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# A Four-Step Semi-Implicit Midpoint Approximation Scheme for Fixed Point with Applications

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**Abstract.** This paper aims to put forward and design a four-step semi-implicit approximation scheme to work out the fixed point of a contractive mapping. Convergence analysis and stability of the proposed scheme is incorporated under some mild assumptions. Finally, the significance and applications of the proposed scheme and theoretical findings are proven by exploring a general quasi-variational inequality and a nonlinear fractional differential equation.

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## 1. Introduction

Throughout this paper, we presume that $\Omega \neq \phi$ is a subset of a Banach space $\mathfrak{X}, \mathscr{R}$ signifies the set of real numbers and $\Xi(\Psi) = \{\varrho \in \Omega : \Psi\varrho = \varrho\}$, the set of fixed points of the mapping $\Psi$. A mapping $\Psi : \Omega \to \Omega$ is referred to as contraction if $\exists \kappa \in [0, 1)$ such that $\|\Psi\varrho - \Psi\varsigma\| \leq \kappa\|\varrho - \varsigma\|, \forall \varrho, \varsigma \in \Omega$ and non-expansive for $\kappa = 1$. Non-expansive mappings are crucial generalized notion of contraction mappings and fundamental tools in the theory of fixed points, see, [48]. Clearly, $\Xi(\Psi)$ for a non-expansive self mapping $\Psi$ on a bounded, closed and convex subset $\Omega$ is non-empty, see, [10]. Detailed information on non-expansive mappings and related results can be found in [16, 38].

Non-expansive mappings play vital role in the journey of nonlinear analysis and have been employed to deal various problems of nonlinear analysis such as variational inequality, optimization, equilibrium and initial value problems. In fact, a non-expansive self-mapping on a complete metric space not necessarily owns a fixed point.

**Example 1.** *[37] Consider a closed and bounded subset* $\Omega = \{\varrho = (\varrho_1, \varrho_2, \cdots) : \varrho_k \geq 0, \forall k, \sum_{k=1}^{\infty} \varrho_k = 1\}$ *of a Banach space* $\mathfrak{X}$ *of all real absolutely summable sequences* $(l^1, \|\cdot\|_1)$*. Then the non-expansive mapping* $\Psi : \Omega \to \Omega$ *described by* $\Psi(\varrho) = (0, \varrho_1, \varrho_2, \cdot)$ *does not admit a fixed point.*

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Also, unlike the contraction mappings, the Picard sequence may not converge to a fixed point of a non-expansive mapping. These facts motivated the researchers to explore the mappings which own fixed points over such spaces. A class of weak contractions also known as almost contraction mappings (ACM) was brought into existence by Berinde [8] which is defined below:

**Definition 1.** *A mapping* $\Psi : \Omega \to \Omega$ *is called ACM if for some* $\kappa \geq 0, \exists \tau \in (0,1)$ *so that*

$$\|\Psi(\varrho) - \Psi(\varsigma)\| \leq \tau\|\varrho - \varsigma\| + \kappa\|\varrho - \Psi(\varrho)\|, \forall \varrho, \varsigma \in \Omega. \tag{1}$$

Osilike obtained contractive condition (1) by extending the work of Rhoades [39] and the author proved numerous stability results for (1). If $\kappa = 2\delta, \tau = \delta$, then ACM coincides with Zamfirescu contraction [7, 19], where

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}, \alpha \in [0,1), \beta, \gamma \in [0, 0.5].$$

Further, Imoru and Olantiwo [22] generalized the mapping defined in (1) by involving monotonic increasing function and defined as under:

**Definition 2.** *A mapping* $\Psi : \Omega \to \Omega$ *is referred to as contractive-like if there exists a strictly increasing continuous function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *and* $\tau \in [0,1)$ *so that*

$$\|\Psi(\varrho) - \Psi(\varsigma)\| \leq g(\|\varrho - \Psi(\varrho)\|) + \tau\|\varrho - \varsigma\|, \forall \varrho, \varsigma \in \Omega. \tag{2}$$

The contractive condition in (2) is much broader which include several contractive conditions, see, [7, 19, 35, 39, 40]. If $gu = \kappa u$, where $\kappa \geq 0$ then (2) coincides with (1). Further, for $\kappa = m\tau, m = (1-\tau)^{-1}, 0 \leq \tau < 1$, we acquire the contractive condition due to Rhoades [40]. Further, if $\kappa u = 0$, then (2) becomes

$$\|\Psi(\varrho) - \Psi(\varsigma)\| \leq \tau\|\varrho - \varsigma\|, \tau \in [0,1), \forall \varrho, \varsigma \in \Omega, \tag{3}$$

which is considered by Berinde [7], and Harder and Hicks [19].

In past few years, a tremendous interest has been shown to the fixed point theory which has become most versatile and applicable area of research. Several problems which we encounter in real-world including zeros of monotone operators, ODEs, PDEs, integral equations, VIs, etc., can be reformulated as a fixed point problem. Owing to the significance of fixed point theory, numerous approaches have been carried out to deal with fixed point problems. Among these approaches, iterative approximation is one of the most handy and applicable tools for exploring nonlinear problems. In recent time, several new iterative schemes have been designed and employed. One of the most common schemes for investigating fixed points is named as Mann iterative scheme [28]:

$$\begin{cases} \varrho_0 \in \Omega, \\ \varrho_{k+1} = (1-\alpha_k)\varrho_k + \alpha_k \Psi(\varrho_k), k \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_k\} \in [0, 1]$ and $\Psi : \Omega \to \Omega$ is a non-expansive mapping. In 1974, Ishikawa [23] approximated the fixed points by designing the scheme as under:

$$\begin{cases} \varrho_0 \in \Omega, \\ \sigma_k = (1-\beta_k)\varrho_k + \beta_k \Psi(\varrho_k), \\ \varrho_{k+1} = (1-\alpha_k)\varrho_k + \alpha_k \Psi(\sigma_k), k \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{\alpha_k\}, \{\beta_k\} \in [0, 1]$. Further, Noor[33] posed a three-step scheme which comprises Mann [28] and Ishikawa [23] schemes and expressed as under:

$$\begin{cases} \varrho_0 \in \Omega, \\ \rho_k = (1-\gamma_k)\varrho_k + \gamma_k \Psi(\varrho_k), \\ \sigma_k = (1-\beta_k)\varrho_k + \beta_k \Psi(\rho_k), \\ \varrho_{k+1} = (1-\alpha_k)\varrho_k + \alpha_k \Psi(\sigma_k), k \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \in [0, 1]$. Among the numerous iterative methods posed so far, a few common and intensively used schemes include $S$-iteration [41], $M$-iteration [49], Normal-S [43], Picard-Ishikawa scheme [34], etc.. Recently, Okeke et al. [15] contrived an efficient four step iterative scheme:

$$\begin{cases} \varrho_0 \in \Omega, \\ \varrho_{k+1} = \Psi(\varsigma_k), \\ \varsigma_k = \Psi[(1-\alpha_k)\vartheta_k + \alpha_k \Psi(\vartheta_k)], \\ \vartheta_k = (1-\beta_k)\Psi(\varrho_k) + \beta_k \Psi(\varepsilon_k), \\ \varepsilon_k = (1-\gamma_k)\varrho_k + \gamma_k \Psi(\varrho_k), k \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \subset [0, 1]$. The authors approximated the fixed point of a contraction mapping in a uniformly convex Banach space and proved the stability of the proposed scheme. Additionally, the weak convergence for Suzuki's generalized non-expansive mapping was analyzed. The efficiency of the scheme was demonstrated by illustrative example and comparing some known schemes.

