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Statistical Inference Based on Censored Data of Entropy for Lomax Distribution

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Abstract. The current research centers around entropy. This paper investigates the estimation of entropy for the Lomax (Lo) distribution using an adaptive progressive Type-II censored data. The entropy maximum likelihood estimate is computed and the bootstrap confidence intervals of entropy are displayed, approximate confidence intervals are constructed using the asymptotic normality of maximum likelihood estimation and the observed Fisher information matrix. The Bayes entropy estimator is demonstrated using the symmetric and asymmetric loss functions. To further assess the performance of the entropy estimators, particularly under various loss functions, such as linear exponential and squared error, the posterior distribution was calculated. Then, using Monte Carlo simulations, various approaches are compared to identify the believable intervals of the entropy's highest posterior density. Lastly, the recommended methods are illustrated using a numerical example.

2020 Mathematics Subject Classifications: 62F10, 62F15, 62F40

Key Words and Phrases: Lomax model, Entropy; Maximum likelihood estimator, Bootstrap resampling, Adaptive progressive Type-II censored data, Bayes estimator, symmetric and asymmetric loss functions

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1. Introduction

Entropy is a quantitative measure of uncertainty within a probability distribution. Entropy is a measure that reflects the expected value of information contained within a random variable. A higher entropy value signifies a lower information content within the observed data. The measurement of entropy is a significant issue in numerous areas, including statistics, economics, information technology, physics, and the analysis of biological phenomena. For example, entropy can be used to evaluate the probability distribution of electric charge among atoms under specific conditions. This wide applicability has led to extensive research on entropy. The concept of entropy, as a measure of information that

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provides a quantitative measure of uncertainty, was introduced by Shannon [1]. Wong and Chan [2] demonstrated that entropy decreases when data is ordered. AboEleneen [3] simplified the entropy calculation and derived recurrence relations using progressively Type-II censored samples. Sunoj et al. [4] introduced a Shannon entropy function based on quantiles and investigated the properties of the residual entropy function. Seo et al. [5] obtained an entropy estimator using upper record values from the generalized half-logistic distribution. Cho et al. [6] discussed an estimation of the entropy for a Rayleigh distribution based on doubly-generalized Type-II hybrid censored samples. The entropy under Type-II censored data and doubly generalized Type-II hybrid censoring are examined by Zhao et al. [7]. Cho et al. [8] investigated Bayesian estimators of entropy for the Weibull distribution using a generalized progressive hybrid censoring scheme. Similarly, Lee [9] studied maximum likelihood and Bayesian estimators of entropy for the inverse Weibull distribution under a generalized progressive hybrid censoring scheme. Almohaimeed [10] derived an exact expression for entropy information within both types of progressively hybrid censored data, applying it to the exponential distribution. Chacko and Asha [11] explored entropy estimation for the generalized exponential distribution using record values. Hassan and Zaky [12] focused on obtaining the maximum likelihood estimator of Shannon entropy for the inverse Weibull distribution using multiple censored data. Wang and Gui [13] examined certain entropy inferences based on Type-II progressive for Burr Type-XI distribution.

The ideal method for gathering data depends on the test's cost and duration. Common and basic filtering methods are referred to in the literature as censoring schemes (CSs) of Type-I and Type-II. The test duration is suggested under Type-I CS, however, there are an arbitrary number of failures r, $0 < r < n$, but the test time is random. However, the test time T in the Type-II CS is arbitrary and might be $T \to \infty$. Nevertheless, the removal of units at locations other than the experiment's termination point is not flexible enough to be permitted by the traditional Type-I and Type-II censoring procedures. One of the more general filtering techniques known as progressive CSs that were developed as a result of a consequence of this inflexibility is the development of the progressive Type-II censoring scheme. Refer to Balakrishnan and Aggarwala [14]. The progressive Type-II censoring model's operation is demonstrated in the following. For each of the n devices that undergo a life test, write $X_1,\ X_2,\ ...,\ X_n$ as its corresponding lifetime. The Progressive Type-II censoring scheme $R = R_1,\ R_2,\ ...,\ R_r$, and the number of units observed r ($r < n$) are determined before the experiment $0 < R_i, i = 1, 2, ..., r$ and $n = r + \sum_{i=1}^{r-1} R_i$. The remaining R_i units are arbitrarily removed from the experiment once the ith failure has been located. The experiment will continue as long as this guideline is adhered to, ending when failures are identified. Consequently, $X_{i:\ r:\ n}$, $i = 1, 2, ..., r$ are the Type-II progressive right censoring's observed statistics. Ng et al. [15] propose an adaptive Type-II progressive censoring to shorten the test duration overall and increase the efficacy of statistical inference. Here's how this plan works: Think about conducting a life test on n identical units. The observed number of failures r ($r < n$) is predetermined, and the test length is permitted to exceed the predefined period T. The description of the progressive censorship R is eliminated from the test once the ith failure is observed during the life test. The r fully observed lifetimes are represented by the notation $X_{i:\ r:\ n}$, $i = 1, 2, ..., r$. If the rth failure time occurs before time T (i.e. $X_{i:\ r:\ r} < T$), the test ends at time $X_{i:\ r:\ n}$ using the same progressive censoring technique $(R_1,\ R_2,\ ...,\ R_r)$. In this case, $R_r = n - r - \sum_{i=1}^{r-1} R_i$. If Jth failure time happens before time T, i.e., $X_{J:\ r:\ r} < T < X_{J+1:\ r:\ r}$, $(1 < J < r-1)$. In the case of failures, the number of units progressively removed from the test is then modified by setting $R_{J+1} = R_{J+2} = ... = R_{r-1} = 0$. At time R_r, any leftover units R_r are then eliminated, where $R_r = n - r - \sum_{i=1}^{J} R_i$. Thus, the progressive censored scheme that performs best in this situation is $(R_1,\ R_2,\ ...,\ R_J, 0, 0, 0, n - r - \sum_{i=1}^{J} R_i)$. We use X_i in this study rather than $X_{i:r:n}$,$i = 1, 2, ..., r$. The observed data under-considered censoring strategy may be represented by one of the two scenarios below:

Case 1: $(X_1, R_1), (X_2, R_2), ..., (X_r, R_r)$, if $X_r < T$, where $R_r = n - r - \sum_{i=1}^{J-1} R_i$,

Case 2: $(X_1, R_1), (X_2, R_2), ..., (X_J, R_J), (X_{J+1}, 0), ...(X_{r-1}, 0), (X_r, R_r)$, if $X_J < T < X_{J+1}$.

