EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5732
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Characterizations of δ_1-$\beta_{\mathcal{I}}$-Paracompactness Concerning an Ideal

Chawalit Boonpok[1], Areeyuth Sama-ae[2,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. Al-Jarrah presented and examined the idea of β_1-paracompactness in topological spaces, whereas Qahis extended the original idea of β_1-paracompact spaces by further developing and investigating the idea of β_1-paracompact spaces with respect to an ideal. This work analyzes the properties, subsets, and subspaces of δ_1-$\beta_{\mathcal{I}}$-paracompact spaces, which are wider in scope than the β_1-paracompact spaces delineated by Qahis. Furthermore, we investigate the invariants of δ_1-$\beta_{\mathcal{I}}$-paracompact spaces via the view of functions.

2020 Mathematics Subject Classifications: 54A05, 54B05, 54C08
Key Words and Phrases: Ideal topological space, δ_1-$\beta_{\mathcal{I}}$-paracompact, δ-$\beta_{\mathcal{I}}$-open

1. Introduction

Paracompact spaces, established considerably later than the two earlier classes, are seen to be one of the most important classes of topological spaces, concurrently generalizing both metrizable and compact spaces. Paracompact spaces were promptly recognized by topologists and analysts. A paracompact space in mathematics is a topological space where every open cover has an open refinement that is locally finite. The notion of spaces was initially developed by Dieudonné [10] in 1944. In 1969, Singal and Arya [24] introduced a novel notion of paracompactness termed almost paracompactness, which is a weaker form of standard paracompactness that defines its fundamental topological properties. A Hausdorff space is paracompact if and only if it allows partitions of unity that are subservient to any open cover. Every paracompact Hausdorff space is normal, as referenced in [12]. Various forms of generalized paracompactness in literature, including S-paracompactness [4], P_3-paracompactness [9], and β-paracompactness [5], are evaluated. In 2006, Al-Zoubi [4] employed semi-open sets to characterize S-paracompact spaces, a generalization of paracompact spaces, and analyzed the relationships between these spaces. Li and Song [18]

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5732

Email addresses: chawalit.b@msu.ac.th (C. Boonpok), areeyuth.s@psu.ac.th (A. Sama-ae)

developed a Hausdorff S-paracompact space that is not paracompact and looked at other characterizations of S-paracompact spaces. In 2013, Demir and Ozbakir [9] proposed a diminished variant of expandable and paracompact spaces, termed β-expandable spaces and β-paracompact spaces, respectively. Every β-paracompact space is a β-expandable space, as demonstrated in a provided argument. Yildirim et al. [25] came up with the idea of β-paracompactness in an ideal topological space and compared it to other types of paracompactness that are already known. In 2024, Alrababah et al. [6] studied the notion, attributes, and theorems associated with Dparacompact spaces.

In 1930, Kuratowski proposed the concept of an ideal topological space [17]. Jankovic and Hamlett [15] have performed an investigation and offered a thorough explanation of the basic characteristics related to ideal topological spaces. In addition to elucidating the concept of I-open sets, they conducted exhaustive research on topologies that make use of ideals. A thorough investigation of the idea of I-open sets was carried out by Abd. El-Monsef et al. [1]. The concept of I_g-closed sets was originally presented by Dontchev et al. [11] in the year 1999. There was an early proposal made by Abd. El-Monsef et al. [2] about the concept of the s-local function. Khan and Noiri [16] subsequently performed an analysis on this concept.

The concept of paracompactness in respect to an ideal was originally presented by Zahid [27], and it was subsequently investigated by Hamlett et al. [13]. An investigation was studied by Sathiyasundari and Renukadevi [23] to investigate the idea of I-paracompactness and to assess its properties. The concept of I-paracompact spaces has been expanded to include some conclusions that were obtained from the concept of paracompact spaces. Within the context of ideal topological spaces, Sanabria et al. [22] conducted an investigation into the concept of S-paracompactness. The focus of their research was on the development and investigation of a new category of spaces, which they referred to as I-S-paracompact spaces. These spaces were constructed inside the framework of an ideal topological space. Spaces that are both S-paracompact and I-paracompact are included in this class. In 2016, Al-Jarrah [3] introduced the concept of β_1-paracompactness, employing the definition of β-open as follows: A topological space (X, τ) is β_1-paracompact if every β-open cover of X has a locally finite open refinement. In 2019, Qahis [21] developed a novel class of β_1-paracompact spaces related to an ideal, analyzing their characterizations and exploring the corresponding invariants. This study presents a new classification of β_1-paracompact spaces that is wider than that suggested by Qahis.

2. Preliminaries

In this article, the notation (X, τ) denotes a topological space without any assumptions on separation axioms. For a subset A of a topological space (X, τ), $CI(A)$ represents the closure of A in (X, τ), whereas $Int(A)$ signifies the interior of A in (X, τ). An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X that fulfills the following criteria:

(i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$, and (*ii*) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space $(X, \tau, \mathcal{I})$ is characterized as a topological space (X, τ) paired with an ideal $\mathcal{I}$ on the set X. The collection of all subsets of X is represented as $P(X)$. A set operator $(.)^* : P(X) \to P(X)$, characterized as a local function [17], is defined with respect to τ and $\mathcal{I}$: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for all $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. We simply write A^* instead of $A^*(\mathcal{I}, \tau)$. A topology $\tau^*(\mathcal{I})$, or τ^* for brevity that is finer than τ may be created for any ideal topological space, defined by $\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau$ and $I \in \mathcal{I}\}$. Nevertheless, $\beta(\mathcal{I}, \tau)$ does not uniformly define a topology. Furthermore, $Cl^*(A) = A \cup A^*$ provides a Kuratowski closure operator for τ^*. We simply write for τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\beta(\mathcal{I}, \tau) = \tau^*$, then we say $\mathcal{I}$ is τ-simple [15]. If $(X, \tau, \mathcal{I})$ satisfies this condition, then $\mathcal{I}$ is said to be compatible [15] or $\mathcal{I}$ is said to be τ-local. Given an ideal topological space $(X, \tau, \mathcal{I})$, we say $\mathcal{I}$ is $\mathcal{I}$-codense if $\mathcal{I} \cap \tau = \{\emptyset\}$.