A mapping $\Psi$ in a Banach space $\mathscr{X}$ with domain $\mathcal{D}(\Psi)$ and range $\mathcal{R}(\Psi)$ is referred to as accretive, if

$$\langle \Psi\varrho - \Psi\varsigma, J(\varrho - \varsigma)\rangle \geq 0, \forall \varrho, \varsigma \in \mathcal{D}(\Psi),$$

where $J : \mathscr{X} \to \mathscr{X}^*$ is the duality mapping and $\Psi$ is referred to as monotone, if

$$\langle \Psi\varrho - \Psi\varsigma, \varrho - \varsigma\rangle \geq 0, \forall \varrho, \varsigma \in \mathcal{D}(\Psi).$$

If $\mathscr{X} = \mathscr{H}$, a Hilbert space, then both the concepts are identical in the sense of Minty [30] and Browder [10]. On the contrary, number of real-life problems appearing in science and engineering can be studied by formulating as a model of the following initial-value problem (IVP):

$$\frac{d\varrho}{dt} = \Psi(\varrho); \varrho(0) = \varrho_0. \tag{8}$$

Since the accretive and monotone mappings are connected to the evolution model (8) and this relation makes these mappings quite fruitful and applicable. A fundamental result documented by [10] affirms that (8) admits a solution when $\Psi$ is locally Lipshitzian and accretive on $\mathscr{X}$. Additionally, estimating a zero of nonlinear mapping $\Psi$, i.e., $0 \in \Psi\varrho$ is a significant and powerful tool in approximation theory because solutions of elliptic differential equations, optimization problems, inclusion problems, fixed point problems can be obtained as a model of inclusion problem $0 \in \Psi\varrho$ which is identical to the equilibrium state: $\frac{d\varrho}{dt} = 0, \Psi(\varrho) = 0$, see, [10, 11].

To obtain a numerical solution is challenging task when the involved mapping $\Psi$ is not continuous. Several researchers obtained numerical solutions of (8) by approximation approaches, see, Mustafa [47], Duffull and Hegarty [13], Khorasani and Adibi [25]. One of the fundamental and impressive techniques is implicit midpoint rule (IMR):

$$\frac{1}{\mu}(\varrho_{k+1} - \varrho_k) = \Psi\Big(\frac{\varrho_{k+1} + \varrho_k}{2}\Big), \tag{9}$$

where $\mu > 0$ is a step-size. The sequence $\{\varrho_k\}$ induced by (9) converges to the exact solution of (8) under modest assumptions, see, [3, 5]. If $\Psi$ is expressed as $\Psi(\varrho) := \Gamma(\varrho) - \varrho$, then the IVP (8) transformed into

$$\varrho^{'} = \varrho - \Gamma(\varrho), \ \ \varrho(0) = \varrho_0 \tag{10}$$

and the IMR (9) becomes:

$$\frac{1}{\mu}(\varrho_{k+1} - \varrho_k) = \Big[\frac{\varrho_{k+1} + \varrho_k}{2} - \Gamma\Big(\frac{\varrho_{k+1} + \varrho_k}{2}\Big)\Big], \tag{11}$$

In [26], the authors deployed the fact that equilibrium associated to (10) is identical to the fixed point $\varrho = \Gamma(\varrho)$, which compelled the authors to design the following fixed point implicit iterative scheme:

$$\varrho_{k+1} = (1 - \alpha_k)\varrho_k + \alpha_k \Gamma\Big(\frac{\varrho_{k+1} + \varrho_k}{2}\Big), \tag{12}$$

where $\{\alpha_k\} \subset (0, 1)$ and $\Gamma : \mathscr{H} \to \mathscr{H}$ is nonexpansive. The authors carried out weak convergence results by taking some modest assumptions into consideration. Same fact motivated, Xu et al. [20] to design the following implicit midpoint method using viscosity technique for non-expansive mapping:

$$\varrho_{k+1} = \alpha_k \psi(\varrho_k) + (1 - \alpha_k)\Gamma\Big(\frac{\varrho_{k+1} + \varrho_k}{2}\Big), \tag{13}$$

where, $\Gamma$ is non-expansive and $\psi$ is contraction mapping. More precisely, following result was proved.

**Theorem 1.** *Let $\Omega \neq \emptyset$ be a closed convex set in a Hilbert space $\mathscr{H}$. Suppose that $\Gamma : \Omega \to \Omega$ is a non-expansive and $\psi : \Omega \to \Omega$ is a contraction mapping. If $\{\alpha_k\}$ complies with the following preassumptions:*

$(P_1)$ $\lim_{k\to\infty} \alpha_k = 0$; $(P_2)$ $\sum_{k=0}^{\infty} \alpha_k = \infty$; $(P_3)$ $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$.
*Then* $\{\varrho_k\}_{k=1}^{\infty}$ *produced by (13) converges to* $\varrho \in Fix(\Gamma)$ *and* $\varrho$ *solves the following variational inequality:*

$$\langle (I-\tau)\psi, \varrho - \tau\rangle \geq 0, \forall \varrho \in Fix(\Gamma).$$

Luo et al. [36] obtained the results of Xu et al. [20] in uniformly smooth Banach space. For more details on implicit schemes, we refer, [12, 21, 31, 51].

Motivated and encouraged by the earlier revealed results and iterative process (7), we propose and design a four-step semi-implicit approximation scheme (14) to work out the fixed point of a contractive mapping. The accomplishment of the task is performed as mentioned herein: Second section begins with the designing of a semi-implicit mid point scheme followed by some basic results. Convergence of the planned is analyzed to explore a fixed point of a contractive mapping and the uniqueness of the solution is established. Further, the stability of the designed scheme is discussed. In the third section, we discuss the significance and applicability of our designed scheme. A general quasi-variational inequality and a fractional differential equation are investigated by employing our designed scheme. The concluding comments and expected future research plans are outlined in the last section.

## 2. Iterative Scheme and Convergence

Let $\emptyset \neq \Omega$ be a closed convex subset of a Banach space $\mathfrak{X}$ equipped with norm $\|\cdot\|$. Suppose the mapping $\Psi : \Omega \to \Omega$ satisfies contractive condition (2). Based on the iterative scheme (7), we are interested to suggest and analyze the following semi-implicit midpoint scheme ($SIMPS$) as under:

$$\begin{cases} \varrho_{k+1} = \Psi(\sigma_k), \\ \sigma_k = \Psi\Big[(1-\alpha_k)\Big(\dfrac{\sigma_k + \vartheta_k}{2}\Big) + \alpha_k \Psi\Big(\dfrac{\sigma_k+\vartheta_k}{2}\Big)\Big], \\ \vartheta_k = (1-\beta_k)\Psi\Big(\dfrac{\vartheta_k+\varrho_k}{2}\Big) + \beta_k\Psi\Big(\dfrac{\vartheta_k+\theta_k}{2}\Big), \\ \theta_k = (1-\gamma_k)\Big(\dfrac{\varrho_k+\theta_k}{2}\Big) + \gamma_k\Psi\Big(\dfrac{\varrho_k+\theta_k}{2}\Big)\Big], \end{cases} \tag{14}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \subseteq (0,1)$.

**Definition 3.** *[9] Let* $\{\varphi_k\} \subset \Omega$ *be an arbitrary sequence. An iterative scheme* $\varrho_{k+1} = \Lambda(\Psi, \varrho_k)$ *so as* $\{\varrho_k\} \to \varrho \in \Xi(\Psi)$ *is said to be* $\Psi$*-stable. If for* $\mu_k = \|\varphi_{k+1} - \Lambda(\Psi, \varphi_k)\|$, $\lim_{k\to\infty} \mu_k = 0$ *if and only if* $\lim_{k\to\infty} \varphi_k = \varrho$.

**Lemma 1.** *[50] Suppose the nonnegative real sequences* $\{\varrho_k\}_{k=1}^{\infty}$ *and* $\{\varsigma_k\}_{k=1}^{\infty}$ *satisfy*

$$\varrho_{k+1} \leq (1-p_k)\varrho_k + \varsigma_k,$$

*where* $p_k \in (0,1)$, $\sum_{k=1}^{\infty} p_k = \infty$ *and* $\lim_{k\to\infty} \dfrac{\varsigma_k}{p_k} = 0$. *Then* $\lim_{k\to\infty} \varrho_k = 0$.