It should be noted that Type-II and Type-II progressive censoring systems are based on the adaptive Type-II censored scheme. Although the adaptive The Type-II censored scheme is reduced to a Type-II censoring scheme. No units will be removed if $T = 0, J = 0$, and if $T = \infty, J = m$, the adaptive Type-II censored scheme is the same as the Type-II progressive censored scheme with the pre-fixed progressive censored scheme $R_i (i = 1, 2, ..., m)$, survival units will be randomly removed during the trial. Discussions about the adaptive Type-II censored system have been ongoing. As an example, Sobhi and Soliman [16] examined exponentiated Weibull hazard functions, reliability, and parameter estimation. Using the Type-II censored adaptive technique, Nassar and Abo-Kasem [17] created Bayes and machine learning estimations for the inverse Weibull distribution's unknown parameters. According to the adaptive Type-II censored scheme, Sewailem and Baklizi [18] looked at the Bayes and ML estimates for the log-logistic distribution parameters. Xu and Gui [19] proposed the adaptive Type-II censored technique to calculate entropy estimations for inverse Weibull distributions. Panahi and Moradi [20] computed the parameters of an exponentiated inverted Rayleigh model. Chen and Gui's [21] study focused on Chen's adaptive progressive Type-II censoring model. They considered the Kumarswamy-exponential distribution in Mohan and Chacko's [22] adaptive progressive Type-II censoring algorithm. Hora et al. [23] examined the Bayesian and classical outcomes for unknown parameters of the inverse Lomax distribution under the adaptive progressive Type-II censoring scheme. Amein et al. [24] investigated several estimation techniques using the adaptive Type-II progressive censored sample from the Gompertz distribution. By setting $X_i = X_{r:\ r:\ n}$, $i = 1, ..., r$. To keep things simple, the following can be used to express the probability function adaptive Type-II progressive censored data:

$$\ell(\underline{x} \mid \underline{\Omega}) = c_J \prod_{i=1}^{r} f(x_i \mid \underline{\Omega}) \prod_{i=1}^{J} (S(x_i \mid \underline{\Omega}))^{R_i} (S(x_r \mid \underline{\Omega}))^{R^*}, \tag{1}$$

where $c_J = \prod_{i=1}^{r} [n - i + 1 - \sum_{k=1}^{\min\{i-1,J\}} R_k]$, and the vector of the unknown parameters is denoted by $\underline{\Omega} = (\beta, \xi)$.

This study's main goal is to examine the adaptive progressively Type-II from the entropy of Lo distribution. since there aren't many relevant works that discuss this subject. It discussed as follows:

- The entropy function is estimated using the maximum likelihood approach.
- We calculate the maximum likelihood estimator for the parameters and entropy numerically, which considers the Newton-Rapshon.
- We discuss interval estimation using the bootstrap approach and the approximation information matrix methods.
- assuming that the scale and shape parameters in the model each follow different Gamma priors, the Metropolis-Hasting method is then used to get the Bayes estimators and associated credible intervals.
- Lastly, the performance of estimations utilizing Monte Carlo simulation is assessed using mean squared error. On the other hand, average length and probability coverage are used in interval estimate.

The rest of the document is structured as follows: The model and its underlying assumptions are presented in Section 2. Section 2 provided the maximum likelihood estimate (MLE) and Bayesian analysis applying the Markov Chain Monte Carlo (MCMC), the squared error (SE) function and the linear-exponential (LINEX) loss function. Section 2, the estimated approximate confidence intervals derived from the MLEs, the bootstrap interval and the highest posterior density (HPD) credible interval are derived. We do a simulated analysis and simulate a data set to illustrate the estimate methods covered in this paper and examine actual data in Section 3. Section 4 contains the final remarks.

2. Methodology

In this particular instance, unit lifetime has a Lo distribution in the model that is being developed. The point estimates of entropy are created using the Bayesian and MLE approaches. Additionally, interval estimators are developed using bootstrap methods, HPD credible intervals, and the asymptotic property of MLEs.

2.1. The Lomax distribution model

If X has the Lo distribution Next, the probability density function (PDF) and cumulative distribution function (CDF) of X are given by:

$$f(x) = \beta\, \xi^{\beta}(\xi + x)^{-\beta-1} \quad , \quad \beta \ , \ \xi, x > 0, \tag{2}$$

$$F(x) = 1 - \xi^{\beta}(\xi + x)^{-\beta}, \quad \beta \ , \ \xi, x > 0, \tag{3}$$

where the scale parameter is represented by ξ and the shape parameter by β. This distribution was presented as a model for corporate failure data by Lo [25]. Chahkandi and Ganjali [26] state that it is a member of the class of distributions with declining failure rates. When the researcher suspects a heavy-tailed distribution in the population, it offers a very good substitute for popular lifespan distributions including in Gamma Weibull, or exponential (see [27]. This distribution's genesis, various features, and application areas are described in [28, 29]. The literature contains a wide range of applications for the Lo distribution. For instance, it has been widely utilized for life testing and dependability modeling; take Balkema and de Haan [30] as an example. Ahsanullah [31] evaluated the Lo's record values distribution. Balakrishnan and Ahsanullah [32] proposed some recurring linkages between record value moments from the Lo distribution. Using both complete and censored samples, a number of authors have tackled inferential problems for the Lo distribution. Childs et al. [33] have looked at nonidentical right-truncated Lo random variables' order statistics. Howlader and Hossain [34] looked at Bayesian estimation of the Lo distribution's survival function. Ghitany et al. [35] took into account the expanded Lo distribution and Marshall-Olkin method. Elfattah et al. [36] used the progressive Type-I censoring using the Lo distribution to determine the Bayesian and non-Baysian estimators for the same sample size. ML and approximation ML predictors derived from the Pareto distribution using multistage progressive filtering, as well as best linear unbiased predictors, were among the several failure time predictors covered by Raqab et al. [37]. Using progressive Type-II censoring, the optimal censoring technique for determining the parameters of the Lo distribution was investigated by Cramer and Schmiedt [38]. Asgharzadeh and Valiollahi [39] used progressive Type-II censoring to create the Bayesian estimator of the scale parameter of the Lo distribution.

2.2. Modeling

The famous Shannon information entropy is defined using the first two moments of a random variable X, as follows:

$$H(f) = H(X) = -\int_0^\infty f(x)\log f(x)dx, \tag{4}$$

where $f(x)$ is the PDF of a continuous random variable X.

Theorem

CDF requires that X be a random variable, its entropy would be

$$H(f) = (\beta+1)[\log(\xi) + \frac{1}{\beta}] - \log(\beta\xi^\beta). \tag{5}$$

Proof.

$$\begin{aligned} H(f) &= -\int_0^\infty \beta\ \xi^\beta(\xi+x)^{-\beta-1}[\log(\beta\ \xi^\beta) - (\beta+1)\log(\xi+x)]dx \\ &= \int_0^\infty \beta(\beta+1)\ \xi^\beta(\xi+x)^{-\beta-1}\log(\xi+x)dx - \int_0^\infty \beta\ \xi^\beta(\xi+x)^{-\beta-1}\log(\beta\ \xi^\beta)dx \end{aligned}$$

$$= (\beta+1)[\log(\xi) + \frac{1}{\beta}] - \log(\beta\ \xi^{\beta}), \tag{6}$$

where

$$\int_0^\infty \beta\ \xi^{\beta}(\xi+x)^{-\beta-1}\log(\beta\ \xi^{\beta})dx = \int_0^\infty f(x)\log(\beta\ \xi^{\beta})dx = \log(\beta\ \xi^{\beta}). \tag{7}$$

