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Semitotal Roman Domination in Graphs

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Abstract. Let G be a nontrivial graph without isolated vertices. A function $f : V(G) \to \{0, 1, 2\}$ is a semitotal Roman dominating function of G if for each $v \in V(G)$ with $f(v) = 0$, there exists $u \in V(G)$ for which $f(u) = 2$ and $uv \in E(G)$ and for each $v \in V(G)$ with $f(v) \neq 0$, there exists $u \in V(G)$ for which $f(u) \neq 0$ and $d_G(u, v) \leq 2$. The minimum weight $\omega_G(f) = \sum_{u \in V(G)} f(u)$ of a semitotal Roman dominating function f of G is the *semitotal Roman domination number* of G, denoted by $\gamma_{t2R}(G)$. In this paper, we initiate the study of semitotal Roman domination. We characterize graphs G with small values of $\gamma_{t2R}(G)$ and solve some realization problems with other existing related concepts. We also investigate the semitotal Roman domination in the join, corona and complimentary prism of graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Semitotal Roman dominating function, semitotal Roman domination number, Roman dominating function, Roman domination number, total Roman dominating function, and total Roman domination number

1. Introduction

Semitotal domination and Roman domination are two among intriguing concepts in domination in graphs. Both extend the classical notion of domination by incorporating additional structural or functional constraints, offering alternative perspective and applications.

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The semitotal domination in graph was introduced by W. Goddard et al. in [9], and it bridges the gap between total domination and traditional domination. A semitotal dominating set S requires that S is a dominating set, but unlike the total domination, only requires that each vertex in S is of distance 1 or 2 from at least one vertex in S. It has practical applications in network resilience and communication.

Roman domination, which draws inspiration from the Roman empire's defense strategies, was introduced in 2004 by E.J. Cockayne et al. [8]. In this context, a Roman dominating function assigns weights (0, 1, or 2) to the vertices of a graph such that every vertex with weight 0 has an adjacent vertex with weight 2. It reflects a scenario where resources (represented by weights) are allocated to ensure the protection of vulnerable nodes, offering insights into resource optimization and strategic planning in networks. Since its introduction, Roman domination has become a very active area of research (see for example [1], [6], [8], [11], [16], [17], [18]). Part of its development was the introduction of the total Roman domination in 2016 by A. Ahangar et al. [1], some of the follow-up studies of which can be found in [15] and [16].

In this paper, we introduce and initiate the study of the semitotal Roman domination. We explore graphs G with values of $\gamma_{t2R}(G)$ equal to 2, 3 or 4, and we solve some realization problems involving the concept with other existing related concepts. We also investigate the semitotal Roman domination in the join, corona and complimentary prism of graphs.

All throughout this paper, by a graph G we mean simple and undirected. We refer to [5] for all graph terminologies we used but are not defined here. As usual, the symbols $V(G)$ and $E(G)$ denote the *vertex set* and *edge set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ and $|E(G)|$ are the *order* and *size*, respectively, of G.

Given two graphs G and H with disjoint vertex sets, the *join* $G+H$ of G and H is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$. The *corona* $G \circ H$ of graph G and H is obtained by taking one copy of G of order n and n copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. The *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$.

For $u \in V(G)$, the *open neighborhood* of u is the set $N_G(u)$ of all vertices adjacent to u. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. For $S \subseteq V(G)$,the *open neighborhood* of S is the set $N_G(S) = \cup_{u\in S} N_G(u)$. The *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. A dominating set S of G with $|S| = \gamma(G)$ is called a γ*-set* of G. The authors always refer to [4] for the introduction and more comprehensive discussion of the development of the concept of domination in graphs.

Provided that G has no isolated vertices, a set $S \subseteq V(G)$ is a *total dominating set* in G if for every $v \in V(G)$, there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a total dominating set in G, denoted by $\gamma_t(G)$, is the total domination number of G. A total dominating set S in G with $|S| = \gamma_t(G)$ is called a γ_t-*set* of G. M. Henning and A. Yeo provides in [12] a very comprehensive discussion of total domination, including its history and the succeeding developments.