**Theorem 2.** *Let* $\emptyset \neq \Omega \subseteq \mathscr{X}$ *be a closed convex bounded set and* $\Psi : \Omega \to \Omega$ *satisfies (2). If* $\Xi(\Psi) \neq \emptyset$*, then* $\{\varrho_k\}_{k=1}^{\infty}$ *initiated by SIMPS (14) converges strongly to* $\varrho \in \Xi(\Psi)$*.*

*Proof.* Suppose that $\varrho \in \Xi(\Psi)$. Then, it results from the last formulation of (14) that

$$\begin{aligned}
\|\theta_k - \varrho\| &= \Big\|(1-\gamma_k)\Big(\frac{\varrho_k+\theta_k}{2}\Big) + \gamma_k \Psi\Big(\frac{\varrho_k+\theta_k}{2}\Big)\Big] - \varrho\Big\| \\
&\leq (1-\gamma_k)\Big\|\frac{\varrho_k+\theta_k}{2} - \varrho\Big\| + \gamma_k\Big\|\Psi\Big(\frac{\varrho_k+\theta_k}{2}\Big) - \varrho\Big\| \\
&= (1-\gamma_k)\Big\|\frac{\varrho_k+\theta_k}{2} - \varrho\Big\| + \gamma_k\Big\|\Psi(\varrho) - \Psi\Big(\frac{\varrho_k+\theta_k}{2}\Big)\Big\| \\
&\leq (1-\gamma_k)\Big\|\frac{\varrho_k+\theta_k}{2} - \varrho\Big\| + \gamma_k\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\varrho_k+\theta_k}{2}\Big) - \varrho\Big\|\Big] \\
&\leq \frac{\varepsilon_k}{2}(\|\varrho_k - \varrho\| + \|\theta_k - \varrho\|),
\end{aligned}$$

where $\varepsilon_k = (1 - \gamma_k + \tau\gamma_k)$, which turns after simplification into

$$\|\theta_k - \varrho\| \leq \frac{\varepsilon_k}{(2-\varepsilon_k)}\|\varrho_k - \varrho\|. \tag{15}$$

Again it yields from scheme (14) that

$$\begin{aligned}
\|\vartheta_k - \varrho\| &= \Big\|(1-\beta_k)\Psi\Big(\frac{\vartheta_k+\varrho_k}{2}\Big) + \beta_k\Psi\Big(\frac{\vartheta_k+\theta_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\beta_k)\Big\|\Psi\Big(\frac{\vartheta_k+\varrho_k}{2}\Big) - \varrho\Big\| + \beta_k\Big\|\Psi\Big(\frac{\vartheta_k+\theta_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\beta_k)\Big\|\Psi(\varrho) - \Psi\Big(\frac{\vartheta_k+\varrho_k}{2}\Big)\Big\| + \beta_k\Big\|\Psi(\varrho) - \Psi\Big(\frac{\vartheta_k+\theta_k}{2}\Big)\Big\| \\
&\leq (1-\beta_k)\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\vartheta_k+\varrho_k}{2}\Big) - \varrho\Big\|\Big] \\
&\quad + \beta_k\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\vartheta_k+\theta_k}{2}\Big) - \varrho\Big\|\Big] \\
&\leq (1-\beta_k)\tau\Big\|\Big(\frac{\vartheta_k+\varrho_k}{2}\Big) - \varrho\Big\| + \beta_k\tau\Big\|\Big(\frac{\vartheta_k+\theta_k}{2}\Big) - \varrho\Big\| \\
&\leq \frac{\tau}{2}\|\vartheta_k - \varrho\| + \frac{\tau}{2}[(1-\beta_k)\|\varrho_k - \varrho\| + \beta_k\|\theta_k - \varrho\|],
\end{aligned}$$

which yields into

$$\|\vartheta_k - \varrho\| \leq \frac{\tau}{(2-\tau)}[(1-\beta_k)\|\varrho_k - \varrho\| + \beta_k\|\theta_k - \varrho\|]. \tag{16}$$

Combining (15) and (16), one gets

$$\|\vartheta_k - \varrho\| \leq \frac{\tau}{(2-\tau)}\Big[1 - \beta_k\Big(1 - \frac{\varepsilon_k}{(2-\varepsilon_k)}\Big)\Big]\|\varrho_k - \varrho\|. \tag{17}$$

Further, the second formulation of $SIMPS$ (14) yields

$$\begin{aligned}\|\sigma_k - \varrho\| &= \Big\|\Psi\Big[(1-\alpha_k)\Big(\frac{\sigma_k+\vartheta_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\sigma_k+\vartheta_k}{2}\Big)\Big] - \varrho\Big\| \\ &= \Big\|\Psi(\varrho) - \Psi\Big[(1-\alpha_k)\Big(\frac{\sigma_k+\vartheta_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\sigma_k+\vartheta_k}{2}\Big)\Big]\Big\| \\ &\leq g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big[(1-\alpha_k)\Big(\frac{\sigma_k+\vartheta_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\sigma_k+\vartheta_k}{2}\Big)\Big] - \varrho\Big\| \\ &\leq \tau(1-\alpha_k)\Big\|\Big(\frac{\sigma_k+\vartheta_k}{2}\Big) - \varrho\Big\| + \tau\alpha_k\Big\|\Psi(\varrho) - \Psi\Big(\frac{\sigma_k+\vartheta_k}{2}\Big)\Big\| \\ &\leq \tau[1-\alpha_k(1-\tau)]\Big\|\Big(\frac{\sigma_k+\vartheta_k}{2}\Big) - \varrho\Big\| \\ &= \frac{\pi_k}{2}[\|\sigma_k - \varrho\| + \|\vartheta_k - \varrho\|].\end{aligned}$$

Thus, we acquire

$$\|\sigma_k - \varrho\| \leq \frac{\pi_k}{(2-\pi_k)}\|\vartheta_k - \varrho\|, \tag{18}$$

where $\pi_k = \tau[1-\alpha_k(1-\tau)]$. The straightforward calculation after taking the assumptions $\{\alpha_k\}_{k=1}^{\infty} \in (0,1)$, $\tau \in [0,1)$ and (17) into play leads $\varepsilon_k \in [0,1)$. Thus, we acquire $1 - \beta_k\Big(1 - \frac{\varepsilon_k}{2-\varepsilon_k}\Big) \leq 1$ and hence,

$$\|\sigma_k - \varrho\| \leq \frac{\pi_k}{(2-\pi_k)}\frac{\tau}{(2-\tau)}\|\varrho_k - \varrho\|. \tag{19}$$

Finally, the first formulation of $SIMPS$ (14) along with (19) turns into

$$\begin{aligned}\|\varrho_{k+1} - \varrho\| &= \|\Psi(\sigma_k) - \varrho\| \\ &= \|\Psi(\varrho) - \Psi(\sigma_k)\| \\ &\leq g(\|\varrho - \Psi(\varrho)\|) + \tau\|\sigma_k - \varrho\| \\ &\leq (1-\hat{\ell}_k)\|\varrho_k - \varrho\|,\end{aligned} \tag{20}$$

where,

$$\hat{\ell_k} = \frac{(2-\tau)(2-\pi_k) - \tau^2\pi_k}{(2-\tau)(2-\pi_k)}. \tag{21}$$

Since $\pi_k = \tau(1-\alpha_k+\tau\alpha_k), \tau \in [0,1)$ and $\{\alpha_k\}_{k=1}^{\infty} \subseteq (0,1)$ yields $\pi_k \leq \tau$. Thus, we obtain

$$\begin{aligned}\hat{\ell}_k &\geq \frac{1}{4}[(2-\tau)(2-\pi_k) - \tau^2\pi_k] \\ &\geq \frac{1}{4}[1+(1-\tau)][1+(1-\tau)] - \tau^3 \\ &> 0.\end{aligned}$$

Further, $1 - \hat{\ell}_k = \frac{\tau^2\pi_k}{(2-\tau)(2-\pi_k)} \geq 0$ and $\sum_{k=0}^{\infty}\hat{\ell}_k = \infty$. Utilizing Lemma 1, it follows from (20) that $\lim_{k\to\infty}\|\varrho_k - \varrho\| = 0$. Next, we manifest the uniqueness of $\varrho$, suppose that

$\varrho_1, \varrho_2 \in \Omega$ so that $\varrho_1 \neq \varrho_2$ and $\varrho_1, \varrho_2 \in \Xi(\Psi)$. Then

$$\begin{aligned} \|\varrho_1 - \varrho_2\| &= \|\Psi(\varrho_1) - \Psi(\varrho_2)\| \\ &\leq g(\|\varrho_1 - \Psi(\varrho_1)\|) + \tau\|\varrho_1 - \varrho_2\| \\ &= \tau\|\varrho_1 - \varrho_2\|. \end{aligned} \tag{22}$$

Since $\tau \in [0, 1)$, then (22) gives $\|\varrho_1 - \varrho_2\| = 0$ and consequently $\varrho_1 = \varrho_2$.