Using integration by parts, we have:

$$\begin{aligned}\int_0^\infty \beta(\beta+1)\ \xi^{\beta}(\xi+x)^{-(\beta+1)}\log(\xi+x)dx &= (\beta+1)\log(\xi) + \int_0^\infty (\beta+1)\ \xi^{\beta}(\xi+x)^{-(\beta+1)}dx \\ &= (\beta+1)\log(\xi) + \int_0^\infty \frac{(\beta+1)}{\beta}\ f(x)dx \\ &= (\beta+1)\log(\xi) + \frac{(\beta+1)}{\beta} \\ &= (\beta+1)[\log(\xi) + \frac{1}{\beta}]. \end{aligned} \tag{8}$$

Consequently, the following expression must be used to represent the Shannon entropy of the Lo distribution, which depends on the parameters β and ξ.

$$H(f) = (\beta+1)[\log(\xi) + \frac{1}{\beta}] - \log(\beta\ \xi^{\beta}). \tag{9}$$

2.3. Point Estimation

2.3.1. Entropy Maximum Likelihood Estimator

To determine the point estimation, for the Lo distribution, let $\underline{x} = (x_{1:r:n} < x_{2:r:n} < \ldots < x_{r:r:n})$ be an adaptive Type-II progressive censored order statistic. The likelihood function is provided by Eqs. (1), (2), and (3), which is

$$\ell\left(\beta,\ \xi \mid \underline{x}\right) = \prod_{i=1}^{r}\beta\ \xi^{\beta}(\xi+x_i)^{-\beta-1} \times \prod_{i=1}^{J}\left(\xi^{\beta}(\xi+x_i)^{-\beta}\right)^{R_i} \times \left(\xi^{\beta}(\xi+x_r)^{-\beta}\right)^{R_r^*}, \tag{10}$$

where,

$$R_r^* = n - r - \sum_{i=1}^{J} R_i, \qquad i = 1, 2, \ldots, r. \tag{11}$$

This allows us to express the likelihood function's logarithm as

$$L\left(\beta,\ \xi \mid \underline{x}\right) = r\log\beta + (r + \sum_{i=1}^{J} R_i)\beta\log\xi - (\beta+1)\sum_{i=1}^{r}\log(\xi+x_i)$$

$$-\beta\sum_{i=1}^{J}R_i\log(\xi+x_i)-\beta R_r^*\log(\xi+x_r). \tag{12}$$

The likelihood formulas are computed as

$$\frac{\partial L\left(\beta,\ \xi\mid \underline{x}\right)}{\partial\beta}=\frac{r}{\beta}+(r+\sum_{i=1}^{J}R_i)\log\xi-\sum_{i=1}^{r}\log(\xi+x_i)-\sum_{i=1}^{J}R_i\log(\xi+x_i)-R_r^*\log(\xi+x_r)=0, \tag{13}$$

$$\frac{\partial L\left(\beta,\ \xi\mid \underline{x}\right)}{\partial\xi}=\frac{(r+\sum\limits_{i=1}^{J}R_i)\beta}{\xi}-(\beta+1)\sum_{i=1}^{r}\frac{1}{(\xi+x_i)}-\beta\sum_{i=1}^{J}\frac{R_i}{(\xi+x_i)}-\beta\frac{R_r^*}{(\xi+x_r)}=0, \tag{14}$$

The nonlinear Eqs. (13) and (14) Newton's method can be used to numerically solve it and find the parameters' MLEs β and ξ. Thus, the MLE of $H(f)$ equals

$$\hat{H}(f)=-\log(\hat{\beta}\ \hat{\xi}^{\hat{\beta}})+(\hat{\beta}+1)[\log(\hat{\xi})+\frac{1}{\hat{\beta}}], \tag{15}$$

where $\hat{H}(f)$ as given in Eq. (5) after replacing β and ξ by $\hat{\beta}$ and $\hat{\xi}$, respectively.

2.3.2. Entropy Bayes Estimations Using Markov Chain Monte Carlo

This paper with using an adaptive progressive Type-II censored data computes the Bayesian estimate of entropy where the informative prior is driven by Gamma priors under both symmetric and asymmetric loss functions. When computing the Bayes estimators, although any other loss function can be readily included, we typically assume a SE loss function. Non-informative previous distribution can be used in some circumstances when we lack adequate prior knowledge.

(1) Prior Distribution and Corresponding Posterior Distribution

This is especially valid for our research. The likelihood function will then determine the joint posterior density. Independent Gamma priors for the parameters explain the prior data. Considering the parameters β and ξ to have independent Gamma priors, we obtain the Bayes estimation of entropy in a manner that

$$\pi\left(\beta\right)\sim\beta^{a-1}e^{-b\beta}\ \text{ and }\pi\left(\xi\right)\sim\xi^{c-1}e^{-d\xi}, \tag{16}$$

when we know the hyper-parameters (a,b) and (c,d). According to Eqs. (10) and (16), the joint posterior density of β and ξ given the data is

$$\pi(\beta,\ \xi|\underline{x})=\frac{\pi\left(\beta\right)\pi\left(\xi\right)\ell(\beta,\ \xi|\underline{x})}{\int_\beta\int_\xi\pi\left(\beta\right)\pi\left(\xi\right)\ell(\beta,\ \xi|\underline{x})d\beta d\xi}. \tag{17}$$

$$\pi(\beta,\ \xi|\underline{x})\propto\beta^{a+r-1}\xi^{c+\beta(r+\sum\limits_{i=1}^{r}R_i)+R_r^*)-1}e^{-b\beta}e^{-d\xi}e^{-(\beta+1)\sum\limits_{i=1}^{r}\log(\xi+x_i)}$$

$$\times\, e^{-\beta \sum\limits_{i=1}^{J} R_i \log(\xi+x_i)} e^{-\beta R_r^* \log(\xi+x_r)}. \tag{18}$$

In general, the integration shown by Eq. (18) is more challenging and does not allow for closed-form formulations, particularly when dealing with high-dimensional causes. Consequently, approximation techniques like numerical integration and the Lindely approximation can be used. However, the key techniques like MCMC rely on building. The posterior distribution based on empirical data, It can be accomplished instead by modeling a large sample of the posterior distribution. With a variety of algorithms, the more broad Metropolis-Hastings (M-H) algorithm can be used with Gibbs sampling or Gibbs sampling methods, (see Ahmed [40], Ahmed and Mustafa [41], and Ahmed et al. [42]). The importance sampling technique is another. The posterior distribution provided by the following is an expression for Eq. (18):

$$\pi(\beta,\ \xi|\underline{x}) \propto \pi_\beta(\beta|\xi,\underline{x})\pi_\xi(\xi|\beta,\underline{x}), \tag{19}$$

where

$$\pi(\beta,\ \xi|\underline{x}) \propto \beta^{a+r-1}\xi^{c+\beta(r+(\sum\limits_{i=1}^{r} R_i)+R_r^*)-1} e^{-b\beta} e^{-d\xi} \times e^{-(\beta+1)\sum\limits_{i=1}^{r}\log(\xi+x_i)} e^{-\beta\sum\limits_{i=1}^{J} R_i\log(\xi+x_i)} e^{-\beta R_r^*\log(\xi+x_r)}, \tag{20}$$