A set $S \subseteq V(G)$ is a *semitotal dominating set* in G if S is a dominating set in G such that for every $x \in S$, there exists $y \in S\backslash\{x\}$ for which $d_G(x, y) \leq 2$. The smallest cardinality of a semitotal dominating set in G, denoted by $\gamma_{t2}(G)$, is called a semitotal domination number in G. A semitotal dominating set S in G with cardinality $\gamma_{t2}(G)$ is called a γ_{t2}-*set* of G. It was introduced in [9] and further studied in [1, 10, 13, 14, 16].

A *Roman dominating function* (RDF) of G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Provided G has no isolated vertices, a *total Roman dominating function* of G is a dominating Roman function $f : V(G) \to \{0, 1, 2\}$ such that for every $v \in V(G)$ with $f(v) \neq 0$, there exists $u \in V(G)$ with $f(u) \neq 0$ and $uv \in V(G)$. The *Roman domination number* (resp. *total Roman domination number*) of G is the minimum weight $\omega_G(f) = \sum_{v\in V(G)} f(v)$ of an RDF (resp. TRDF) of G. We write $f \in RDF(G)$ and $f \in TRDF(G)$ to mean that f is an RDF and TRDF, respectively, of G. An RDF (resp. TRDF) with $\omega_G(f) = \gamma_R(G)$ is referred to as a γ_R-*function* (resp. γ_{tR}-*function*) of G.

2. The semitotal Roman domination

We start by introducing a semitotal Roman dominating function. A function $f : V(G) \to \{0, 1, 2\}$ is a *semitotal Roman dominating function* of a graph G, and write $f \in SRDF(G)$, if each of the following holds:

(*i*) For each $v \in V(G)$ with $f(v) = 0$, there exists $u \in V(G)$ with $f(u) = 2$ and $uv \in E(G)$;

(*ii*) For each $v \in V(G)$ with $f(v) \neq 0$, there exists $u \in V(G)$ for which $f(u) \neq 0$ and $d_G(u, v) \leq 2$.

The minimum weight $\omega_G(f) = \sum_{u\in V(G)} f(u)$ of a semitotal Roman dominating function f of G is the *semitotal Roman domination number* of G, denoted by $\gamma_{t2R}(G)$.

As usual, we write $f = (V_0, V_1, V_2)$ for a function $f : V(G) \to \{0, 1, 2\}$ of a graph G where $V_i = \{v \in V(G) : f(v) = i \text{ for each } i \in \{0, 1, 2\}\}$. More precisely, $f \in SRDF(G)$ if and only if $V_2 \cap N_G(v) \neq \varnothing$ for each $v \in V_0$ and $V_1 \cup V_2$ is a semitotal dominating set of G. In this case, $\omega_G(f) = |V_1| + 2|V_2|$.

Proposition 1. *Let G be any graph. Then G admits a semitotal Roman dominating function if and only if G has no isolated vertices.*

Proposition 2. *For all graphs G without isolated vertices,*

$$\gamma_{t2}(G) \leq \gamma_R(G) \leq \gamma_{t2R}(G) \leq \min\{\gamma_{tR}(G), 2\gamma_{t2}(G)\}. \tag{1}$$

Proof. Clearly, a total Roman dominating function is a semitotal Roman dominating function and a semitotal Roman dominating function of G is a Roman dominating function. Thus, $\gamma_R(G) \leq \gamma_{t2R}(G) \leq \gamma_{tR}(G)$. Also, if $S \subseteq V(G)$ is a semitotal dominating set of G, then $f = (V(G) \setminus S, \varnothing, S) \in SRDF(G)$, showing that $\gamma_{t2R}(G) \leq 2\gamma_{t2}(G)$. Now, let $f = (V_0, V_1, V_2) \in RDF(G)$. Then $S = V_1 \cup V_2$ is a total dominating set of G. We claim that $V_1 \subseteq N_G(S, 2)$. Let $x \in V_1$. Suppose that $d_G(x, y) \geq 3$ for all $y \in S \setminus \{x\}$. Since x is not an isolated vertex, there exists $z \in N_G(x) \cap V_0$. Then there exists $w \in V_2$ for which $zw \in E(G)$, implying that $d_G(x, w) \leq 2$, a contradiction. This establishes the claim, which implies that $S \setminus N_G(S, 2) \subseteq V_2$. For each $u \in S \setminus N_G(S, 2)$, pick one $v_u \in V(G)$ for which $d_G(u, v_u) \leq 2$. If $S^* = \{v_u : u \in S \setminus N_G(S, 2)\}$, then $S \cup S^*$ is a semitotal dominating set of G with $|S^*| \leq |V_2|$. Thus, $\gamma_{t2}(G) \leq |S| + |S^*| \leq |V_1| + 2|V_2| = \gamma_R(G)$. □