**Theorem 3.** *Suppose that $\emptyset \neq \Omega \subseteq \mathscr{X}$ is a closed convex bounded set and the mapping $\Psi : \Omega \to \Omega$ satisfies (2). If $\varrho \in \Xi(\Psi)$, then $\{\varrho_k\}_{k=1}^{\infty}$ initiated by SIMPS (14) is $\Psi$-stable.*

*Proof.* Let $\{\varphi_k\} \subset \Omega$ be an arbitrary sequence and $\{\varrho_k\}_{k=1}^{\infty}$ initiated by $SIMPS$ (14) is $\varrho_{k+1} = \Lambda(\Psi, \varrho_k)$ such as $\{\varrho_k\} \to \varrho \in \Xi(\Psi)$. Suppose that $\mu_k = \|\varphi_{k+1} - \Lambda(\Psi, \varphi_k)\|$, where $\{\varphi_k\}$ is initiated as under:

$$\begin{cases} \varphi_{k+1} = \Psi(\zeta_k), \\ \zeta_k = \Psi\Big[(1-\alpha_k)\Big(\dfrac{\zeta_k + \xi_k}{2}\Big) + \alpha_k\Psi\Big(\dfrac{\zeta_k + \xi_k}{2}\Big)\Big], \\ \xi_k = (1-\beta_k)\Psi\Big(\dfrac{\xi_k + \varphi_k}{2}\Big) + \beta_k\Psi\Big(\dfrac{\xi_k + \omega_k}{2}\Big), \\ \omega_k = (1-\gamma_k)\Big(\dfrac{\varphi_k + \omega_k}{2}\Big) + \gamma_k\Psi\Big(\dfrac{\varphi_k + \omega_k}{2}\Big)\Big]. \end{cases} \tag{23}$$

To establish the $\Psi$-stability of the scheme (14), we corroborate $\lim\limits_{k\to\infty} \mu_k = 0$ if and only if $\lim\limits_{k\to\infty} \varphi_k = \varrho$. Assume that $\lim\limits_{k\to\infty} \mu_k = 0$. By utilizing the triangle inequality, we acquire

$$\begin{aligned} \|\varphi_{k+1} - \varrho\| &= \|\varphi_{k+1} - \Lambda(\Psi, \varphi_k) + \Lambda(\Psi, \varphi_k) - \varrho\| \\ &\leq \|\varphi_{k+1} - \Lambda(\Psi, \varphi_k)\| + \|\Lambda(\Psi, \varphi_k) - \varrho\| \\ &\leq \mu_k + \|\varphi_{k+1} - \varrho\| \\ &= \mu_k + \|\Psi(\zeta_k) - \varrho\| \\ &\leq \mu_k + g(\|\varrho - \Psi(\varrho)\|) + \tau\|\zeta_k - \varrho\| \\ &= \mu_k + \tau\|\zeta_k - \varrho\|. \end{aligned} \tag{24}$$

Again, from second equation of (23), we estimate

$$\begin{aligned}
\|\zeta_k - \varrho\| &= \Big\|\Psi\Big[(1-\alpha_k)\Big(\frac{\zeta_k+\xi_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\zeta_k+\xi_k}{2}\Big)\Big] - \varrho\Big\| \\
&= \Big\|\Phi(\varrho) - \Psi\Big[(1-\alpha_k)\Big(\frac{\zeta_k+\xi_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\zeta_k+\xi_k}{2}\Big)\Big]\Big\| \\
&\leq g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|(1-\alpha_k)\Big(\frac{\zeta_k+\xi_k}{2}\Big) + \alpha_k\Psi\Big(\frac{\zeta_k+\xi_k}{2}\Big) - \varrho\Big\| \\
&\leq \tau(1-\alpha_k)\Big\|\Big(\frac{\zeta_k+\xi_k}{2}\Big) - \varrho\Big\| + \tau\alpha_k\Big\|\Psi\Big(\frac{\zeta_k+\xi_k}{2}\Big) - \varrho\Big\| \\
&\leq \tau(1-\alpha_k)\Big\|\Big(\frac{\zeta_k+\xi_k}{2}\Big) - \varrho\Big\| + \tau\alpha_k\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\zeta_k+\xi_k}{2}\Big) - \varrho\Big\|\Big] \\
&\leq \frac{\tau}{2}[1-\alpha_k(1-\tau)][\|\zeta_k - \varrho\| + \|\xi_k - \varrho\|] \\
&= \frac{\pi_k}{2}[\|\zeta_k - \varrho\| + \|\xi_k - \varrho\|]
\end{aligned}$$

which turns into

$$\|\zeta_k - \varrho\| \leq \frac{\pi_k}{(2-\pi_k)}\|\xi_k - \varrho\|. \tag{25}$$

$$\begin{aligned}
\|\xi_k - \varrho\| &= \Big\|(1-\beta_k)\Psi\Big(\frac{\xi_k+\varphi_k}{2}\Big) + \beta_k\Psi\Big(\frac{\xi_k+\omega_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\beta_k)\Big\|\Psi\Big(\frac{\xi_k+\varphi_k}{2}\Big) - \varrho\Big\| + \beta_k\Big\|\Psi\Big(\frac{\xi_k+\omega_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\beta_k)\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\xi_k+\varphi_k}{2}\Big) - \varrho\Big\|\Big] \\
&\quad + \beta_k\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\xi_k+\omega_k}{2}\Big) - \varrho\Big\|\Big] \\
&\leq \frac{\tau}{2}\|\xi_k - \rho\| + \frac{\tau}{2}[(1-\beta_k)\|\varphi_k - \varrho\| + \beta_k\|\omega_k - \varrho\|]
\end{aligned}$$

which turns into

$$\|\xi_k - \varrho\| \leq \frac{\tau}{(2-\tau)}[(1-\beta_k)\|\varphi_k - \varrho\| + \beta_k\|\omega_k - \varrho\|], \tag{26}$$

and

$$\begin{aligned}
\|\omega_k - \varrho\| &= \Big\|(1-\gamma_k)\Big(\frac{\varphi_k+\omega_k}{2}\Big) + \gamma_k\Psi\Big(\frac{\varphi_k+\omega_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\gamma_k)\Big\|\Big(\frac{\varphi_k+\omega_k}{2}\Big) - \varrho\Big\| + \gamma_k\Big\|\Psi\Big(\frac{\varphi_k+\omega_k}{2}\Big) - \varrho\Big\| \\
&\leq (1-\gamma_k)\Big\|\Big(\frac{\varphi_k+\omega_k}{2}\Big) - \varrho\Big\| + \gamma_k\Big[g(\|\varrho - \Psi(\varrho)\|) + \tau\Big\|\Big(\frac{\varphi_k+\omega_k}{2}\Big) - \varrho\Big\|\Big] \\
&\leq \frac{\varepsilon_k}{2}[\|\varphi_k - \varrho\| + \|\omega_k - \varrho\|]
\end{aligned}$$

which turns into

$$\|\omega_k - \varrho\| \leq \frac{\varepsilon_k}{(2-\varepsilon_k)}\|\varphi_k - \varrho\|, \tag{27}$$