$$\pi_\beta(\beta|\xi,\underline{x}) \propto \quad \beta^{a+n-1} e^{-\beta(b-(r+(\sum\limits_{i=1}^{r} R_i)+R_r^*)\log(\xi)+\sum\limits_{i=1}^{r}\log(\xi+x_i)+\sum\limits_{i=1}^{J} R_i\log(\xi+x_i)+R_r^*\log(\xi+x_r))}, \tag{21}$$

$$\pi_\xi(\xi|\beta,\underline{x}) \propto \xi^{c+\beta(r+(\sum\limits_{i=1}^{r} R_i)+R_r^*)-1} e^{-\sum\limits_{i=1}^{r}\log(\xi+x_i)-d\xi} e^{-\beta(\sum\limits_{i=1}^{J} R_i\log(\xi+x_i)+R_r^*\log(\xi+x_r)+\sum\limits_{i=1}^{r}\log(\xi+x_i))}. \tag{22}$$

(2) Loss Function

Bayesian methods are increasingly popular in reliability research and estimation. Selecting a single value to represent the estimate of an unknown parameter requires defining a loss function. Bayesian estimation typically employs two types of loss functions: the symmetric SE loss function and the asymmetric LINEX loss function, introduced by Varian [43]). The SE loss function provides the foundation for evaluating estimators' performance in a lot of studies.

$$\hat{\underline{\Omega}}_{SE} = \int_{\underline{\Omega}} \underline{\Omega}\pi(\beta,\ \xi|\underline{x})d\beta d\xi. \tag{23}$$

Because this function is symmetric, overestimation and underestimation are given equal weight. An overestimate is far more dangerous than an underestimate, similar to how functions for failure rate and reliability are estimated. Varian invented LINEX loss function, this is one of the most widely used asymmetric loss functions [43]. This function rises roughly to zero on one side and is roughly linear on the other. Varian [43] introduced a representation of the LINEX loss function can be expressed in the manner shown below, assuming that the minimal loss happens at $\hat{\Omega} = \Omega$

$$L_1(\Delta) \propto e^{c\Delta} - c\Delta - 1,\ c \neq 0, \tag{24}$$

where $\underline{\hat{\Omega}}$ is the estimate of $\underline{\Omega}$ and $\Delta = (\underline{\hat{\Omega}} - \underline{\Omega})$. The degree and direction of symmetry are expressed by the magnitude of c. When $c > 0$, overestimation is more dangerous than underestimation, while the opposite is true if $c < 0$. The LINEX loss reduces to the SE loss when c is close to zero. The LINEX loss function of Eq. (24) has a posteriors-expectation of

$$E_{\underline{\Omega}}(L_1(\underline{\hat{\Omega}} - \underline{\Omega})) \propto e^{c\underline{\hat{\Omega}}} E_{\underline{\Omega}}(e^{(-c\underline{\Omega})}) - c(\underline{\hat{\Omega}} - E_{\underline{\Omega}}(\underline{\Omega}) - 1. \tag{25}$$

The value of $\underline{\hat{\Omega}}$ under the LINEX loss function, the Bayes estimator is

$$\underline{\hat{\Omega}}_{LINEX} = -\frac{1}{c}\log(E_{\underline{\Omega}}[e^{(-c\underline{\Omega})}]). \tag{26}$$

In this way $E_{\underline{\Omega}}(e^{(-c\underline{\Omega})})$ exists, after creating β and ξ, the Bayes estimates of entropy is calculated from the posterior density functions. Given adaptive progressive Type-II censored data, the MCMC algorithm will execute the subsequent procedures:

Step 1: Choose an initial guess for (β**,** ξ), denoted by ($\beta^{(0)}$**,** $\xi^{(0)}$).

Step 2: Assign $w = 1$.

Step 3: Using Gamma $(r+a, (b-(r+(\sum_{i=1}^{r} R_i)+R_r^*)\log(\xi)+\sum_{i=1}^{r}\log(\xi+x_i)+\sum_{i=1}^{J} R_i\log(\xi+x_i)+R_n^*\log(\xi+x_r))\ )$ to generate $\beta^{(w)}$.

Step 4: Applying M-H to $\pi_\xi(\xi|\beta, \underline{x})$ to produce $\xi^{(w)}$, with the $N(\xi^{(w-1)}, var(\hat{\xi}))$ proposal distribution.

M-H algorithm

1. Create $\xi^{(*)}$ as a starting point, for which $\pi_\xi\left(\xi^{(w-1)}, var(\hat{\xi})\right)$.

2. Determine the likelihood of acceptance

$$\rho_\xi = \min\left[1, \frac{\pi_\xi\left(\xi^*, var(\hat{\xi})\right)}{\pi_\xi\left(\xi^{(w-1)}, var(\hat{\xi})\right)}\right]. \tag{27}$$

and generate $U \sim U\,(0,1)$.

3. If the suggestion is accepted by $U \leq \rho_\xi$, then set $\xi^{(*)} = \xi^{(w)}$. If not, turn down the proposal

Step 5: Determine $H_1(f)$ using Eq. (5).

Step 6: Assign $w = w + 1$.

Step 7: Steps $3 - 5$ N should be repeated

Step 8: Determine R 's Bayes estimator regarding the SE loss function as

$$\hat{H}_{SE}(f) = \frac{\sum\limits_{i=M+1}^{N} H^{(i)}(f)}{N - M}, \tag{28}$$

with respect to the LINEX loss function, $H(f)$'s Bayes estimator as

$$\hat{H}_{LINEX}(f) = -\frac{1}{c}\log\left[\frac{\sum\limits_{i=M+1}^{N} e^{-cH^{(i)}(f)}}{N - M}\right], \tag{29}$$

where M is burned in.

2.3.3. Interval Estimation

(1) Asymptotic confidence intervals

For MLE, the asymptotic normality is used to construct the parameters asymptotic confidence intervals (ACIs). The Fisher information matrix defines the negative expectation of the second derivatives of the log-likelihood function aboutthe the model parameters. The anticipation of the second derivative is typically more significant in more circumstances. The observed Fisher information matrix therefore provides an appropriate approximation that may be utilized to build interval estimation in the manner described below. The model parameter vector's second derivative $\underline{\Omega} = (\beta,\ \xi)$ log-likelihood function is I_0 with using an adaptive progressive Type-II censored. data:

$$I_0(\underline{\Omega}) = \left[-\frac{\partial^2 \ell\,(\beta,\ \xi \mid \underline{x})}{\partial\beta\partial\ \xi}\right], \tag{30}$$

$$\frac{\partial^2 L\,(\beta,\ \xi \mid \underline{x})}{\partial\beta^2} = -\frac{r}{\beta^2}, \tag{31}$$

$$\frac{\partial^2 L\,(\beta,\ \xi \mid \underline{x})}{\partial\beta\partial\xi} = \frac{\partial^2 L\,(\beta,\ \xi \mid \underline{x})}{\partial\xi\partial\beta} = \frac{(r + \sum\limits_{i=1}^{J} R_i)}{\xi} - \sum_{i=1}^{r}\frac{1}{(\xi + x_i)} - \sum_{i=1}^{J}\frac{R_i}{(\xi + x_i)} - \frac{R_r^*}{(\xi + x_i)}, \tag{32}$$

$$\frac{\partial^2 L\,(\beta,\ \xi \mid \underline{x})}{\partial\xi^2} = \frac{(r + \sum\limits_{i=1}^{J} R_i)\beta}{\xi^2} + (\beta + 1)\sum_{i=1}^{n}\frac{1}{(\xi + x_i)^2} + \beta\sum_{i=1}^{J}\frac{R_i}{(\xi + x_i)^2} + \frac{\beta R_r^*}{(\xi + x_i)^2}. \tag{33}$$