The following examples show that the inequalities in Equation 1 are sharp. For all $n \geq 1$,

$$\gamma_R(K_{1,n}) < \gamma_{t2R}(K_{1,n}) = 3 = \gamma_{tR}(K_{1,n}) < 2\gamma_{t2}(K_{1,n}).$$

On the other hand,

$$\gamma_R(P_5) = \gamma_{t2R}(P_5) = 2\gamma_{t2}(P_5) = 4 < \gamma_{tR}(P_5).$$

Proposition 3. *Let G be any graph without isolated vertices, and let $f = (V_0, V_1, V_2)$ be a γ_{t2R}-function in G. Then each of the following holds:*

(i) $V_0 = \varnothing$ if and only if $V_2 = \varnothing$; and

(ii) $V_1 = \varnothing$ if and only if V_2 is a γ_{t2}-set in G.

Proof. Clearly, if $V_2 = \varnothing$, then $V_0 = \varnothing$. Conversely, if $V_0 = \varnothing$ and $V_2 \neq \varnothing$, then $g = (V_0, V_1 \cup V_2, \varnothing) \in SRDF(G)$ with $\omega_G(g) = |V_1| + |V_2| < \omega_G(f)$, a contradiction. This proves (i).

If $V_1 = \varnothing$, then $\gamma_{t2R}(G) = 2|V_2| \geq 2\gamma_{t2}(G)$. Then Inequality 1 completes the equation $\gamma_{t2R}(G) = 2\gamma_{t2}(G)$. Consequently, $|V_2| = \gamma_{t2}(G)$ and V_2 is a γ_{t2}-set of G. Conversely, suppose that V_2 is a γ_{t2}-set of G. Then Inequality 1 yields $|V_1| + 2\gamma_{t2}(G) = \gamma_{t2R}(G) \leq 2\gamma_{t2}(G)$, showing that $V_1 = \varnothing$. This completes (ii). □

Proposition 4. *Let G be a graph without isolated vertices. Then*

(i) $\gamma_{t2R}(G) = \gamma_{t2}(G)$ if and only if G has a γ_{t2R}-function $f = (V_0, V_1, V_2)$ for which $V_0 = \varnothing$; and

(ii) $\gamma_{t2R}(G) = 2\gamma_{t2}(G)$ if and only if G has a γ_{t2R}-function $f = (V_0, V_1, V_2)$ for which V_2 is a γ_{t2}-set of G.

Proof. For (i), suppose that $\gamma_{t2R}(G) = \gamma_{t2}(G)$, and let $f = (V_0, V_1, V_2)$ be a γ_{t2R}-function of G. Then $|V_1 + |V_2| = |V_1| + 2|V_2|$, showing that $V_2 = \varnothing$. The converse follows from Proposition 3.

To prove (ii), if $\gamma_{t2R}(G) = 2\gamma_{t2}(G)$ and $f = (V_0, V_1, V_2)$ is a γ_{t2R}- function of G, then $|V_1| + 2|V_2| = 2|V_1| + 2|V_2|$. Necessarily, $V_1 = \varnothing$. By Proposition 3, V_2 is a γ_{t2}-set of G. Conversely, by Proposition 3, $V_1 = \varnothing$ and $\gamma_{t2R}(G) = 2|V_2| = 2\gamma_{t2}(G)$. □

Proposition 5. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{t2R}(G) = 2$ if and only if $G = K_2$.*

Proof. Let $G = K_2$, then $\gamma_{t2R}(G) = 2$. Conversely, assume that $\gamma_{t2R}(G) = 2$ and let $f = (V_0, V_1, V_2)$ be a γ_{t2R}-function of G. Since $V_1 \cup V_2$ is a semitotal dominating set, $2 \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = 2$. Necessarily, $V_2 = \varnothing$ and, by Proposition 3, $V_0 = \varnothing$. Therefore, $V(G) = V_1$, that is, $G = K_2$. □