Definition 1. [14] *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. A point $x \in X$ is called a $\delta_{\mathcal{I}}$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for every neighborhood U of x. The $\delta Cl_{\mathcal{I}}(A)$ represents the $\delta_{\mathcal{I}}$-closure of A, which is the set of all $\delta_{\mathcal{I}}$-cluster points of A. A subset A of X is called a $\delta_{\mathcal{I}}$-closed [26] if $\delta Cl_{\mathcal{I}}(A) = A$, and the complement of a $\delta_{\mathcal{I}}$-closed set is called a $\delta_{\mathcal{I}}$-open set. The union of all $\delta_{\mathcal{I}}$-open sets included in A is the $\delta_{\mathcal{I}}$-interior of A, which will be represented by $\delta Int_{\mathcal{I}}(A)$.*

Lemma 1. [14] *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements are true:*

(i) If $A \subset B$ then $\delta Cl_{\mathcal{I}}(A) \subset \delta Cl_{\mathcal{I}}(B)$.

(ii) If A is an open set, then $\delta Cl_{\mathcal{I}}(A) = A$.

(iii) If A is a closed set, then $\delta Int_{\mathcal{I}}(A) = A$.

Definition 2. [14] *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called δ-$\beta_{\mathcal{I}}$-open if $A \subset Cl(Int(\delta Cl_{\mathcal{I}}(A)))$ and is called δ-$\beta_{\mathcal{I}}$-closed if $Int(Cl(\delta Int_{\mathcal{I}}(A))) \subset A$.*

Definition 3. [14] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The union of all δ-$\beta_{\mathcal{I}}$-open sets contained in A is called the δ-$\beta_{\mathcal{I}}$-interior of A denoted by δ-$\beta Int_{\mathcal{I}}(A)$. The intersection of all δ-$\beta_{\mathcal{I}}$-closed sets containing A is called the δ-$\beta_{\mathcal{I}}$-closure of A denoted by δ-$\beta Cl_{\mathcal{I}}(A)$.*

The following lemma is easily derived from the definition 3.

Lemma 2. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements are true:*

(i) δ-$\beta Cl_{\mathcal{I}}(A) \subset Cl(A)$.

(ii) If A is open, then A is δ-$\beta_{\mathcal{I}}$-open.

(iii) If A is closed, then A is δ-$\beta_{\mathcal{I}}$-closed and δ-$\beta Cl_{\mathcal{I}}(A) = A$.

A collection $\mathcal{V}$ of subsets of a topological space X is said to be locally finite if each point $x \in X$ has a neighborhood U that contains x and U intersects only finitely many of the sets in the collection $\mathcal{V}$. The upcoming theorems will use the following lemmas.

Lemma 3. [7] *The union of a finite family of locally finite collection of sets in a topological space is a locally finite family of sets.*

Lemma 4. [8] *If $\{U_\alpha : \alpha \in \Lambda\}$ is a locally finite family of subsets in a topological space X and if $V_\alpha \subset U_\alpha$ for all $\alpha \in \Lambda$, then the family $\{V_\alpha : \alpha \in \Lambda\}$ is a locally finite in X.*

Lemma 5. [13] *Let (X, τ) and (Y, σ) be topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is a continuous surjective function and $\{U_\alpha : \alpha \in \Lambda\}$ is a locally finite in Y, then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a locally finite in X.*

Let (X, τ) and (Y, σ) denote topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is called almost closed [20] if for any regular closed set F in X, the image $f(F)$ is closed in Y. A subset K of the space X is defined as N-closed relative to X [20] if every cover of K by regular open sets of X possesses a finite subcover.

Lemma 6. [19] *Let (X, τ) and (Y, σ) be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be almost closed surjection with N-closed point inverse. If $\{U_\alpha : \alpha \in \Lambda\}$ is a locally finite open cover of X, then $\{f(U_\alpha) : \alpha \in \Lambda\}$ is a locally finite cover of Y.*

3. δ_1-$\beta_{\mathcal{I}}$-paracompactness of spaces and subsets

This section discusses the concept of δ_1-$\beta_{\mathcal{I}}$-paracompactness, a less strict form of β_1-paracompactness examined by Qahis [21], followed by an exploration of its characterization. Al-Jarrah [3] defined β_1-paracompactness as follows: A topological space (X, τ) is called β_1-paracompact if every β-open cover of X has a locally finite open refinement. Qahis [21] expanded the notion of β_1-paracompactness to β_1-paracompactness concerning an ideal as follows: An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\beta_1\mathcal{I}$-paracompact if every β-open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. Utilizing δ-$\beta_{\mathcal{I}}$-open sets, we establish a new type of paracompactness that is weaker than the one that Qahis developed.

Definition 4. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ_1-$\beta_{\mathcal{I}}$-paracompact if every δ-$\beta_{\mathcal{I}}$-open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ (not necessarily a cover) such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. The collection $\mathcal{V}$ of subsets of X such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover of X.*

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ_1-$\beta_{\mathcal{I}}$-paracompact relative to X if for every δ-$\beta_{\mathcal{I}}$-open cover $\mathcal{U}$ of A has a locally finite open refinement $\mathcal{V}$ such that $A - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$.

We have the following result based on the definition that was previously given.

Theorem 1. *If an ideal topological space $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then it is $\beta_1\mathcal{I}$-paracompact.*

Proof. The theorem is established as every β-open cover of X serves as a δ-$\beta_{\mathcal{I}}$-open cover.

Corollary 1. *If an ideal topological space $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then it is paracompact.*

Proof. Since any $\beta_1\mathcal{I}$-paracompact space is always a paracompact space, the corollary is obtained.

Consider the ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $\mathcal{I} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Hence, the set of all δ-$\beta_{\mathcal{I}}$-open sets of X is $\{A : A \subset X\}$. Every δ-$\beta_{\mathcal{I}}$-open cover $\mathcal{U}$ of X possesses a locally finite open refinement $\{\{1\}\}$, such that $X - \{1\} = \{2, 3\} \in \mathcal{I}$. Consequently, $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact; but, (X, τ) is not β_1-paracompact, as there exists a β-open cover $\{\{1, 2\}, \{1, 3\}\}$ of (X, τ) that lacks a locally finite open refinement [21].

In the subsequent theorem, we discuss a space endowed with two topologies; hence, to avoid ambiguity, we must redefine the concept of local finiteness. A collection $\mathcal{V}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be τ-locally finite if for each $x \in X$, there exists an open set $U \in \tau$ such that $x \in U$ and U intersects with at most finitely many elements of $\mathcal{V}$. As stated in [14], the intersection of any two δ-$\beta_{\mathcal{I}}$-open sets is not necessarily a δ-$\beta_{\mathcal{I}}$-open set; therefore, this assumption must be made in the subsequent theorem.