Next, $I_0(\underline{\Omega})$, the observed information matrix, identifies the vector parameter $\underline{\Omega}$'s ML estimate. It is now possible to distribute the MLE asymptotic distribution theory as $I_0^{-1}(\hat{\underline{\Omega}})$ is the variance covariance matrix for the mean $\underline{\Omega}$ of a bivariate normal distribution satisfies standard regularity requirements let

$$\hat{\underline{\Omega}} \longrightarrow N(\underline{\Omega}, I_0^{-1}(\hat{\Omega})).$$

Using the delta approach, to obtain an approximate estimation of the entropy variance, allow

$$A = \left(\frac{\partial H(f)}{\partial \beta}, \frac{\partial H(f)}{\partial \xi} \right), \tag{34}$$

where the first derivatives of the $H(f)$ in relation to the parameters β and ξ are $\frac{\partial H(f)}{\partial \beta}$ and $\frac{\partial H(f)}{\partial \xi}$:

$$H(f) = -\log(\beta\ \xi^{\beta}) + (\beta+1)[\log(\xi) + \frac{1}{\beta}], \tag{35}$$

$$\begin{aligned} \frac{\partial H(f)}{\partial \beta} &= -\frac{\xi^{\beta} + \beta \xi^{\beta} \log(\xi)}{\beta\ \xi^{\beta}} + \log(\xi) + \frac{1}{\beta} - \frac{(\beta+1)}{\beta^2} \\ &= -\frac{1}{\beta} - \log(\xi) + \log(\xi) + \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta^2} \\ &= -\frac{\beta+1}{\beta^2}, \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial H(f)}{\partial \xi} &= -\frac{\beta^2 \xi^{\beta-1}}{\beta\ \xi^{\beta}} + \frac{(\beta+1)}{\xi} \\ &= -\frac{\beta}{\xi} + \frac{(\beta+1)}{\xi} = \frac{1}{\xi}, \end{aligned} \tag{37}$$

$H(\hat{f})$'s estimated asymptotic variance is provided by

$$var(H(\hat{f})) \longrightarrow \left[A I_0^{-1} A^t \right] |_{(\hat{\beta},\ \hat{\xi})}.$$

where A^t is the transpose of A, the MLE $H(\hat{f})$ of $H(f)$ has an asymptotic distribution. As a result, the asymptotic $100(1-\alpha)\%$ confidence interval for $H(f)$ is as follows:

$$H(\hat{f}) \pm Z_{\alpha/2} \sqrt{var(H(\hat{f}))}. \tag{38}$$

(2) Bootstrap confidence intervals

This part uses the percentile interval to create intervals of confidence derived by applying the parametric bootstrap method to the unknown parameters using an adaptive progressive Type-II censored data, see Efron [44] for more information about bootstrap confidence intervals. To get a bootstrap sample, the following algorithm is developed:

1. Get MLEs $\hat{\beta}$, $\hat{\xi}$ and $\hat{H}(f)$ from the initial two samples $\{x_1, x_2, ..., x_n\}$.

2. The bootstrap estimate of β, ξ and $H(f)$ is calculated by generating a bootstrap sample $\{x_1^*, x_2^*, ..., x_n^*\}$ using $\hat{\beta}$, $\hat{\xi}$ and $\hat{H}(f)$.

3. To acquire the bootstrap samples, repeat steps (1) through (2) N times, making sure that each estimate is arranged in ascending order: $\{ \hat{H}_{M+1}(f),\ \hat{H}_{M+2}(f),\ ...,\ \hat{H}_{N-M}(f)\}$

The approximate confidence interval for $H\hat{(}f)$ at $100(1-\alpha)\%$ is provided by

$$(\hat{H}_{i\left((N-M)\frac{\alpha}{2}\right)}(f),\ \hat{H}_{i\left((N-M)(1-\frac{\alpha}{2})\right)}(f)),\ \mathrm{i} = M+1, ...N.$$

(3) Credible confidence intervals

For a random quantity, a posterior or 100 $(1-\alpha)\%$ Bayesian credible interval The interval denoted by $\underline{\Omega}$ is where the posterior probability $(1-\alpha)$ that $\underline{\Omega}$ is situated.

The procedure below is used to get reliable $H(f)$ confidence intervals.

Algorithm (2)

1. In algorithm (1), repeat steps (1) through (5).

2. The Bayesian credible interval for the $H(f)$ is then established using the generated MCMC samples using the algorithm Chen and Shao [45] developed. The posterior sample is arranged as follows: $\hat{H}_{M+1}(f)$, $\hat{H}_{M+2}(f)$, ..., $\hat{H}_{N-M}(f)$. The $100(1-\alpha)\%$ HPD credible intervals for $H(f)$ are obtained by

$$(\hat{H}_{i\left((N-M)\frac{\alpha}{2}\right)}(f),\ \hat{H}_{i\left((N-M)(1-\frac{\alpha}{2})\right)}(f)),\ \mathrm{i} = M+1, ...N. \tag{39}$$

where $\alpha/2$ displays the standard normal values with probability tailed α.

3. Numerical Application

3.1. Simulation study

This subsection compares and evaluates the various estimating techniques using a Monte Carlo simulation study. The features and effectiveness of each estimate were also evaluated. In this part, we compare the effectiveness of the different methods considering different sample sizes and values of parameters using example findings derived from Monte Carlo simulations. We evaluate how well the Bayes and MLE estimates perform about the mean SE (MSE). Regarding coverage percentages (CP) and average confidence lengths, we contrast the confidence intervals derived using the HPD credible intervals, bootstrap, and MLE using asymptotic distributions. The parameter values that we used were $\beta = 0.8$, $\xi = 0.3$ and $\beta = 1.5$, $\xi = 0.5$. The sample sizes that we utilized were as follows: $(n, r) = (80, 50), (80, 60), (100, 60), (100, 80)$. For the informative hyper-parameters $a = b = c = d = 1$. The LINEX loss function is asymmetric, with the parameter c determining the direction of this asymmetry. For $c > 0$, overestimation incurs a higher cost than underestimation, the opposite is true for $c < 0$. As c approaches zero, the LINEX loss function exhibits symmetry and approximates the SE loss. We applied the LINEX loss function with c=$\pm$0.05to obtain Bayesian estimates. As outlined in Section 2, posterior analysis was performed using a hybrid MCMC algorithm employing Gibbs sampling and M-H. The MCMC algorithm was initialized using the MLEs of the parameters β and ξ. we provide the average mean, the MSE of the MLE, and Bayes estimates of entropy that were related to, and $\tau = 0.3$ and $\tau = 1$ and different CS R were taken into consideration, three different CS were utilized in this study, namely:

scheme I: $R_1 = n - r$, $R_i = 0$ % for $i \neq 1$.

scheme II: $R_{\frac{r+1}{2}} = n - r$, $R_i = 0$ for $i \neq \frac{r+1}{2}$; if r odd, and
$R_{\frac{r}{2}} = n - r$, $R_i = 0$ for $i \neq \frac{r}{2}$; if r even.

scheme III: $R_r = n - r$, $R_i = 0$ for i$\neq r$.