Proposition 6. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{t2R}(G) = 3$ if and only if either $\gamma(G) = 1$ or there exist u and v for which $N_G[v] = V(G) \setminus \{u\}$ and $d_G(u, v) = 2$.*

Proof. Suppose that $\gamma_{t2R}(G) = 3$, and let $f = (V_0, V_1, V_2)$ be a γ_{t2R}-function of G. If $|V_1| = 3$, then since G is connected, $G = K_3$ or $G = P_3$. In any case, $\gamma(G) = 1$. Suppose that $|V_2| = 1 = |V_1|$, say $V_2 = \{v\}$ and $V_1 = \{u\}$. Then $V_0 \neq \varnothing$ and $V_0 = V(G) \setminus \{u\} \subseteq N_G[v]$. If $uv \in E(G)$, then $N_G[v] = V(G)$ and $\gamma(G) = 1$. Otherwise, $d_G(u, v) = 2$.

Conversely, suppose that either there exists $v \in V(G)$ for which $N_G[v] = V(G)$ or there exist $u, v \in V(G)$ such that $V(G) \setminus \{u\} = N_G[v]$ and $d_G(u, v) = 2$. In any case, let $V_0 = V(G) \setminus \{u, v\}$, $V_1 = \{u\}$ and $V_2 = \{v\}$. Then $f \in SRDF(G)$ so that $\gamma_{t2R}(G) \leq \omega_G(f) = 3$. Since $n \geq 3$, Proposition 5 implies that $\gamma_{t2R}(G) = 3$. □

Proposition 7. *Let G be a connected graph. Then $\gamma_{t2R}(G) = 4$ if and only if $\gamma(G) \geq 2$ and either*

- (i) *$\gamma_{t2}(G) = 2$ and for all γ_{t2}-sets $\{u, v\}$ of G, $N_G(u) \setminus N_G(v) \neq \varnothing$ and $N_G(v) \setminus N_G(u) \neq \varnothing$; or*
- (ii) *$\gamma_{t2}(G) = 3$ and G has distinct vertices u, v and w for which $V(G) \setminus \{u, w\} = N_G[v]$ and $d_G(u, v) = 2$ and $d_G(w, v) = 2$.*

Proof. Suppose that $\gamma_{t2R}(G) = 4$. By Proposition 5 and Proposition 6, $\gamma(G) \geq 2$. Let $f = (V_0, V_1, V_2)$ is a γ_{t2R}-function of G. In view of Proposition 3, if $V_2 = \varnothing$, then $|V(G)| = 4$ and $\gamma_{t2R}(G) \leq 3$, a contradiction. Thus, either $|V_2| = 2$ and $V_1 = \varnothing$ or $|V_2| = 1$ and $|V_1| = 2$. By Proposition 3, the former implies that $\gamma_{t2}(G) = |V_2| = 2$ and, by Proposition 6, (i) holds. Now, suppose that $V_2 = \{v\}$ and $V_1 = \{u, w\}$. Since $V_1 \cup V_2$ is a semitotal dominating set of G, $\gamma_{t2}(G) \leq 3$. If $\gamma_{t2}(G) = 2$, then (i) holds. Suppose that $\gamma_{t2}(G) = 3$. Since $f \in SRDF(G)$, $V(G) \setminus \{u, w\} \subseteq N_G[v]$. Suppose that $uv \in E(G)$. If $d_G(u, w) = 1$, then $f^* = (V_0 \cup \{u\}, \{w\}, \{v\}) \in SRDF(G)$ with $\omega_G(f^*) = 3$, a contradiction. If $d_G(u, w) \geq 2$ and $P = [u = x_1, x_2, \ldots, x_k = w]$ is a u-w geodesic in G, then $x_j \in V_0$ for all $j \in \{2, 3, \ldots, k-1\}$. Since $x_{k-1}v \in E(G)$, $d_G(w, v) = 2$. Thus, $f^{**} = (V_0 \cup \{u\}, \{w\}, \{v\}) \in SRDF(G)$ with $\omega_G(f^{**}) = 3$, a contradiction. Thus,

$uv \notin E(G)$. Similarly, $wv \notin E(G)$. Therefore, $N_G[v] = V(G) \setminus \{u, w\}$. Suppose that $d_G(u,v) \geq 3$, and let P be a v-u geodesic in G. Since $N_G[v] = V(G) \setminus \{u, w\}$, P has length 3 and w lies on P so that $d_G(u,v) = 2$ and $uw \in E(G)$. Then $\{u, v\}$ is a γ_{t2}-set of G, a contradiction. Therefore, $d_G(u,v) = 2 = d_G(w,v)$. Thus, (ii) holds.