Theorem 2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is codense and τ-simple, $(X, \tau^*, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, any δ-$\beta_{\mathcal{I}}$-open set in $(X, \tau, \mathcal{I})$ is a δ-$\beta_{\mathcal{I}}$-open set in $(X, \tau^*, \mathcal{I})$, and the intersection of two δ-$\beta_{\mathcal{I}}$-open sets in $(X, \tau, \mathcal{I})$ remains a δ-$\beta_{\mathcal{I}}$-open set in $(X, \tau, \mathcal{I})$, then every δ-$\beta_{\mathcal{I}}$-open cover of $(X, \tau, \mathcal{I})$ has a locally finite δ-$\beta_{\mathcal{I}}$-open $\mathcal{I}$-cover refinement.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda_1\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of $(X, \tau, \mathcal{I})$. Then, $\mathcal{U}$ is a δ-$\beta_{\mathcal{I}}$-open cover of $(X, \tau^*, \mathcal{I})$. Since $(X, \tau^*, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{U}$ has τ^*-locally finite refinement $\mathcal{V} = \{G_\lambda : \lambda \in \Lambda_2\}$ of open sets in $(X, \tau^*, \mathcal{I})$ such that $X - \cup\{G_\lambda : \lambda \in \Lambda_2\} \in \mathcal{I}$, where $G_\lambda = V_\lambda - I_\lambda$, $V_\lambda \in \tau$ and $I_\lambda \in \mathcal{I}$ for all $\lambda \in \Lambda_2$. Because $\mathcal{V}$ is τ^*-locally finite, for every $x \in X$, there exists $H \in \tau^*$ containing x such that $H \cap G_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Given that $\mathcal{I}$ is τ-simple, $H = U - I$, for some $U \in \tau$ and $I \in \mathcal{I}$. Hence, $(U - I) \cap G_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$, which implies that $(U \cap V_\lambda) - (I \cup I_\lambda) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. As $\mathcal{I}$ is codense, $U \cap V_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$, and therefore $U \cap (V_\lambda \cap U_\alpha) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ and for all $\alpha \in \Lambda_1$. Consequently, $\mathcal{W} = \{V_\lambda \cap U_\alpha : \alpha \in \Lambda_1, \lambda \in \Lambda_2\}$ is τ-locally finite, and $\mathcal{W}$ is a δ-$\beta_{\mathcal{I}}$-open refinement of $\mathcal{U}$, following assumption. Next, we will show that $\mathcal{W}$ refines $\mathcal{U}$. Because $\mathcal{V}$ refines $\mathcal{U}$, for every $G_\lambda \in \mathcal{V}$, there exists $U_\alpha \in \mathcal{U}$ such that $G_\lambda \subset U_\alpha$. Therefore $G_\lambda = U_\alpha \cap G_\lambda = U_\alpha \cap (V_\lambda - I_\lambda) = (V_\lambda \cap U_\alpha) - I_\lambda \subset V_\lambda \cap U_\alpha \subset U_\alpha$. It implies that $X - \cup\{V_\lambda \cap U_\alpha : \alpha \in \Lambda_1, \lambda \in \Lambda_2\} \subset X - \cup\{G_\lambda : \lambda \in \Lambda_2\} \in \mathcal{I}$, which implies that $X - \cup\{V_\lambda \cap U_\alpha : \alpha \in \Lambda_1, \lambda \in \Lambda_2\} \in \mathcal{I}$.

The following lemma is required for the proof of Proposition 1.

Lemma 7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. The following statements are true:*

(i) $x \in \delta\text{-}\beta Cl_{\mathcal{I}}(A)$ *if and only if* $G \cap A \neq \emptyset$ *for every* $\delta\text{-}\beta_{\mathcal{I}}$*-open set* G *containing* x.

(ii) $G \cap \delta\text{-}\beta Cl_{\mathcal{I}}(A) = \emptyset$ *if and only if* $G \cap A = \emptyset$, *for every* $\delta\text{-}\beta_{\mathcal{I}}$*-open set* G.

Proof. (i) It is derived from Theorem 1 in [14].
(ii) This is a consequence of (i).

Proposition 1. *Let* $(X, \tau, \mathcal{I})$ *be a* $\delta_1\text{-}\beta_{\mathcal{I}}$*-paracompact space. If for any* $\delta\text{-}\beta_{\mathcal{I}}$*-open set* U *that contains* x*, there exists a* $\delta\text{-}\beta_{\mathcal{I}}$*-open set* V *such that* $x \in V \subset \delta\text{-}\beta Cl_{\mathcal{I}}(V) \subset U$*, then every* $\delta\text{-}\beta_{\mathcal{I}}$*-open cover of* X *has a* $\delta\text{-}\beta_{\mathcal{I}}$*-locally finite* $\delta\text{-}\beta_{\mathcal{I}}$*-closed* $\mathcal{I}$*-cover refinement.*

Proof. Let $\mathcal{U}$ be a $\delta\text{-}\beta_{\mathcal{I}}$-open cover of X. For each $x \in X$, let U_x be a $\delta\text{-}\beta_{\mathcal{I}}$-open set in $\mathcal{U}$ containing x. By assumption, there exists a $\delta\text{-}\beta_{\mathcal{I}}$-open set V_x such that $x \in V_x \subset \delta\text{-}\beta Cl_{\mathcal{I}}(V_x) \subset U_x$. Thus $\mathcal{V} = \{V_x : x \in X\}$ is a $\delta\text{-}\beta_{\mathcal{I}}$-open cover refinement of $\mathcal{U}$. As $(X, \tau, \mathcal{I})$ is $\delta_1\text{-}\beta_{\mathcal{I}}$-paracompact, there exists a locally finite open refinement $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ which refines $\mathcal{V}$ and $X - \cup\{H_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. By (ii) of Lemma 7, $\mathcal{H}_1 = \{\delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha) : \alpha \in \Lambda\}$ is $\delta\text{-}\beta_{\mathcal{I}}$-locally finite. As $H_\alpha \subset \delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha)$ for all $\alpha \in \Lambda$, $X - \cup\{\delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha) : \alpha \in \Lambda\} \subset X - \cup\{H_\alpha : \alpha \in \Lambda\}$, and it therefore implies that $X - \cup\{\delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha) : \alpha \in \Lambda\} \in \mathcal{I}$. Hence, $\mathcal{H}_1$ is an $\mathcal{I}$-cover. Next, we shall verify that $\mathcal{H}_1$ refines $\mathcal{U}$. Let $\delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha) \in \mathcal{H}_1$. Since $\mathcal{H}$ refines $\mathcal{V}$, there exists $V_x \in \mathcal{V}$ such that $H_\alpha \subset V_x$, it implies that $\delta\text{-}\beta Cl_{\mathcal{I}}(H_\alpha) \subset \delta\text{-}\beta Cl_{\mathcal{I}}(V_x) \subset U_x$. Consequenty, $\mathcal{H}_1$ refines $\mathcal{U}$. Therefore, the proposition has been established.