Mathematica 10 is used for all computations. Tables 3 and 4 (which provide the CP and average length of the 95% asymptotic, bootstrap, and HPD credible intervals of entropy) are determined., whereas Tables 1and 2 display the averaged mean and MSEs of the estimates in parenthesis. For the MCMC approach, we choose $N = 11000$ with $M = 1000$ as the burn-in time period. The results of the simulation investigation are shown in Tables 1 - 4. As effect sample size grows, MSEs fall and CP approaches the suggested value. The Bayes estimation shows lower MSEs with respect to the LINEX loss function than with respect to the SE loss function. The mean length and MSEs of the LINEX loss function drop as c increases.

Based on the results presented in Tables $1 - 4$, we draw the following conclusions:

- Superiority of Bayesian Estimation: Bayesian estimation, implemented through the MCMC technique, consistently outperforms MLEs and bootstrap estimates in terms of MSE for the parameter $H(f)$.

- Effect of Sample Size: With fixed values of τ and n, increased effective sample size leads to a reduction in MSEs and convergence of CP towards the nominal value.

- Impact of Censoring Parameters: When τ is fixed, increasing the values of n and r generally results in a decrease in the MSEs of all estimators. Conversely, increasing τ while keeping n and r fixed tends to increase the MSEs in most cases.

- LINEX vs. SE Loss: Bayesian estimation exhibits lower MSEs when using the LINEX loss function compared to the SE loss function.

- Efficiency of Estimators: All point estimators are highly efficient, as indicated by their very small average MSEs, which approach zero as n and r increase.

- LINEX Loss Parameter and MSE: The mean length and MSEs of the LINEX loss function decrease with increasing values of c**.**

- Overall Performance: The simulation results demonstrate that Bayesian estimations, particularly those under the LINEX loss function with $c = 0.05$, achieve the best

MSEs for estimating $H(f)$, surpassing other methods. Furthermore, the Bayes credible intervals exhibit lower widths and higher CP compared to other methods, as reflected in the mean lengths and CPs presented in Tables $1-4$.

While the Bayesian estimators demonstrate superior performance, the simulations indicate that all point and interval estimation methods considered are efficient. The Bayesian technique is a particularly advantageous choice when prior knowledge is available.

Table 1: Mean and MSE of the ML, bootstrap and Bayes of the entropy $H(f)$ at $\tau = 0.3$.

			ML		Boot		Bayes					
							SE		LINEX			
									$c=-0.05$		$c=0.05$	
n	r	CS	mean	MSE	mean	MSE	mean	MSE	mean	MSE	mean	MSE
						$\beta=0.8,\ \xi=0.3$						
80	50	I	1.8927	0.3037	1.9067	0.2401	1.9249	0.2043	1.9210	0.2010	1.9097	0.1867
		II	1.7892	0.3104	1.9210	0.2548	1.9357	0.2107	1.9657	0.2149	1.9687	0.1987
		III	1.9075	0.3110	1.8689	0.2550	1.8590	0.2110	1.9230	0.2207	1.8802	0.2070
80	60	I	1.9204	0.3007	1.8832	0.2204	1.9016	0.2108	1.9197	0.1893	1.8940	0.1804
		II	1.8247	0.2970	1.9670	0.2420	1.8999	0.2220	1.8967	0.1801	1.8034	0.1896
		III	1.9378	0.3049	1.8349	0.2507	1.9773	0.2247	1.9270	0.1896	1.8034	0.1901
100	60	I	1.9107	0.2980	1.9110	0.2701	1.9054	0.2152	1.9130	0.1991	1.9046	0.1846
		II	1.9637	0.2976	1.8369	0.2657	1.9370	0.2210	1.9678	0.1901	1.9078	0.1891
		III	1.8934	0.3000	1.9331	0.2789	1.9207	0.2225	1.9875	0.1638	1.9011	0.1920
100	80	I	1.9042	0.2807	1.9110	0.2562	1.9049	0.2124	1.8903	0.1937	1.8904	0.1864
		II	1.9678	0.2834	1.9678	0.2489	1.9815	0.2210	1.9680	0.1987	1.8863	0.1899
		III	1.8963	0.2975	1.8675	0.2553	1.9822	0.2217	1.8650	0.2018	1.8960	0.2014
						$\beta=1.5,\ \xi=0.5$						
80	50	I	1.1007	0.2934	1.0160	0.2347	1.1006	0.2318	0.9987	0.2057	0.9896	0.2012
		II	1.0890	0.3048	1.0789	0.2344	1.0463	0.2486	1.0132	0.2132	0.9681	0.2014
		III	0.9982	0.3155	1.0100	0.2355	1.0040	0.24440	0.9861	0.2204	0.9600	0.2093
80	60	I	0.9870	0.2807	0.9908	0.2429	1.1003	0.2310	1.0108	0.2027	1.1091	0.2009
		II	0.8670	0.2991	0.9837	0.2341	1.1240	0.2341	1.0855	0.2045	1.0041	0.2014
		III	0.9970	0.2998	0.9981	0.2458	1.1104	0.2333	1.0052	0.2109	1.0348	0.2045
100	60	I	1.0107	0.2908	1.1084	0.2307	1.0091	0.2019	1.0940	0.2007	1.0207	0.1893
		II	0.9678	0.3045	1.0733	0.2540	0.9840	0.2000	1.0054	0.2041	1.0547	0.1937
		III	0.9705	0.3004	1.0145	0.2556	0.9912	0.2104	0.9931	0.2210	1.0024	0.1995
100	80	I	0.9987	0.2827	0.9890	0.2320	0.9807	0.2087	0.9937	0.2011	0.9819	0.1708
		II	0.9830	0.2984	0.9930	0.2447	0.9800	0.2104	0.9824	0.2080	0.9870	0.1801
		III	0.9378	0.2963	0.9670	0.2555	0.9933	0.2122	0.9687	0.2100	0.9967	0.1833

:

Table 2: Mean and MSE of the ML, bootstrap and Bayes of the entropy $H(f)$ at $\tau = 1$.