where $\varepsilon_k = (1-\gamma_k+\tau\gamma_k)$. By implementing back substitution from (25)-(27), (24) becomes

$$\|\varphi_{k+1} - \varrho\| \le \mu_k + (1 - \hat{\ell}_k)\|\varphi_k - \varrho\|, \tag{28}$$

where $\hat{\ell}_k$ is identical as given in (21). By availing the assumption $\lim_{k\to\infty} \mu_k = 0$, Lemma 1 yeilds $\|\varphi_k - \varrho\| \to 0$ as $k \to \infty$, i.e., $\lim_{k\to\infty} \varphi_k = \varrho$. Conversely, assume that $\lim_{k\to\infty} \varphi_k = \varrho$, and following the same procedure, we achieve

$$\begin{aligned}\mu_k &= \|\varphi_{k+1} - \Lambda(\Psi, \varphi_k)\| \\ &= \|\varphi_{k+1} - \varrho + \varrho - \Lambda(\Psi, \varphi_k)\| \\ &\le \|\varphi_{k+1} - \varrho\| + \|\Lambda(\Psi, \varphi_k) - \varrho\| \\ &\le \|\varphi_{k+1} - \varrho\| + (1 - \hat{\ell}_k)\|\varphi_k - \varrho\|.\end{aligned}$$

Appealing to the assumption $\lim_{k\to\infty} \varphi_k = \varrho$, it follows that $\lim_{k\to\infty} \mu_k = 0$. Hence, $SIMPS$ (14) is $\Psi$-stable.

## 3. Applications

In this section, we shall explore and examine a general quasi-variational inequality and a nonlinear fractional differential equation by employing our outlined semi-implicit midpoint scheme.

### 3.1. General quasi-variational inequality

Let $\mathscr{H}$ be a Hilbert space over $\mathscr{R}$ and $\mathcal{C}(\mathscr{H})$, the collection of non empty closed convex subsets of $\mathscr{H}$. We contemplate the problem to observe an element $\varrho \in \mathscr{H} : \psi(\varrho) \in \mathcal{C}(\varrho)$ so that

$$\langle \Psi(\varrho), \psi(\varsigma) - \psi(\varrho)\rangle \ge 0, \forall \varsigma \in \mathscr{H}, \psi(\varsigma) \in \mathcal{C}(\varrho), \tag{29}$$

where $\Psi, \psi : \mathscr{H} \to \mathscr{H}$ be (not necessarily) linear mappings, and the set-valued mapping $\mathcal{C} : \mathscr{H} \to 2^{\mathscr{H}}$ assigns each element $\varrho \in \mathscr{H}$, a closed convex subset $\mathcal{C}(\varrho)$ of $\mathscr{H}$. The inequality (29) is called the generalized quasi variational inequality ($GQVI$) and we signify its solution set by ✠$(\mathcal{C}(\varrho), \Psi, \psi)$. In fact, a quasi-variational inequality ($QVI$) is a kind of modified variational inequality in which the constraint set varies with the variable. Numerous economic and engineering problems, such as Nash equilibrium problems, control and optimization, operations research, etc., are recognized to be well suited for modeling and analysis using $QVIs$. $GQVI$ (29) can be seen as a unified problem and consists several considerably significant problems as special cases which are listed as under.

(i) For $\mathcal{C}(\varrho) = \mathcal{C}$, $GQVI$ (29) is identical to the following general variational inequality which was set forth by Noor [32].

$$\langle \Psi(\varrho), \psi(\varsigma) - \psi(\varrho)\rangle \ge 0, \forall \varsigma \in \mathscr{H}, \psi(\varsigma) \in \mathcal{C}. \tag{30}$$

(ii) Further for $\psi = I$, Problem (30) becomes the classical variational inequality introduced by Stampacchia[45].

(iii) If $\psi = I$, GQVI (29) turns into the following classical quasi-variational inequality introduced in [6]:

$$\langle \Psi(\varrho), \varsigma - \varrho \rangle \geq 0, \forall \varsigma \in \mathcal{C}(\varrho). \tag{31}$$

(iv) For $\varrho_0 \in \mathscr{H}$, the dual cone of $\mathcal{C}(\varrho_0) \subset \mathscr{H}$ is described by

$$\bar{\mathcal{C}}(\varrho_0) = \{\varrho \in \mathscr{H} : \langle \varrho, \varsigma \rangle \geq 0, \forall \varsigma \in \mathcal{C}(\varrho_0)\}.$$

Then problem (30) turns into a general complementarity problem of discerning an element $\varrho \in \mathscr{H}$ so that

$$\langle \Psi(\varrho), \psi(\varrho) \rangle \geq 0, \psi(\varrho) \in \mathcal{C}(\varrho) \text{ and } \Psi(\varrho) \in \bar{\mathcal{C}}(\varrho). \tag{32}$$

Now, to bring off the required goal, we accumulate a few supplementary results and definitions below.

**Lemma 2.** *[4] If for any $\varsigma \in \mathscr{H}, \varrho \in \mathcal{C}(\varrho)$, the implicit projection $\mathcal{P}_{\mathcal{C}(\varrho)} : \mathscr{H} \to \mathcal{C}(\varrho) \subset \mathscr{H}$ obeys the inequality $\langle \kappa - \omega, \varpi - \kappa \rangle \geq 0$ if and only if $\mathcal{P}_{\mathcal{C}(\varrho)}(\omega) = \kappa, \forall \varpi \in \mathcal{C}(\varrho)$.*

Next, we shall design the following fixed point problem associated to $GQVI$ (29) by imposing the Lemma 2.

**Lemma 3.** *An element $\varrho \in \mathscr{H} : \psi(\varrho) \in \mathcal{C}(\varrho)$ solves GQVI (29) if and only if $\varrho \in \Xi(\Pi)$, where $\Pi(\varrho) = \varrho - \psi(\varrho) + \mathcal{P}_{\mathcal{C}(\varrho)}[\psi(\varrho) - \lambda\Psi(\varrho)]$ and $\lambda > 0$ is a constant.*

*Proof.* Assume that $\varrho \in ✠(\mathcal{C}(\varrho), \Psi, \psi)$ then $\langle \Psi(\varrho), \psi(\varsigma) - \psi(\varrho) \rangle \geq 0, \forall \varsigma \in \mathscr{H} : \psi(\varsigma) \in \mathcal{C}(\varrho)$. By making use of Lemma 2, we acquire $\Pi_{\mathcal{C}(\varrho)}[\psi(\varrho) - \lambda\Psi(\varrho)] = \psi(\varrho)$. So, $\varrho \in \Xi(\Pi)$. On the other side, assume that $\varrho \in \Xi(\Pi)$ then for all $\varrho \in \mathscr{H} : \psi(\varrho) \in \mathcal{C}(\varrho)$, we obtain $\Pi(\varrho) = \varrho$, thus, one can write $\psi(\varrho) = \Pi_{\mathcal{C}(\varrho)}[\psi(\varrho) - \lambda\Psi(\varrho)]$. Again, by the virtue of Lemma 2, we get $\langle \Psi(\varrho), \psi(\varsigma) - \psi(\varrho) \rangle \geq 0, \forall \varsigma \in \mathscr{H} : \psi(\varsigma) \in \mathcal{C}(\varrho)$, i.e., $\varrho \in ✠(\mathcal{C}(\varrho), \Psi, \psi)$.