Theorem 3. *If an ideal topological space* $(X, \tau, \mathcal{I})$ *is* $\delta_1\text{-}\beta_{\mathcal{I}}$*-paracompact, then* $(X, \tau^*, \mathcal{I})$ *is* $\delta_1\text{-}\beta_{\mathcal{I}}$*-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda_1\}$ be a $\delta\text{-}\beta_{\mathcal{I}}$-open cover of $(X, \tau^*, \mathcal{I})$, where $U_\alpha = V_\alpha - I_\alpha$, $V_\alpha \in \tau$ and $I_\alpha \in \mathcal{I}$ for all $\alpha \in \Lambda_1$. Then, $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda_1\}$ is a $\delta\text{-}\beta_{\mathcal{I}}$-open cover of $(X, \tau, \mathcal{I})$ and therefore there exists a τ-locally finite open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda_2\}$ such that $X - \cup\{W_\lambda : \lambda \in \Lambda_2\} \in \mathcal{I}$, as $(X, \tau, \mathcal{I})$ is $\delta_1\text{-}\beta_{\mathcal{I}}$-paracompact. Now, we have $\{W_\lambda \cap I_{\alpha'} : \lambda \in \Lambda_2\}$, for some $\alpha' \in \Lambda_1$, is a set of subset of $\mathcal{I}$ and hence, by assumption, $\cup_{\lambda \in \Lambda_2}(W_\lambda \cap I_{\alpha'}) \in \mathcal{I}$. Hence,

$$X - \cup_{\lambda \in \Lambda_2}(W_\lambda - I_{\alpha'}) \subset (X - \cup_{\lambda \in \Lambda_2} W_\lambda) \cup (\cup_{\lambda \in \Lambda_2} W_\lambda \cap I_{\alpha'}) \in \mathcal{I},$$

which implies that $X - \cup_{\lambda \in \Lambda_2}(W_\lambda - I_{\alpha'}) \in \mathcal{I}$. As $\mathcal{W}$ is τ-locally finite, $\mathcal{W}' = \{W_\lambda - I_{\alpha'} : \lambda \in \Lambda_2\}$ is τ-locally finite. Because τ^* is finer than τ, $\mathcal{W}'$ is a τ^*-locally finite τ^*-open which refines $\mathcal{U}$. Consequently, $(X, \tau^*, \mathcal{I})$ is $\delta_1\text{-}\beta_{\mathcal{I}}$-paracompact.

A topological space (X, τ) is a T_2 space if any two distinct points x and y in X, there exist disjoint open neighborhoods U and V such that $x \in U$ and $y \in V$. The following theorem establishes a characteristic of a $\delta_1\text{-}\beta_{\mathcal{I}}$-paracompact subset within a T_2 ideal topological space X.

Theorem 4. *If an ideal topological space* $(X, \tau, \mathcal{I})$ *is a* T_2 *space and* A *is* $\delta_1\text{-}\beta_{\mathcal{I}}$*-paracompact relative to* X*, then* A *is closed in* $(X, \tau^*, \mathcal{I})$.

Proof. We shall verify that $A^* \subset A$. Suppose $x \notin A$. As $(X, \tau, \mathcal{I})$ is T_2, for each $y \in A$, there exists an open set U_y such that $y \in U_y$ and $x \notin Cl(U_y)$. Thus, the family $\mathcal{U} = \{U_y : y \in A\}$ is an open cover of A, and hence it is a δ-$\beta_{\mathcal{I}}$-open cover of A. As A is δ_1-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{U}$ has a τ-locally finite open refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of $\mathcal{U}$ such that $A - \cup\{V_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. Now $x \notin Cl(V_\alpha)$ for all α implies that $x \notin \cup\{Cl(V_\alpha) : \alpha \in \Lambda\}$. Because of $\cup\{Cl(V_\alpha) : \alpha \in \Lambda\} = Cl(\cup\{V_\alpha : \alpha \in \Lambda\})$, x is not in $Cl(\cup\{V_\alpha : \alpha \in \Lambda\})$. Let $G_1 = X - Cl(\cup\{V_\alpha : \alpha \in \Lambda\})$ and $G_2 = A - Cl(\cup\{V_\alpha : \alpha \in \Lambda\})$. We know that $G_1 \in \tau$, $G_2 \in \mathcal{I}$, $(G_1 - G_2) \cap A = \emptyset$, and $x \in G_1 - G_2 \in \tau^*$, it follows that $x \notin A^*$. Consequently, A^* is closed in the topological space (X, τ^*). The theorem has been shown.

Theorem 5. *If an ideal topological space $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact and $A \subset X$ is δ-$\beta_{\mathcal{I}}$-closed in X, then A is δ_1-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of A. As $X - A$ is a δ-$\beta_{\mathcal{I}}$-open subset of X, $\mathcal{H} = \{U_\alpha : \alpha \in \Lambda\} \cup \{X - A\}$ is a δ-$\beta_{\mathcal{I}}$-open cover of X. By assumption, $\mathcal{H}$ has a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}$ such that for each $\lambda \in \Lambda_1$, $V_\lambda \subset U_\alpha$ for some $\alpha \in \Lambda$, $V \subset X - A$ and $X - (\cup\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}) \in \mathcal{I}$. As $A - \cup\{V_\lambda : \lambda \in \Lambda_1\} = A \cap X - \cup\{V_\lambda : \lambda \in \Lambda_1\} = A \cap X - \cup(\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}) \subset X - \cup(\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\})$, we have $A - \cup\{V_\lambda : \lambda \in \Lambda_1\} \in \mathcal{I}$. For any $\lambda \in \Lambda_1$, there exists $\alpha \in \Lambda$ such that $V_\lambda \subset U_\alpha$, showing that $\{V_\lambda : \lambda \in \Lambda_1\}$ represents a locally finite open refinement of $\mathcal{U}$. This demonstrates that A is δ_1-$\beta_{\mathcal{I}}$-paracompact.