Conversely, assume that $\gamma(G) \geq 2$. Suppose that (i) holds. In view of Proposition 6, $\gamma_{t2R}(G) \geq 4$. Now let $S = \{u, v\}$ be a γ_{t2}-set of G. Since $f = (V(G) \setminus \{u, v\}, \varnothing, \{u, v\}) \in SRDF(G)$, $\gamma_{t2R}(G) \leq \omega_G(f) \leq 4$. Now, suppose that (ii) holds. Since $\gamma_{t2}(G) = 3$, $\gamma_{t2R}(G) \geq 4$ by Proposition 6. Moreover, since $f = (V(G) \setminus \{u, w\}, \{u, w\}, \{v\}) \in SRDF(G)$, $\gamma_{t2R}(G) \leq w_G(f) = 4$. In any case, $\gamma_{t2R}(G) = 4$, □

Proposition 6 asserts that if $\gamma_{t2R}(G) = 3$, then $\gamma_{t2}(G) = 2$. Proposition 7 also provides that the converse need not be true. In particular, for G_2 in Figure 1, $\gamma_{t2}(G_2) = 2$ but $\gamma_{t2R}(G) = 4$ by Proposition 7.

The graph G_1 in Figure 1 provides a graph which perfectly satisfies condition (ii) of Proposition 7.

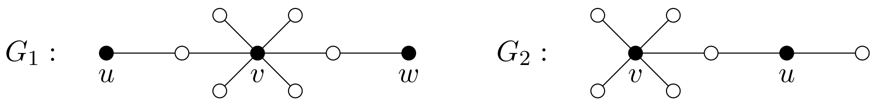

Figure 1: Graphs satisfying the conditions of Proposition 7

Proposition 8. *For each positive integer n and integer $0 \leq k \leq n$, there exists a connected graph G for which $\gamma_R(G) = 4n$ and $\gamma_{t2R}(G) = 4n + k$. Consequently, the difference $\gamma_{t2R}(\cdot) - \gamma_R(\cdot)$ can be made arbitrarily large.*

Proof. We consider the following cases:

Case 1: Suppose that $k = 0$. Consider the corona $G = P_{2n} \circ \overline{K_4}$ provided in Figure 2. Then $\gamma_R(G) = 4n = \gamma_{t2R}(G)$.

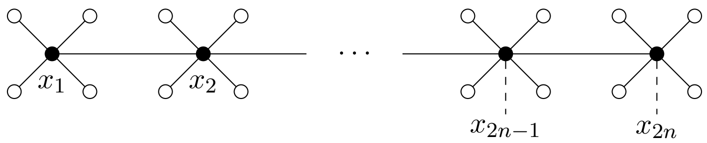

Figure 2: The graph $P_{2n} \circ \overline{K_4}$

Case 2: Suppose that $k = n$. In this case take the graph G as in Figure 3 obtained from $P_{6n-2} = [x_1, x_2, \ldots, x_{6n-2}]$ by joining $\overline{K_4}$ with x_{3j-2} for all $j = 1, 2, \ldots, 2n$. Then $\gamma_R(G) = 4n$ and $\gamma_{t2R}(G) = 4n + k$.

Case 3: Finally, suppose that $1 \leq k \leq n - 1$. Consider the graph G obtained from $P_{2n+4k} = [x_1, x_2, \ldots, 2n + 4k]$ by joining $\overline{K_4}$ with x_j for all $j \in \{1, 2, \ldots, 2n - 2k\} \cup \{2n - 2k + 3l : l = 1, 2, \ldots, 2k\}$ (see Figure 4 below). Then $\gamma_R(G) = 2(2n - 2k) + 4k = 4n$ and $\gamma_{t2R}(G) = 2(2n - 2k) + (4k + k) = 4n + k$.