			ML		Boot		Bayes					
							SE		LINEX			
									$c = -0.05$		$c = 0.05$	
n	r	CS	mean	MSE	mean	MSE	mean	MSE	mean	MSE	mean	MSE
$\beta = 0.8,\ \xi = 0.3$												
80	50	I	1.9245	0.3248	1.9307	0.2528	1.9370	0.2148	1.9325	0.2021	1.9207	0.1933
		II	1.8934	0.3269	1.9427	0.2657	1.9904	0.2117	1.9638	0.2234	1.8965	0.2140
		III	1.9250	0.3350	1.9670	0.2751	1.9680	0.2280	1.9740	0.2518	1.9968	0.2284
80	60	I	1.9441	0.3087	1.9583	0.2145	1.9529	0.2083	1.9516	0.1932	1.9900	0.1812
		II	1.9801	0.3088	1.9811	0.2734	1.9932	0.2183	1.9924	0.2067	1.9960	0.2154
		III	1.9217	0.3157	1.9330	0.2964	1.9821	0.2518	1.9872	0.2146	1.9854	0.2214
100	60	I	1.9960	0.3007	1.9423	0.2845	1.9668	0.2008	1.9450	0.1918	1.9484	0.1700
		II	1.9470	0.3096	1.8854	0.2995	1.9993	0.2537	1.9934	0.2056	1.8934	0.1991
		III	1.8990	0.3099	1.9127	0.2998	1.9478	0.2850	1.9870	0.2349	1.9956	0.2175
100	80	I	1.9807	0.2756	1.9657	0.2450	1.9880	0.2014	1.9704	0.1824	1.9815	0.1634
		II	1.9967	0.2967	1.9218	0.2750	1.9825	0.2527	1.9962	0.2014	1.8863	0.1993
		III	1.9670	0.2998	1.8968	0.2934	1.9934	0.2631	1.9278	0.2527	1.8960	0.2427
$\beta = 1.5,\ \xi = 0.5$												
80	50	I	1.0128	0.3049	0.9907	0.2608	0.9838	0.2546	0.9751	0.2145	0.9910	0.2001
		II	1.0213	0.3157	1.0053	0.2772	1.0041	0.2751	0.9845	0.2548	0.9958	0.2172
		III	1.0184	0.3199	1.0048	0.2934	0.9934	0.2281	0.9995	0.2648	0.9938	0.2510
80	60	I	0.9967	0.2907	0.9972	0.2560	1.0033	0.2470	1.0087	0.2137	1.0159	0.2001
		II	0.9937	0.3048	0.9990	0.2647	1.0082	0.2510	1.0047	0.2227	1.0051	0.2247
		III	0.9853	0.3089	0.9580	0.2780	1.0544	0.2630	1.0208	0.2638	1.0028	0.2599
100	60	I	1.0073	0.3007	0.9677	0.2478	1.0408	0.2128	1.0210	0.2084	1.0138	0.1901
		II	1.0128	0.3108	1.0247	0.2529	1.0084	0.2257	1.0249	0.2452	1.0521	0.2400
		III	1.0174	0.3127	1.0004	0.2660	1.0899	0.2712	0.9886	0.2234	1.0066	0.2310
100	80	I	0.9821	0.2927	0.9933	0.2438	0.9914	0.2123	0.9969	0.2058	0.9916	0.1884
		II	0.9968	0.3046	0.9657	0.2570	0.9937	0.2524	0.9954	0.2634	0.9760	0.2430
		III	0.9931	0.3086	0.9980	0.2668	0.9920	0.2554	0.9964	0.2754	0.9931	0.2465

Table 3: Mean length and CP of ML, bootstrap and Bayes of the entropy at $\tau = 0.3$.

			MLE		Bootstrap		Bayes	
n	r	CS	Mean Length	CP	Mean Length	CP	Mean Length	CP
$\beta = 0.8,\ \xi = 0.3$								
80	50	I	1.5720	0.9460	1.4906	0.9280	1.1185	0.9500
		II	1.5107	0.9320	1.5208	0.9160	1.0982	0.9340
		III	1.5073	0.9160	1.5107	0.9080	1.1130	0.9320
80	60	I	1.4870	0.9500	1.4458	0.9500	1.0253	0.9640
		II	1.5420	0.9340	1.4283	0.9340	1.2104	0.9560
		III	1.4630	0.9280	1.5012	0.9320	1.2113	0.9460
100	60	I	1.5004	0.9500	1.5296	0.9520	1.2104	0.9760
		II	1.5133	0.9320	1.5183	0.9460	1.0214	0.9500
		III	1.4820	0.9560	1.4170	0.9320	1.3240	0.9320
100	80	I	1.3879	0.9520	1.3084	0.9560	1.0245	0.9460
		II	1.4452	0.9340	1.4406	0.9520	1.3201	0.9500
		III	1.5480	0.9200	1.4120	0.9500	1.3004	0.9720
$\beta = 1.5,\ \xi = 0.5$								
80	50	I	0.9140	0.9360	0.9210	0.9360	0.9245	0.9460
		II	0.9128	0.9140	1.0053	0.9040	09543	0.9240
		III	0.9348	0.9280	0.9210	0.9160	0.9180	0.9320
80	60	I	1.1250	0.9480	0.9207	0.9240	0.9324	0.9500
		II	1.0899	0.9360	0.9990	0.9300	0.9638	0.9340
		III	1.0233	0.9280	0.9207	0.9320	0.9109	0.9460
100	60	I	0.9967	0.9540	0.9087	0.9460	0.9137	0.9420
		II	1.0018	0.9480	1.0247	0.9300	0.9126	0.9460
		III	0.9421	0.9320	0.9087	0.9320	0.9137	0.9500
100	80	I	0.9453	0.9280	0.9248	0.9560	0.9218	0.9740
		II	0.9638	0.9160	0.9657	0.9460	0.9875	0.9700
		III	0.9453	0.9280	0.9248	0.9340	0.9974	0.9640

Table 4: Mean length and CP of ML, bootstrap and Bayes of the entropy at $\tau = 1$.