Theorem 6. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A is δ_1-$\beta_{\mathcal{I}}$-paracompact and B is δ-$\beta_{\mathcal{I}}$-closed in X, then $A \cap B$ is δ_1-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of $A \cap B$. Since $X - B$ is a δ-$\beta_{\mathcal{I}}$-open subset in X, $\mathcal{U}_1 = \{U_\alpha : \alpha \in \Lambda\} \cup \{X - B\}$ is a δ-$\beta_{\mathcal{I}}$-open cover of A. Since A is δ_1-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{U}_1$ has a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}$ such that for each $\lambda \in \Lambda_1$, $V_\lambda \subset U_\alpha$ for some $\alpha \in \Lambda$, $V \subset X - B$ and $A - (\cup\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}) \in \mathcal{I}$. As $A \cap B - \cup\{V_\lambda : \lambda \in \Lambda_1\} = A \cap B - \cup(\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\}) \subset A - \cup(\{V_\lambda : \lambda \in \Lambda_1\} \cup \{V\})$, we have $A \cap B - \cup\{V_\lambda : \lambda \in \Lambda_1\} \in \mathcal{I}$. For any $\lambda \in \Lambda_1$, there exists a $\alpha \in \Lambda$ such that $V_\lambda \subset U_\alpha$, indicating that the collection $\{V_\lambda : \lambda \in \Lambda_1\}$ constitutes a locally finite open refinement of $\mathcal{U}$. This suggests that $A \cap B$ is δ_1-$\beta_{\mathcal{I}}$-paracompact.

Theorem 7. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A and B are δ_1-$\beta_{\mathcal{I}}$-paracompact in X, then $A \cup B$ is δ_1-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of $A \cup B$. It follows that $\mathcal{U}$ is an δ-$\beta_{\mathcal{I}}$-open cover of A and B. By hypothesis, there are locally finite open families $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda_1\}$ of A and $\mathcal{W} = \{W_\mu : \mu \in \Lambda_2\}$ of B which refine $\mathcal{U}$ such that $A - \cup\{V_\lambda : \lambda \in \Lambda_1\} \in \mathcal{I}$ and $B - \cup\{W_\mu : \mu \in \Lambda_2\} \in \mathcal{I}$. It suggests that $A - \cup\{V_\lambda : \lambda \in \Lambda_1\} = I_1$ and $B - \cup\{W_\mu : \mu \in \Lambda_2\} = I_2$, for some $I_1, I_2 \in \mathcal{I}$. Hence,

$$A \cup B \subset (\cup\{V_\lambda : \lambda \in \Lambda_1\} \cup I_1) \cup (\cup\{W_\mu : \mu \in \Lambda_2\} \cup I_2)$$

$$= \cup\{V_\lambda \cup W_\mu : \lambda \in \Lambda_1, \mu \in \Lambda_2\} \cup (I_1 \cup I_2),$$

which implies that $A \cup B \subset \cup\{V_\lambda \cup W_\mu : \lambda \in \Lambda_1, \mu \in \Lambda_2\} \in \mathcal{I}$. As $\mathcal{V}$ and $\mathcal{W}$ are locally finite, for any point $x \in X$ there exist δ-$\beta_{\mathcal{I}}$-open sets G_1 and G_2 such that both G_1 and G_2 intersect at most finitely members of $\mathcal{U}$ and $\mathcal{V}$, respectively. Thus, at most finitely many members of $\{V_\lambda \cup W_\mu : \lambda \in \Lambda_1, \mu \in \Lambda_2\}$, does $G_1 \cap G_2$ intersect. Therefore, $A \cup B$ is δ_1-$\beta_{\mathcal{I}}$-paracompact.

4. Preserving δ_1-$\beta_{\mathcal{I}}$-paracompactness

In this section, we will illustrate how δ_1-$\beta_{\mathcal{I}}$-paracompactness is maintained under specific conditions. We start by introducing the subsequent definition.

Definition 5. *Let $(X, \tau, \mathcal{I})$, and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be a function.*

(i) f is called δ-$\beta_{\mathcal{I}}$-irresolute if $f^{-1}(V)$ is a δ-$\beta_{\mathcal{I}}$-open set in X for every δ-$\beta_{\mathcal{J}}$-open set V in Y.

(ii) f is called δ-$\beta_{\mathcal{I}}$-open if $f(U)$ is a δ-$\beta_{\mathcal{J}}$-open set in Y for every δ-$\beta_{\mathcal{I}}$-open set U in X.

(iii) f is called δ-$\beta_{\mathcal{I}}$-closed if $f(U)$ is a δ-$\beta_{\mathcal{J}}$-closed set in Y for every δ-$\beta_{\mathcal{I}}$-closed set U in X.

Note that $f^{-1}(\mathcal{J})$ is an ideal on X if $f : X \to Y$ is a function, (X, τ) is a topological space, and (Y, τ') is a topological space with an ideal $\mathcal{J}$. Furthermore, given that f is surjective and X possesses an ideal $\mathcal{I}$, $f(\mathcal{I})$ is an ideal on Y.

The proof of Theorem 8 requires the following lemma.

Lemma 8. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be surjective. Then f is δ-$\beta_{\mathcal{I}}$-closed if and only if for every $y \in Y$ and for every δ-$\beta_{\mathcal{I}}$-open set U in X that contains $\{f^{-1}(y)\}$, there exists a δ-$\beta_{\mathcal{J}}$-open set V containing y in Y such that $f^{-1}(V) \subset U$.*

Proof. Let $y \in Y$ and U be a δ-$\beta_{\mathcal{I}}$-open set in X such that $\{f^{-1}(y)\} \subset U$. We have that $V = Y - f(X - U)$ is a δ-$\beta_{\mathcal{I}}$-open set such that $y \in V$ and $f^{-1}(V) \subset U$. Subsequently, the necessity is verified. Sufficiency will now be demonstrated. Let F be a δ-$\beta_{\mathcal{I}}$-closed subset of X and $y \in Y - f(F)$. Consequently, $\{f^{-1}(y)\} \subset X - F$. According to the hypothesis, there exists a δ-$\beta_{\mathcal{J}}$-open set V_y such that $f^{-1}(V_y) \subset X - F$, thereby suggesting that $y \in V_y \subset Y - f(F)$. Therefore, $Y - f(F) = \cup\{V_y : y \in Y\}$ forms a δ-$\beta_{\mathcal{I}}$-open set in Y. Thus, $f(F)$ represents a δ-$\beta_{\mathcal{I}}$-closed set. We conclude that f is δ-$\beta_{\mathcal{I}}$-closed.