Now, we take the following assumption into account to accomplish the required goal.
**Assumption A:** For given elements $\varrho, \tau, \varsigma \in \mathscr{H}$ and $\kappa > 0$, $\mathcal{P}_{\mathcal{C}}$ obeys the following inequality

$$\|\mathcal{P}_{\mathcal{C}(\tau)}(\varsigma) - \mathcal{P}_{\mathcal{C}(\varrho)}(\varsigma)\| \leq \kappa\|\tau - \varrho\|.$$

**Definition 4.** *A mapping $\Psi : \mathscr{H} \to \mathscr{H}$ is called*

- *$\rho_1$-strongly monotone if $\exists \rho_1 > 0$ so that*

$$\langle \Psi(\varrho) - \Psi(\varsigma), \varrho - \varsigma \rangle \geq \rho_1\|\varrho - \varsigma\|^2, \forall \varrho, \varsigma \in \mathscr{H};$$

- *relaxed $(u, v)$-cocoercive, if $\exists u, v > 0$ so that*

$$\langle \Psi(\varrho) - \Psi(\varsigma), \varrho - \varsigma \rangle \geq (-u)\|\Psi(\varrho) - \Psi(\varsigma)\|^2 + v\|\varrho - \varsigma\|^2, \forall \varrho, \varsigma \in \mathscr{H};$$

- *$\rho_2$-Lipschitz continuous, if $\exists \rho_2 > 0$ so that*

$$\|\Psi(\varrho) - \Psi(\varsigma)\| \leq \rho_2\|\varrho - \varsigma\|, \forall \varrho, \varsigma \in \mathscr{H}.$$

Now, as an application of $SIMPS$ (14), we shall re-structure the following semi-implicit midpoint scheme to find the common solution of $GQVI$ (29) and fixed point of the mapping defined in (2).

**Algorithm 1.** *For given initial point $\varrho_0$, estimate the sequence $\{\varrho_k\}_{k=1}^{\infty}$ by the following implicit iterative scheme:*

$$\begin{cases} \varrho_{k+1} = \Psi[\sigma_k - \psi(\sigma_k) + \mathcal{P}_{\mathcal{C}(\sigma_k)}[\psi(\sigma_k) - \lambda\Psi(\sigma_k)]], \\ \sigma_k = \Psi\Big[(1-\alpha_k)\Big(\dfrac{\sigma_k + \vartheta_k}{2}\Big) + \alpha_k \Psi\Big(\dfrac{\sigma_k + \vartheta_k}{2}\Big)\Big], \\ \vartheta_k = (1-\beta_k)\Psi\Big(\dfrac{\vartheta_k + \varrho_k}{2}\Big) + \beta_k \Psi\Big(\dfrac{\vartheta_k + \theta_k}{2}\Big), \\ \theta_k = (1-\gamma_k)\Big(\dfrac{\varrho_k + \theta_k}{2}\Big) + \gamma_k \Psi\Big(\dfrac{\varrho_k + \theta_k}{2}\Big)\Big], \end{cases} \tag{33}$$

*where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \subseteq (0,1)$.*

**Theorem 4.** *Let $\mathcal{P}_{\mathcal{C}(\varrho)} : \mathscr{H} \to \mathcal{C}(\varrho)$ be a projection mapping and $\Psi, \psi : \mathscr{H} \to \mathscr{H}$ be non-linear mappings so that $\Psi$ satisfies (2) and $\Xi(\Psi) \cap \maltese(\mathcal{C}(\varrho), \Psi, \psi) \neq \emptyset$. Assume that the assumption $\mathbf{A}$ and the following relations hold:*

*($R_1$) $\Psi$ is $l$-Lipschitz continuous and relaxed $(u, v)$-cocoercive and $\psi$ is $t$-Lipschitz continuous and $r$-strongly monotone.*

*($R_2$) The constant $\lambda > 0$ obeys the following relation:*

$$\lambda l^2 \leq \frac{2\lambda v + \Delta(\Delta - 2)}{\lambda + 2u}, \Delta = 2\sqrt{1 - 2r + t^2} + \kappa. \tag{34}$$

*Then $\{\varrho_k\}_{k=1}^{\infty}$ approximated by (33) converges strongly to $\varrho \in \Xi(\Psi) \cap \maltese(\mathcal{C}(\varrho), \Psi, \psi)$.*

*Proof.* Invoking the $l$-Lipschitz continuity and relaxed $(u, v)$-cocoercivity of $\Psi$ yields

$$\begin{aligned} &\|(\sigma_k - \varrho) - \lambda[\Psi(\sigma_k) - \Psi(\varrho)]\|^2 \\ &= \|\sigma_k - \varrho\|^2 - 2\lambda\langle \Psi(\sigma_k) - \Psi(\varrho), \sigma_k - \varrho\rangle + \lambda^2\|\Psi(\sigma_k) - \Psi(\varrho)\|^2 \\ &\leq \|\sigma_k - \varrho\|^2 + 2\lambda u\|\Psi(\sigma_k) - \Psi(\varrho)\|^2 - 2\lambda v\|\sigma_k - \varrho\|^2 + \lambda^2 l^2\|\sigma_k - \varrho\|^2 \\ &\leq \|\sigma_k - \varrho\|^2 + 2\lambda u l^2\|\sigma_k - \varrho\|^2 - 2\lambda v\|\sigma_k - \varrho\|^2 + \lambda^2 l^2\|\sigma_k - \varrho\|^2 \\ &= [1 - 2\lambda(v - ul^2) + \lambda^2 l^2]\|\sigma_k - \varrho\|^2 = \mathbb{B}^2\|\sigma_k - \varrho\|^2. \end{aligned} \tag{35}$$

Employing the $t$-Lipschitz continuity and $r$-strongly monotone property of $\psi$ provide with the relation

$$\begin{aligned} &\|\sigma_k - \varrho - [\psi(\sigma_k) - \psi(\varrho)]\|^2 \\ &= \|\sigma_k - \varrho\|^2 - 2\langle \psi(\sigma_k) - \psi(\varrho), \sigma_k - \varrho\rangle + \|\psi(\sigma_k) - \psi(\varrho)\|^2 \\ &\leq \|\sigma_k - \varrho\|^2 - 2r\|\sigma_k - \varrho\|^2 + t^2\|\sigma_k - \varrho\|^2 \\ &= (1 - 2r + t^2)\|\sigma_k - \varrho\|^2 = \mathbb{A}^2\|\sigma_k - \varrho\|^2. \end{aligned} \tag{36}$$

$$\begin{aligned}
&\|\varrho_{k+1}-\varrho\| = \|\Psi[\sigma_k - \psi(\sigma_k) + \mathcal{P}_{\mathcal{C}(\sigma_k)}[\psi(\sigma_k) - \lambda\Psi(\sigma_k)]] - \varrho\| \\
&= \|\Psi(\varrho) - \Psi[\sigma_k - \psi(\sigma_k) + \mathcal{P}_{\mathcal{C}(\sigma_k)}[\psi(\sigma_k) - \lambda\Psi(\sigma_k)]]\| \\
&\leq g(\|\varrho - \Psi(\varrho)\|) + \tau\|\sigma_k - \psi(\sigma_k) + \mathcal{P}_{\mathcal{C}(\sigma_k)}[\psi(\sigma_k) - \lambda\Psi(\sigma_k)] - \varrho\| \\
&= \tau\|\sigma_k - \psi(\sigma_k) + \mathcal{P}_{\mathcal{C}(\sigma_k)}[\psi(\sigma_k) - \lambda\Psi(\sigma_k)] - [\varrho - \psi(\varrho) + \mathcal{P}_{\mathcal{C}(\varrho)}[\psi(\varrho) - \lambda\Psi(\varrho)]]\| \qquad (37)\\
&\leq \tau\Big(\|\sigma_k - \varrho - [\psi(\sigma_k) - \psi(\varrho)]\| + \|\psi(\sigma_k) - \psi(\varrho) - \lambda[\Psi(\sigma_k) - \Psi(\varrho)]\| + \kappa\|\sigma_k - \varrho\|\Big) \\
&\leq 2\tau\|\sigma_k - \varrho - [\psi(\sigma_k) - \psi(\varrho)]\| + \tau\|(\sigma_k - \varrho) - \lambda[\Psi(\sigma_k) - \Psi(\varrho)]\| + \tau\kappa\|\sigma_k - \varrho\| \\
&\leq \tau(2\mathbb{A} + \mathbb{B} + \kappa)\|\sigma_k - \varrho\|.
\end{aligned}$$

Replicating the process as from (15)-(19) and combining with (37) yields

$$\|\varrho_{k+1} - \varrho\| \leq (1 - \hat{\ell}_n)(2\mathbb{A} + \mathbb{B} + \kappa)\|\varrho_k - \varrho\|, \tag{38}$$

where $\hat{\ell}_k$ is described in (21). Evidently, $(2\mathbb{A} + \mathbb{B} + \kappa) < 1$ from the assumption $(R_2)$. Then (38) turns into

$$\|\varrho_{k+1} - \varrho\| \leq (1 - \hat{\ell}_n)\|\varrho_k - \varrho\|. \tag{39}$$

Thus, from (39) and implementing Lemma 1, we acquire $\lim\limits_{k\to\infty} \|\varrho_k - \varrho\| = 0$.