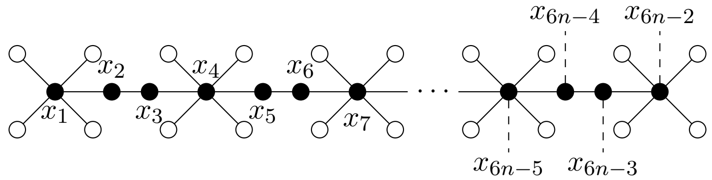

Figure 3: A graph G with $\gamma_R(G) = 4n$ and $\gamma_{t2R}(G) = 4n + n$

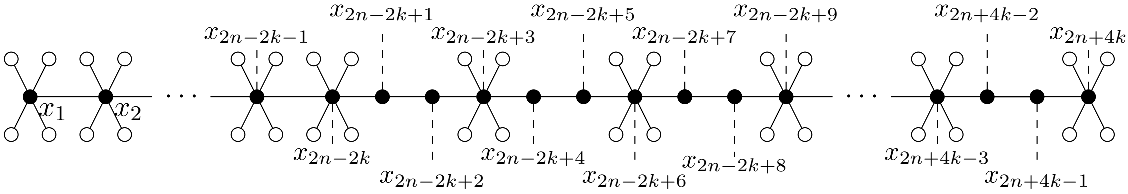

Figure 4: A graph G with $\gamma_R(G) = 4n$ and $\gamma_{t2R}(G) = 4n + k$, $1 \leq k \leq n - 1$

□

Proposition 9. *For each positive integer n and integer $0 \leq k \leq n$, there exists a connected graph G for which $\gamma_{t2R}(G) = 4n$ and $\gamma_{tR}(G) = 4n + k$. Consequently, the difference $\gamma_{tR}(\cdot) - \gamma_{t2R}(\cdot)$ can be made arbitrarily large.*

Proof. For $k = 0$, we take the graph $G = P_{2n} \circ \overline{K_4}$ in Figure 2. For this graph, $\gamma_{t2R}(G) = \gamma_{tR}(G) = 4n$. Suppose that $k = n$. Consider the graph G given in Figure 5 obtained from $P_{4n-1} = [x_1, x_2, \ldots, x_{4n-1}]$ by joining $\overline{K_4}$ with x_{2j-1} for all $j = 1, 2, \ldots, 2n$. Then $\gamma_{t2R}(G) = 2(2n) = 4n$ and $\gamma_{tR}(G) = 2(2n) + n = 4n + k$.

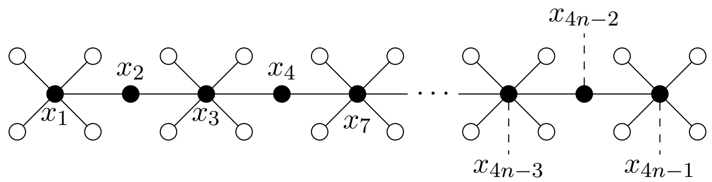

Figure 5: A graph G with $\gamma_R(G) = 4n$ and $\gamma_{t2R}(G) = 4n + n$

Now, suppose that $1 \leq k \leq n-1$. Obtain G as the graph in Figure 6 from $P_{2n+2k+1} = [x_1, x_2, \ldots, x_{2n+2k+1}]$ by joining $\overline{K_4}$ with x_j for all $j \in \{1, 2, \ldots, 2n-2k\} \cup \{2n-2k+2l+1 : l = 1, 2, \ldots, 2k\}$. Then $\gamma_{t2R}(G) = 2(2n - 2k) + 2(2k) = 4n$ and $\gamma_{tR}(G) = 4n + k$.