Next, we provide the characteristics of a function that maps between two ideal topological spaces, where one space conveys identical properties to the other. Initially, we define the concept of δ-$\beta_{\mathcal{I}}$-compactness and offer a lemma utilized in the proof of the theorem 8.

Definition 6. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ-$\beta_{\mathcal{I}}$-compact if every cover $\mathcal{V}$ of δ-$\beta_{\mathcal{I}}$-open sets of X has $V_1, V_2, ..., V_n \in \mathcal{V}$ such that $X \subset V_1 \cup V_2 \cup \cdots \cup V_n$.*

The next theorem describe characterizations of a function that maps from a δ_1-$\beta_{\mathcal{I}}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ to an ideal topological space $(Y, \tau', \mathcal{J})$, ensuring that Y has the same properties as X.

Theorem 8. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces. Suppose that $f : X \to Y$ satisfies the following statements:*

(i) f is open;

(ii) f is δ-$\beta_{\mathcal{I}}$-irresolute;

(iii) f is δ-$\beta_{\mathcal{I}}$-closed;

(iv) f is a surjective function with $\{f^{-1}(y)\}$ is δ-$\beta_{\mathcal{I}}$-compact for every $y \in Y$; and

(v) $f(\mathcal{I}) \subset \mathcal{J}$.

If $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', \mathcal{J})$ is δ_1-$\beta_{\mathcal{J}}$-paracompact.

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of Y. Assuming that f is δ-$\beta_{\mathcal{I}}$-irresolute, it follows that $\mathcal{H} = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ forms a δ-$\beta_{\mathcal{I}}$-open cover of X. Given that X is δ_1-$\beta_{\mathcal{I}}$-paracompact, the collection $\mathcal{H}$ possesses a τ-locally finite refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda_1\}$ such that $X - \cup\{V_\alpha : \alpha \in \Lambda_1\} \in \mathcal{I}$. As f is open, $f(\mathcal{V}) = \{f(V_\alpha) : \alpha \in \Lambda_1\}$ is an open refinement of $\mathcal{U}$ and $Y - \cup\{f(V_\alpha) : \alpha \in \Lambda_1\} \in \mathcal{J}$. Next, we shall verify that $f(\mathcal{V})$ is τ'-locally finite. Let $y \in Y$. Since $\mathcal{V}$ is τ-locally finite, for $x \in \{f^{-1}(y)\}$, there exists an open set G_x containing x such that G_x intersects at most finitely many members of $\mathcal{V}$. Because $\{f^{-1}(y)\}$ is δ-$\beta_{\mathcal{I}}$-compact and $\{G_x : f(x) = y\}$ forms an open cover of $\{f^{-1}(y)\}$, there exists a finite subcollection H_y such that $\{f^{-1}(y)\} \subset \cup H_y$, and $\cup H_y$ intersects at most finitely many members of $\mathcal{V}$. Assuming f is δ-$\beta_{\mathcal{I}}$-closed, by applying Lemma 8, there exists a δ-$\beta_{\mathcal{J}}$-open set W_y containing y such that $f^{-1}(W_y) \subset \cup H_y$. Hence, $f^{-1}(W_y)$ intersects at most finitely many members of $\mathcal{V}$. This indicates that W_y intersects at most finitely many members of $f(\mathcal{V})$. Consequently, since $f(\mathcal{V})$ is a τ'-locally finite in Y, it implies that $(Y, \tau', \mathcal{J})$ is δ_1-$\beta_{\mathcal{J}}$-paracompact.

The subsequent theorem and corollaries establish characterizations of a function that maps from a δ_1-$\beta_{\mathcal{I}}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ to a topological space (Y, τ'), guaranteeing that Y possesses the same characteristics as X.

Theorem 9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and (Y, τ') be a topological space. Suppose that $f : X \to Y$ satisfies the following statements:*

(i) f is open;

(ii) f is δ-$\beta_{\mathcal{I}}$-irresolute;

(iii) f is almost closed; and

(iv) f is a surjective function with N-closed point inverse.

If $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', f(\mathcal{I}))$ is δ_1-$\beta_{f(\mathcal{I})}$-paracompact.

Proof. As $f : X \to Y$ is surjective, $f(\mathcal{I})$ is an ideal on Y. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a δ-$\beta_{f(\mathcal{I})}$-open cover of Y. As f is δ-$\beta_{\mathcal{I}}$-irresolute, $\mathcal{H} = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is a δ-$\beta_{\mathcal{I}}$-open cover of X. Since X is δ_1-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{H}$ has a locally finite open refinement $\mathcal{H}_1 = \{H_\alpha : \alpha \in \Lambda_1\}$ such that $X - \cup\{H_\alpha : \alpha \in \Lambda_1\} \in \mathcal{I}$. Thus, $f(X - \cup\{H_\alpha : \alpha \in \Lambda_1\}) \in f(\mathcal{I})$. We know that $Y - \cup\{f(H_\alpha) : \alpha \in \Lambda_1\} \subset f(X - \cup\{H_\alpha : \alpha \in \Lambda_1\})$, we therefore have that $Y - \cup\{f(H_\alpha) : \alpha \in \Lambda_1\} \in f(\mathcal{I})$. Since f is open, almost closed, surjective with N-closed point inverse, and $\mathcal{H}_1$ is locally finite, $f(\mathcal{H}_1) = \{f(H_\alpha) : \alpha \in \Lambda_1\}$ is locally finite by Lemma 6. Next, we shall verify that $f(\mathcal{H}_1)$ refines $\mathcal{U}$. Let $f(H_\alpha) \in f(\mathcal{H}_1)$. Then $H_\alpha \in \mathcal{H}_1$. As $\mathcal{H}_1$ refines $\mathcal{H}$, there exists $f^{-1}(U_\lambda) \in \mathcal{H}$ such that $H_\alpha \subset f^{-1}(U_\lambda)$ for some $\lambda \in \Lambda$. Therefore, $f(H_\alpha) \subset f(f^{-1}U_\lambda)) \subset U_\lambda$. This shows that $(Y, \tau', f(\mathcal{I}))$ is δ_1-$\beta_{f(\mathcal{I})}$-paracompact.