By taking $\mathcal{C}(\varrho) =: \mathcal{C}$, we deduce the following corollary to estimate the common solution of the $GVI$ (30) and the contractive mapping (2).

**Corollary 1.** *Let $\mathcal{P}_{\mathcal{C}} : \mathscr{H} \to \mathcal{C}$ be a projection mapping and $\Psi, \psi : \mathscr{H} \to \mathscr{H}$ be non-linear mappings so that $\Psi$ satisfies (2) and $\Xi(\Psi) \cap ✠(\mathcal{C}, \Psi, \psi) \neq \emptyset$, where $✠(\mathcal{C}, \Psi, \psi)$ signifies the solution set of the $GVI$ (30). Assume that the following relations hold:*

- *$(v_1)$ $\Psi$ is $l$-Lipschitz continuous and relaxed $(u, v)$-cocoercive and $\psi$ is $t$-Lipschitz continuous and $r$-strongly monotone.*
- *$(v_2)$ The constant $\lambda > 0$ obeys the following relation:*

$$\lambda l^2 \leq \frac{2\lambda v + \Delta(\Delta - 2)}{\lambda + 2u}, \Delta = 2\sqrt{1 - 2r + t^2}. \tag{40}$$

*Then $\{\varrho_k\}_{k=1}^{\infty}$ approximated by (33) converges strongly to $\varrho \in \Xi(\Psi) \cap ✠(\mathcal{C}, \Psi, \psi)$.*

**Example 2.** *Let $l_2 = \{\varrho = (\varrho_0, \varrho_1, \varrho_2, \cdots) : \sum_{k=0}^{\infty} |\varrho_k^2| < \infty, \varrho_k \in \mathscr{R}, \forall n = 0, 1, 2, \cdots\}$ be a Hilbert space with norm $\|\varrho\|_2 = \sqrt{\sum_{k=0}^{\infty} |\varrho_k^2|}$. Define $\Psi, \psi : l_2 \to l_2$ by*

$$\Psi(\varrho) = \Big(\frac{\varrho_0}{3}, 0, 0, \cdots\Big), \text{ and } \psi(\varrho) = \Big(\frac{3\varrho_0}{4}, 0, 0, \cdots\Big), \forall \varrho \in l_2.$$

*Then, for all $\varrho, \omega \in l_2$, we calculate*

$$\begin{aligned}
\langle \Psi(\varrho) - \Psi(\omega), \varrho - \omega \rangle &= \Big\langle \Big(\frac{\varrho_0}{3} - \frac{\omega_0}{3}, 0, 0, \cdots\Big), (\varrho_0 - \omega_0, \varrho_1 - \omega_1, \varrho_2 - \omega_2, \cdots) \Big\rangle \\
&\geq -\frac{1}{3}\|\Psi(\varrho) - \Psi(\omega)\|_2^2 + \frac{1}{3}\|\varrho - \omega\|_2^2, \\
\|\Psi(\varrho) - \Psi(\omega)\|_2 &= \Big\|\frac{\varrho}{3} - \frac{\omega}{3}\Big\|_2 = \frac{1}{3}\|\varrho - \omega\|_2,
\end{aligned}$$

*i.e., $\Psi$ is relaxed $(\frac{1}{3}, \frac{1}{3})$-cocoercive and $\frac{1}{3}$-Lipschitz continuous and*

$$\begin{aligned}\langle\psi(\varrho)-\psi(\omega),\varrho-\omega\rangle &= \Big\langle\Big(\frac{3\varrho_0}{4}-\frac{3\omega_0}{4},0,0,\cdots\Big),(\varrho_0-\omega_0,\varrho_1-\omega_1,\varrho_2-\omega_2,\cdots)\Big\rangle\\ &= \frac{3}{4}\|\varrho-\omega\|_2^2,\\ \|\psi(\varrho)-\psi(\omega)\|_2 &= \Big\|\frac{3\varrho}{4}-\frac{3\omega}{4}\Big\|_2=\frac{3}{4}\|\varrho-\omega\|_2.\end{aligned}$$

*Thus, $\psi$ is $\frac{3}{4}$-strongly monotone and $\frac{3}{4}$-Lipschitz continuous. Also, for $\tau = \dfrac{1}{3}$ and strictly continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, we have*

$$\begin{aligned}&\|\Psi(\varrho)-\Psi(\omega)\|-\tau\|\varrho-\omega\|-g(\|\varrho-\Psi(\varrho)\|)\\ &=\frac{1}{3}|\varrho-\omega|-\frac{1}{3}|\varrho-\omega|-g(|\varrho-\frac{\varrho}{3}|)\\ &=-g\Big(\frac{2\varrho}{3}\Big)\le 0,\end{aligned}$$

*i.e., $\|\Psi(\varrho) - \Psi(\omega)\| \le \tau\|\varrho - \omega\| + g(\|\varrho - \Psi(\varrho)\|)$. Thus, $\Psi$ satisfies (2) and also $\varrho^* = (0, 0, 0, \cdots) \in \Xi(\Psi)$. Now, define $\mathcal{C} : \mathscr{H} \to \mathscr{H}$ by $\mathcal{C}(\varrho) = \mathcal{C}(\{\varrho_n\}) = \{a = \{a_k\} : a_0 \ge \frac{9}{16}\varrho_0, a_k = 0, \forall n = 1, 2, \cdots\}$. Now, for any $\alpha \in [0, 1]$ and $a_0, b_0 \in \mathcal{C}(\varrho)$ gives $\alpha a_0 + (1 - \alpha)b_0 \ge \frac{9}{16}\varrho_0$, thus, $\mathcal{C}(\varrho)$ is convex set. Now, we shall verify that $\mathcal{C}(\varrho)$ is closed. Define $Q : [\frac{9}{16}\varrho_0, \infty) \to \mathcal{C}(\varrho)$ by $Q(s) = (s, 0, 0, \cdots)$. Then $Q$ is well defined and for distinct $a_0, b_0 \in [\frac{9}{16}\varrho_0, \infty)$, we acquire $(a_0, 0, 0, \cdots) \ne (b_0, 0, 0, \cdots)$, i.e., $Q$ is one-to-one and there exists an $a_0 \in [\frac{9}{16}a_0, \infty)$, such that $Q(a_0) = (a_0, 0, 0, \cdots), \forall a = (a_0, 0, 0, \cdots) \in \mathcal{C}(\varrho)$, i.e., $Q$ is onto. Consider the usual metric spaces $(\mathscr{R}, d)$ and $(l_2, d')$, then for all $a, b \in [\frac{9}{16}a_0, \infty)$, we acquire*

$$d'(Q(a),Q(b)) = d'((a_0,0,0,\cdots),(b_0,0,0,\cdots)) = |a-b| = d(a,b).$$

*Thus, $Q$ is continuous. $Q^{-1}$ is also continuous and one-to-one and onto, so $Q$ is homeomorphism. Since $\mathcal{C}(\varrho)$ is the homeomorphic image of a closed set $[\frac{9}{16}a_0, \infty)$, and hence closed. Define $\mathcal{P}_{\mathcal{C}}(\varrho) : \mathscr{H} \to \mathcal{C}(\varrho)$ as under:*

$$\mathcal{P}_{\mathcal{C}(\varrho)}(l_0,l_1,l_2,\cdots)=\begin{cases}(l_0,l_1,l_2,\cdots), & \text{if } (l_0,l_1,l_2,\cdots)\in\mathcal{C}(\varrho)\\ (\frac{9}{16}\varrho_0,0,0,\cdots), & \text{if } (l_0,l_1,l_2,\cdots)\notin\mathcal{C}(\varrho), l_0<\frac{9}{16}\varrho_0\\ (l_0,0,0,\cdots), & \text{if} (l_0,l_1,l_2,\cdots)\notin\mathcal{C}(\varrho), l_0\ge\frac{9}{16}\varrho_0.\end{cases}$$