			MLE		Bootstrap		Bayes	
			$\beta = 0.8,\ \xi = 0.3$					
n	r	CS	Mean Length	CP	Mean Length	CP	Mean Length	CP
80	50	I	1.4210	0.9520	1.5124	0.9340	1.4563	0.9460
		II	1.4860	0.9460	1.4865	0.9240	1.5429	0.9342
		III	1.5240	0.9640	1.5600	0.9160	1.2014	0.9320
80	60	I	1.4148	0.9540	1.4458	0.9500	1.4562	0.9640
		II	1.4538	0.9240	1.4283	0.9340	1.2433	0.9600
		III	1.5264	0.9320	1.4094	0.9480	1.2363	0.9540
100	60	I	1.4984	0.9640	1.5296	0.9520	1.2213	0.9600
		II	1.6433	0.9540	1.5183	0.9460	1.2130	0.9640
		III	1.5047	0.9320	1.4190	0.9480	1.2207	0.9500
100	80	I	1.4529	0.9140	1.3084	0.9560	1.2001	0.9720
		II	1.4637	0.9240	1.4406	0.9520	1.2540	0.9620
		III	1.4258	0.9460	1.2148	0.9500	1.2019	0.9680
			$\beta = 1.5,\ \xi = 0.5$					
80	50	I	0.9120	0.9460	0.9360	0.9460	0.9113	0.9340
		II	0.9248	0.9360	1.1031	0.9340	0.9124	0.9420
		III	0.9224	0.9140	0.9138	0.9160	0.9011	0.9420
80	60	I	1.1010	0.9480	0.9634	0.9540	0.9541	0.9420
		II	1.1210	0.9460	1.0199	0.9340	0.9124	0.9520
		III	1.1022	0.9320	0.9110	0.9460	0.9007	0.9500
100	60	I	0.9245	0.9560	0.9324	0.9640	0.9421	0.9620
		II	1.0143	0.9460	1.0245	0.9420	0.9360	0.9540
		III	0.9510	0.9460	0.9108	0.9460	0.9016	0.9520
100	80	I	0.9123	0.9340	0.9110	0.9640	0.9124	0.9520
		II	0.9112	0.9260	0.9124	0.9340	0.9630	0.9120
		III	0.9210	0.9117	0.9560	0.9340	0.8991	0.9640

3.2. Real-life data analysis

The preceding theoretical results are shown in this subsection using real-life data. According to Nelson [46], the data show how long it requires a fluid to act as an insulator between electrodes to degrade at a voltage of 34 K. This data is 19 is shown as: $\{0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.5, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91$ $,36.71, 72.89\}$, we confirmed that modeling these data with the Lo distribution is feasible, p-values and the Kolmogorov-Smirnov (K-S) test are employed to provide the empirical survival functions as well as the fitted ones, as shown in Figure 1. For the data, the K-S is 0.1479 and p-value is 0.8002. We took $r = 12$, $\tau = 33.91$, $R = \{1^7, 0^5\}$, based on R, the adaptive Type-II progressive censored sample is $\{0.19, 0.78, 1.31, 3.16, 4.15, 4.67,$ $4.85, 6.5, 8.01, 8.27, 33.91, 36.71\}$, based on this data, The MLEs, the Bayes estimates and the 95 % confidence intervals of model parameters and $H(f)$ obtained in Table (4).

Using the first 2000 values as the burn-in, in the Bayes approach for the MCMC technique, we execute the chan 22000. By depicting the generation outcomes of the entire conditional distribution as the generation from the posterior distribution and the Bayesian approach's convergence under MCMC techniques, Figure 2 and Figure 3 demonstrates the quality of posterior generation.

Table 5: MLEs, bootstrap, Bayes estimates under SE and LINEX loss function and 95% interval estimate of the parameters

	β	ξ	$H(f)$
(.)MLE	1.03095	4.44344	1.86196
ACI	(0.2453, 2.1024)	(2.3014, 6.1032)	(0.78336, 3.5073)
(.)boot	1.02051	4.52180	1.9124
boot CI	(3.2563,19.090)	(2.3171, 14.8848)	(0.82458, 3.4087)
(.) Bayes SE	0.999143	4.60977	2.06395
Bayes credible interval	(0.0124, 2.0159)	(2.1042, 7.0989)	(1.36226, 3.07794)
(.) Bayes LINEX at $c = 0.05$	0.9926	4.3179	2.0594
(.) Bayes LINEX at $c = -0.05$	1.0059	4.9495	2.0686

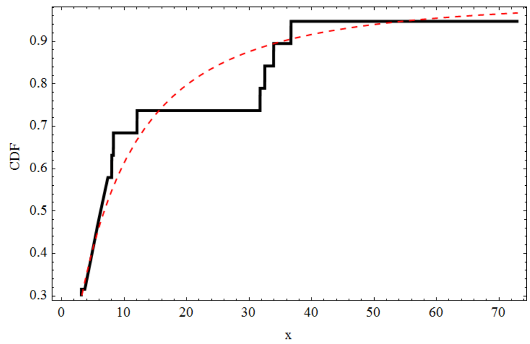

Figure 1: Plots of the fitted CDF function of data and empirical data.

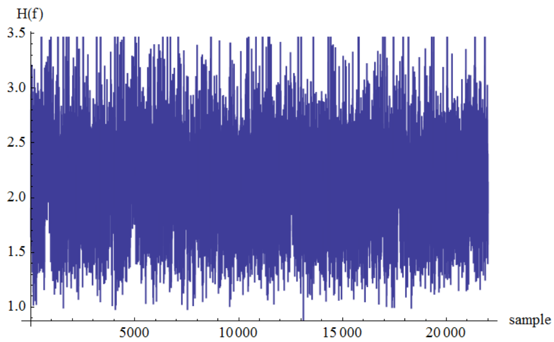

Figure 2: MCMC trace plots of $H(f)$ from data.

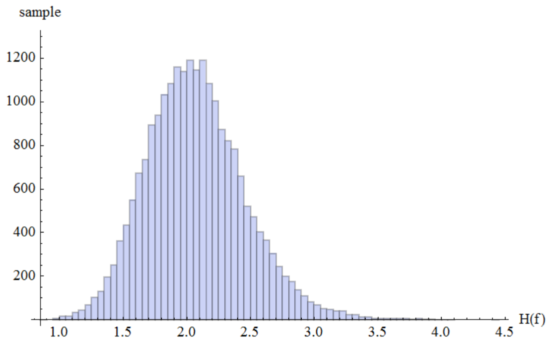

Figure 3: MCMC histogram of $H(f)$ from data.

4. Conclusions

This paper analyzes the entropy of the Lo distribution offers some statistical conclusions and uses adaptive progressively Type-II censoring. The Bayesian approach and classical frequency theory served as the foundation for the development of point and interval estimators. We have looked at the approximate entropy MLE confidence intervals and the Bootstrap entropy confidence intervals using the observed Fisher matrix. Next, we proceed on to the Bayesian approach, which assumes independent Gamma priors and produces the SE and LINEX loss function-based Bayes estimates of entropy. The posterior distributions of some unknown parameters reveal that they deviate from known distributions. To calculate Bayes estimates of entropy with related credible intervals, we use M-H sampling as a component of the Gibbs sampling steps technique. In a simulated study, the effectiveness of each of the previously listed methodologies was then directly compared. The simulation results lead us to the conclusion that when adaptive progressive Type-II is excluded from independent distributions of Lo, the Bayes technique can be used to estimate and generate approximate confidence intervals for unknown parameters. When the Lo distribution was used on real industrial data, it was discovered to be able to properly depict current data to the point where it could be relied upon to analyze comparable genuine data in those domains.

This study opens several significant avenues for future investigation. Areas of particular interest include the design of optimal censoring schemes, the statistical prediction of outcomes under adaptive progressively Type-II censoring, and the extension of inference methods to handle more complex failure models. Applying data mining methodologies to these data may also prove insightful, allowing experts to identify differential patient survival patterns and calculate confidence intervals for survival. These areas warrant further exploration.

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