As any compact set is a N-closed set and any closed map is an almost closed map, by Theorem 9, we have the following corollary.

Corollary 2. *Let a function $f : (X, \tau, \mathcal{I}) \to (Y, \tau')$ be open, δ-$\beta_{\mathcal{I}}$-irresolute, closed, and surjective with compact point inverse. If $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', f(\mathcal{I}))$ is δ_1-$\beta_{f(\mathcal{I})}$-paracompact.*

By observing the proof of Theorem 9, we will obtain the following corollary.

Corollary 3. *Let a function $f : (X, \tau, \mathcal{I}) \to (Y, \tau')$ be open, δ-$\beta_{\mathcal{I}}$-irresolute, almost closed, surjective, and $f(\mathcal{V})$ is a τ'-locally finite in Y for every τ-locally finite $\mathcal{V}$ in X. If $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', f(\mathcal{I}))$ is δ_1-$\beta_{f(\mathcal{I})}$-paracompact.*

The following theorem provides properties of a function that maps from a topological space X to a δ_1-$\beta_{\mathcal{I}}$-paracompact ideal topological space Y guarantees that X exhibits identical characteristics to Y.

Theorem 10. *Let (X, τ) be a topological space and $(Y, \tau', \mathcal{J})$ be an ideal topological space. Suppose that $f : X \to Y$ satisfies the following statements:*

(i) f is δ-$\beta_{\mathcal{I}}$-open;

(ii) f is continuous; and

(iii) f is bijective.

If $(Y, \tau', \mathcal{J})$ is δ_1-$\beta_{\mathcal{J}}$-paracompact, then $(X, \tau, f^{-1}(\mathcal{J}))$ is δ_1-$\beta_{f^{-1}(\mathcal{J})}$-paracompact.

Proof. Let $\mathcal{I} = f^{-1}(\mathcal{J})$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda_1\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of X. As f is δ-$\beta_{\mathcal{I}}$-open, $f(\mathcal{U}) = \{f(U_\alpha) : \alpha \in \Lambda_1\}$ is a δ-$\beta_{\mathcal{J}}$-open cover of Y. By hypothesis, $f(\mathcal{U})$ has a locally finite open refinement $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda_2\}$ such that $Y - \cup\{H_\lambda : \lambda \in \Lambda_2\} \in \mathcal{J}$. It implies that $Y - \cup\{H_\lambda : \lambda \in \Lambda_2\} = J$ for some $J \in \mathcal{J}$, which follows that $Y = \cup\{H_\lambda : \lambda \in \Lambda_2\} \cup J$. Hence, $X = f^{-1}(Y) = f^{-1}(\cup\{H_\lambda : \lambda \in \Lambda_2\} \cup J) =$

$(\cup\{f^{-1}(H_\lambda) : \lambda \in \Lambda_2\}) \cup f^{-1}(J)$. As $f^{-1}(J) \in \mathcal{I}$, $X - \cup\{f^{-1}(H_\lambda) : \lambda \in \Lambda_2\} \in \mathcal{I}$. Consequently, $\mathcal{V} = \{f^{-1}(H_\lambda) : \lambda \in \Lambda_2\}$ is locally finite, as demonstrated by Lemma 5. Now we will verify that $\mathcal{V}$ refines $\mathcal{U}$. Let $f^{-1}(H_\lambda) \in \mathcal{V}$. Hence $H_\lambda \in \mathcal{H}$. Given that $\mathcal{H}$ refines $f(\mathcal{U})$, there exists an element $f(U_\lambda) \in f(\mathcal{U})$ such that $H_\lambda \subset f(U_\lambda)$. This implies that $f^{-1}(H_\lambda) \subset f^{-1}(f(U_\lambda)) = U_\lambda \in \mathcal{U}$. Therefore, X is δ_1-$\beta_{\mathcal{I}}$-paracompact.

The final result of this section presents attributes of a function whereby the inverse image of a δ_1-$\beta_{\mathcal{J}}$-paracompact subset is a δ_1-$\beta_{\mathcal{I}}$-paracompact subset.

Theorem 11. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces. Suppose that $f : X \to Y$ is δ-$\beta_{\mathcal{I}}$-open, continuous, and bijective with $f(\mathcal{I}) = \mathcal{J}$. If $A \subset Y$ is a δ_1-$\beta_{\mathcal{J}}$-paracompact subset of Y, then $f^{-1}(A)$ is a δ_1-$\beta_{\mathcal{I}}$-paracompact subset of X.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a δ-$\beta_{\mathcal{I}}$-open cover of $f^{-1}(A)$ in X. Since f is δ-$\beta_{\mathcal{I}}$-open, $f(\mathcal{U}) = \{f(U_\lambda) : \lambda \in \Lambda\}$ is a δ-$\beta_{\mathcal{I}}$-open cover of A in Y. By hypothesis, $f(\mathcal{U})$ has a locally finite open refinement $\mathcal{H} = \{V_\alpha : \alpha \in \Lambda_1\}$ of A such that $A - \cup\{V_\alpha : \alpha \in \Lambda_1\} \in \mathcal{J}$. Then, $f^{-1}(A) - \cup\{f^{-1}(V_\alpha) : \alpha \in \Lambda_1\} \in f^{-1}(\mathcal{J}) = \mathcal{I}$. As f is continuous, $\mathcal{V} = \{f^{-1}(V_\alpha) : \alpha \in \Lambda_1\}$ is locally finite. Next, we will prove that $\mathcal{V}$ refines $\mathcal{U}$. Let $f^{-1}(V_\alpha) \in \mathcal{V}$. As $\mathcal{H}$ refines $f(\mathcal{U})$, there exists $f(U_\lambda) \in f(\mathcal{U})$ such that $V_\alpha \subset f(U_\lambda)$. It follows that $f^{-1}(V_\alpha) \subset f^{-1}(f(U_\lambda)) = U_\lambda$. Therefore $f^{-1}(A)$ is a δ_1-$\beta_{\mathcal{I}}$-paracompact subset of X.