*Then $\|\mathcal{P}_{\mathcal{C}(a)}(l) - \mathcal{P}_{\mathcal{C}(b)}(l)\| \le \frac{9}{16}\|a - b\|$, i.e., $\mathcal{P}_{\mathcal{C}}$ fulfills assumption* **A**. *Next, we shall explore an element $\varrho^*$ such that $\varrho^* \in \Xi(\Psi) \cap \Xi(\Pi)$. Consider $\varrho^* = (\varrho_0^*, 0, 0, \cdots) : \varrho_0^* \ge 0$. If $\varrho^* > 0$, then*

$$\begin{aligned}\langle\Psi(\varrho^*),\psi(\omega^*)-\psi(\varrho^*)\rangle &= \Big\langle\frac{\varrho^*}{3},\frac{3\omega^*}{4}-\frac{3\varrho^*}{4}\Big\rangle\\ &= \frac{1}{4}\langle(\varrho_0^*,0,0,\cdots),(\omega_0^*-\varrho_0^*,0,0,\cdots)\rangle\\ &< 0,\forall\omega^*=(\omega_0^*,0,0,\cdots)\in\mathcal{C}(\varrho^*).\end{aligned}$$

*On the other hand, for* $\varrho^* = (0, 0, 0, \cdots)$*, we acquire*

$$\langle \Psi(\varrho^*), \psi(\omega^*) - \psi(\varrho^*) \rangle = \langle (0, 0, 0, \cdots), (\omega_0^* - \varrho_0^*, 0, 0, \cdots) \rangle = 0, \forall \omega^* = (\omega_0^*, 0, 0, \cdots) \in \mathcal{C}(\varrho^*).$$

*Thus, for* $\varrho^* = (0, 0, 0, \cdots) \in \Xi(\Psi) \cap \Xi(\Pi)$*.*

### 3.2. Fractional differential equation

The history of fractional calculus can be traced back to the middle of the 19th century from the pure mathematics. But a century later, its substantial and significant applications have been drawn by engineers and physicists in their respective fields. Fractional derivatives are generalization of ordinary derivatives which simultaneously set out the behavior of several physical phenomena. A very much attention have been paid to Fractional differential equations (FDEs) due to their worthy applications in several physical phenomena appearing in engineering, mechanics, economics, biology, etc., see, [27, 29, 46, 52, 53] and references therein and so these equations are widely used in many different domains. Now a days, fixed point theory has become a crucial tool to handle nonlinear problems arising in multi-disciplinary sciences. Its capacity and aptitude to demonstrate the existence and uniqueness of solutions, provides researchers to construct fixed point iterative methods to research and examine FDEs. In recent time, researchers have been explored different classes of FDEs by implementing fundamental tools of fixed point theory, for more details, we refer, [1, 2, 14, 17, 18, 24, 42, 44].

Now, we take $SIMPS$ (14) into account to examine the following Caputo-type nonlinear fractional differential equation (C-NFDE):

$$\begin{cases} {}^{\gamma}\mathscr{D}^{\xi}\varrho(u) + \Phi(u, \varrho(u)) = 0, \\ \varrho(0) = \varrho(1) = 0, 1 < \xi < 2, u \in [0, 1], \end{cases} \tag{41}$$

here, ${}^{\gamma}\mathscr{D}^{\xi}$ signifies a Caputo-fractional derivative of order $\xi$ and $\Phi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $\mathscr{X} = \{\Im : \Im : [0, 1] \to \mathbb{R}\}$ is a real continuous function equipped with supremum norm. The Green's function related to (41) is expressed as under:

$$\mathcal{G}(u, v) = \begin{cases} \frac{1}{\Gamma(\xi)}(u(1-v)^{(\xi-1)} - (u-v)^{(\xi-1)}), & \text{if } 0 \le v \le u \le 1, \\ \frac{u(1-v)^{(\xi-1)}}{\Gamma(\xi)}, & \text{if } 0 \le u \le v \le 1. \end{cases}$$

Now, we proceed to accomplish the goal of this sub-section.

**Theorem 5.** *Let* $\mathscr{X} = \mathcal{C}[0, 1]$ *and the operator* $\Im : \mathscr{X} \to \mathscr{X}$ *is defined by*

$$\Im(\varrho(u)) = \int_0^1 \mathcal{G}(u, v)\Phi(v, h(v))dv, \forall \varrho \in \mathscr{X}.$$

*If,*

$$|\Phi(v, \varrho(v)) - \Phi(v, j(v))| \le g(\varrho - \Im(\varrho)) + \tau|\varrho - j|, \forall v \in [0, 1], \varrho, j \in \mathscr{X}. \tag{42}$$

*Then the scheme (14) associated to* $\Im$ *converges to the solution of C-NFDE (41).*

*Proof.* Evidently, if $\varrho \in \mathscr{X}$ solves (41) iff $\varrho$ solves:

$$\varrho(u) = \int_0^1 \mathcal{G}(u,v)\Phi(v,h(v))dv.$$

Then for all $\varrho, j \in \mathscr{X}$ and $u \in [0,1]$, imposing the assumption (42) and employing the definition of operator $\Im$, we acquire

$$\begin{aligned}
\|\Im(\varrho(u)) - \Im(j(u))\| &= \Big| \int_0^1 \mathcal{G}(u,v)\Phi(v,\varrho(v))dv - \int_0^1 \mathcal{G}(u,v)\Phi(v,j(v))dv \Big| \\
&= \Big| \int_0^1 \mathcal{G}(u,v)[\Phi(v,\varrho(v)) - \Phi(v,j(v))]dv \Big| \\
&\leq \int_0^1 \mathcal{G}(u,v)|\Phi(v,\varrho(v)) - \Phi(v,j(v))|dv \\
&\leq \int_0^1 \mathcal{G}(u,v)[g(|\varrho(v) - \Im(\varrho(v))|) + \tau|\varrho(v) - j(v)|]dv \\
&\leq \sup_{u\in[0,1]} \int_0^1 \mathcal{G}(u,v)[g(|\varrho(v) - \Im(\varrho(v))|) + \tau|\varrho(v) - j(v)|]dv \\
&\leq g(\|\varrho(v) - \Im(\varrho(v))\|) + \tau\|\varrho(v) - j(v)\|,
\end{aligned} \tag{43}$$

which yields

$$\|\Im(\varrho) - \Im(j)\| \leq g(\|\varrho - \Im(\varrho)\|) + \tau\|\varrho - j\|.$$

So, $\Im$ satisfies (2) and by the virtue of the Theorem 2, the sequence initiated by $SIMPS$ (14) converges to an element in $\Xi(\Im)$ which solves C-NFDE (41).

## 4. Conclusions

A four-step semi-implicit midpoint scheme is designed to investigate the fixed point of a contractive mapping under some mild assumptions. Convergence analysis and stability results of the implied scheme are exhibited. Furthermore, we applied our scheme to examine a general quasi-variational inequality. By redesigning the proposed mid-point rule, we inspected a common solution of a contractive mapping and a general quasi-variational inequality. Finally, a nonlinear fractional differential equation is studied by employing $SIMPS$ (14). In future, semi-implicit type schemes could be implemented to explore fixed points of some generalized nonexpansive mappings including Suzuki's generalized nonexpansive mapping, asymptotically and total asymptotically non-expansive mappings. Some nonlinear problems such as variational inequalities, quasi-variational inequalities and inclusions are some worthy future research directions.

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