Politech. Rzeszowskiej Mathematics*, 19:101–105, 1996.

[46] H.M. Srivastava. Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci Technol Trans. Sci.*, 44:327–344, 2020.

[47] H.M. Srivastava, Q.Z. Ahmad, M. Tahir, B. Khan, M. Darus, and N. Khan. Certain subclasses of meromorphically-starlike functions associated with the q-derivative operators. *Ukr. Math. J.*, 73:1260–1273, 2021.

[48] H.M. Srivastava, M. Kamali, and A. Urdaletova. A study of the fekete-szegö functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination condition and associated with the gegenbauer polynomials. *AIMS Mathematics*, 7(2):2568–2584, 2021.

[49] H.M. Srivastava and H.L. Manocha. A treatise on generating functions. *Halsted Press, John Wiley and Sons*, 1984.

[50] H.E. Uçar. Coefficient inequality for $q-$starlike functions. *Applied Mathematical Comput.*, 276:122–126, 2016.

[51] M. Ul-Haq, M. Raza, M. Arif, Q. Khan, and H. Tang. $q-$analogue of differential subordinations. *Mathematics*, 7(724), 2019.

[52] K. Vijaya. Coefficient estimates of bi-univalent bi-bazilevic functions defined by $q-$Ruscheweyh differential operator associated with haradam polynomials. *Palestine Journal of Mathematics*, 11(2):352–361, 2022.