5. Conclusion

This paper examines the characteristics of δ_1-$\beta_{\mathcal{I}}$-paracompact spaces, which are broader in scope than the β_1-paracompact spaces defined by Qahis [21]. Additionally, we investigate the invariants of δ_1-$\beta_{\mathcal{I}}$-paracompactness via functions. We established that if $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(X, \tau^*, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, and that every δ_1-$\beta_{\mathcal{I}}$-closed subset of a δ_1-$\beta_{\mathcal{I}}$-paracompact space is a δ_1-$\beta_{\mathcal{I}}$-paracompact subset. The union of two δ_1-$\beta_{\mathcal{I}}$-paracompact subsets is a δ_1-$\beta_{\mathcal{I}}$-paracompact subset, and the intersection of a δ_1-$\beta_{\mathcal{I}}$-paracompact subset with a δ_1-$\beta_{\mathcal{I}}$-closed set is δ_1-$\beta_{\mathcal{I}}$-paracompact subset. Furthermore, we demonstrate that δ_1-$\beta_{\mathcal{I}}$-paracompactness is preserved under specific conditions: If $f : X \to Y$ is open, δ-$\beta_{\mathcal{I}}$-irresolute, almost closed, surjective with N-closed point inverse, and if $(X, \tau, \mathcal{I})$ is δ_1-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', f(\mathcal{I}))$ is δ_1-$\beta_{f(\mathcal{I})}$-paracompact. If $f : X \to Y$ is δ-$\beta_{\mathcal{I}}$-open, continuous, bijective, and if $(Y, \tau', \mathcal{J})$ is δ_1-$\beta_{\mathcal{J}}$-paracompact, then $(X, \tau, f^{-1}(\mathcal{J}))$ is δ_1-$\beta_{f^{-1}(\mathcal{J})}$-paracompact.

Acknowledgements

The authors would like to express their gratitude to the reviewers for their helpful comments and suggestions that assisted enhance this paper better. This research was supported by the National Science, Research, and Innovation Fund (NSRF) and Prince of Songkla University (Ref. No. SAT6701343S).

References

[1] M. E. Abd.El-Monsef, E. F. Lashien, and A. A. Nasef. On I-open sets and I-continuous functions. *Kyungpook Mathematical Journal*, 32(2):21–30, 1992.

[2] M. E. Abd.El-Monsef, E. F. Lashien, and A. A. Nasef. Some topological operators via ideals. *Kyungpook Mathematical Journal*, 32(2):273284, 1992.

[3] H. H. Al-Jarrah. β_1-paracompact spaces. *Journal of Nonlinear Sciences and Applications*, 9:1728–1734, 2016.

[4] K. Al-Zoubi. S-paracompact spaces. *Acta Mathematica Hungarica*, 110:203–212, 2006.

[5] K. Al-Zoubi and S. Al-Ghour. On P_3-paracompact spaces. *International Journal of Mathematics and Mathematical Sciences*, 2007:1–16, 2007.

[6] R. Alrababah, A. Amourah, J. Salah, and R. Ahmad. Paracompactness in Topological Spaces. *European Journal of Pure and Applied Mathematic*, 17(4):2990–3003, 2024.

[7] A. V. Arkhangelskii and V. I. Ponomarev. *Fundamentals of General Topology: Problems and Exercises.* International Hindustan Publishing, India, 1984.

[8] N. Bourbaki. *General Topology.* Hermann Addison Wesley, Massachusets, 1966.

[9] I. Demir and O.B. Ozbakir. On β-paracompact spaces. *Filomat*, 27(6):971–976, 2013.

[10] J. Dieudonné. Une généralisation des espaces compacts. *Journal de Mathématiques Pures et Appliquées*, 23(9):65–76, 1944.

[11] J. Dontchev, M. Ganster, and T. Noiri. Unified operation approach of generalized closed sets via topological ideals. *Mathematica Japonica*, 49:395–402, 1999.

[12] J. Dugundji. *Topology.* Allyn and Bacon, Boston, 1966.

[13] T. R. Hamlett, D. Rose, and D. Jankovic. Paracompactness with respect to an ideal. *International Journal of Mathematics and Mathematical Sciences*, 20:433–442, 1997.

[14] E. Hatir. On Decompositions of continuity and complete continuity in ideal Topological Spaces. *European Journal of Pure and Applied Mathematics*, 6(3):352–362, 2013.

[15] D. Janković and T. R. Hamlett. New topologies from old via ideals. *The American Mathematical Monthly*, 97(4):295–310, 1990.

[16] M. Khan and T. Noiri. Semi-local functions in ideal topological spaces. *Journal of Advanced Research in Pure Mathematics*, 2(1):36–42, 2010.

[17] K. Kuratowski. *Topology I.* Pańtstwowe Wydawnictwo Naukowe, Warszawa, 1933.

[18] P. Y. Li and Y. K. Song. Some remarks on S-paracompact spaces. *Acta Mathematica Hungarica*, 118:345–355, 2008.

[19] T. Noiri. Completely continuous image of nearly paracompact space. *Matematichki Vesnik*, 29:59–64, 1977.

[20] T. Noiri. A note on inverse-preservations of regular open sets. *Publications de l'Institut Mathématique. Nouvelle Série*, 50:99–102, 1984.

[21] A. Qahis. β_1-paracompact spaces with respect to an ideal. *European Journal of Pure and Applied Mathematics*, 12(1):135–145, 2019.

[22] J. Sanabria, E. Rosas, C. Carpintero, M. Salas-Brown, and O. García. S-paracompactness in ideal topological spaces. *Matematički Vesnik*, 68(3):192–203,

2016.

[23] N. Sathiyasundari and V. Renukadevi. Paracompactness with respect to an ideal. *Filomat*, 20(2):333–339, 2013.

[24] M. K. Singal and S. P. Arya. On nearly paracompact spaces. *Matematički Vesnik*, 6(47):3–16, 1969.

[25] E. D. Yildirim, O. B. Ozbakir, and A. C. Guler. β-paracompactness in ideal topological space. *European Journal of Pure and Applied Mathematics*, 12(2):270–278, 2019.

[26] S. Yuksel, A. Acikgoz, and T. Noiri. On δ-I-continuous functions. *Acta Mathematica Hungarica*, 29:39–51, 2005.

[27] M. I. Zahid. *Para H-closed Spaces, Locally Para H-closed Spaces and Their Minimal Topologies.* PhD thesis, University of Pittsburgh, 1981.