□

Proposition 10. *Let G be a disconnected graph without isolated vertices, and let $C_1, C_2, ..., C_k$ be the components of G. Then*

$$\gamma_{t2R}(G) = \textstyle\sum_{i=1}^{k} \gamma_{t2R}(C_i).$$

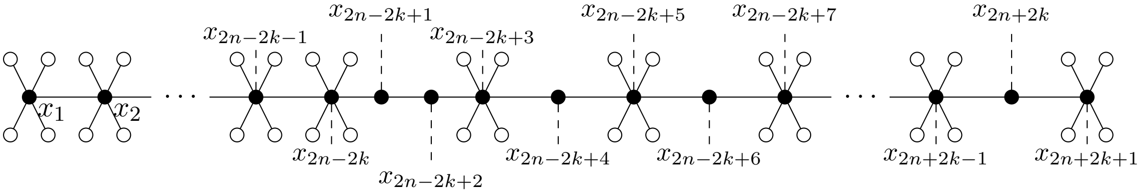

Figure 6: A graph G with $\gamma_{t2R}(G) = 4n$ and $\gamma_{tR}(G) = 4n + k$, $1 \le k \le n - 1$

Proof. Let $f_1, f_2, ..., f_k$ be γ_{t2R}-functions of $C_1, C_2, ..., C_k$, respectively. Then the function $f : V(G) \to \{0, 1, 2\}$ given by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in V(C_1), \\ f_2(x), & \text{if } x \in V(C_2), \\ \vdots & \\ f_k(x), & \text{if } x \in V(C_k), \end{cases}$$

is a SRDF of G. Thus, $\gamma_{t2R}(G) \le \omega_G(f) = \sum_{i=1}^{k} \gamma_{t2R}(C_i)$. Conversely, let f be a γ_{t2R}-function of G. Then, for each $i \in \{1, 2, \ldots, k\}$, the restriction $f|_{C_i}$ of f to C_i is a *SRDF* of C_1, showing $\gamma_{t2R}(C_i) \le \omega_{C_i}(f|_{C_i})$. Thus $\sum_{i=1}^{k} \gamma_{t2R}(C_i) \le \sum_{i=k}^{k} \omega_{C_i}(f) = \omega_G(f) = \gamma_{t2R}(G)$. □

Proposition 11. *For paths, cycles and complete multipartite graphs, we have the following:*

(*i*) *For* $n \ge 2$, $\gamma_{t2R}(P_n) = \begin{cases} n, & n = 2, 3, \\ 3k + r, & n = 4k + r \text{ and } 0 \le r \le 2, \\ 3k + 2, & n = 4k + 3. \end{cases}$

(*ii*) *For* $n \ge 3$, $\gamma_{t2R}(C_n) = \begin{cases} 3, & n = 3, \\ 3k + r, & n = 4k + r \text{ and } 0 \le r \le 2, \\ 3k + 2, & n = 4k + 3. \end{cases}$

(*iii*) *For the complete* k*-partite* $G = K_{n_1,n_2,...,n_k}$ *with* $2 \le n_1 \le n_2 \le \cdots \le n_k$,

$$\gamma_{t2R}(G) = \begin{cases} 3, & \text{if } n_1 = 2; \\ 4, & \text{otherwise.} \end{cases}$$

Proof. For (*i*), if $n = 2, 3$, then the result follows immediately. For $n \ge 4$, write $n = 4k + r$ where $0 \le r \le 3$. We define $f = (V_0, V_1, V_2)$ as follows:

Case 1: Suppose that $r = 0$. Put $V_2 = \{x_{4l-2} : l = 1, 2, \ldots, k\}$, $V_1 = \{x_{4l} : l = 1, 2, \ldots, k\}$ and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$.

Case 2: Suppose that $r = 1$. Put $V_2 = \{x_{4k}, x_{4l-2} : l = 1, 2, \ldots, k\}$, $V_1 = \{x_{4l} : l = 1, 2, \ldots, k-1\}$ and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$.

Case 3: Suppose that $r = 2$. Define $V_2 = \{x_{4l-2} : l = 1, 2, \ldots, k+1\}$, $V_1 = \{x_{4l} : l = 1, 2, \ldots, k\}$ and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$.

Case 4: Finally, suppose that $r = 3$. Define $V_2 = \{v_{4l-2} : l = 1, 2, \ldots, k+1\}$, $V_1 = \{v_{4l} : l = 1, 2, \ldots, k\}$ and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$.

In any case, $f = (V_0, V_1, V_2)$ is a γ_{t2R}-function of P_n with $\omega_{P_n}(f) = 3k+r$ for $0 \leq r \leq 2$ and $\omega_{P_n}(f) = 3k + 2$ when $r = 3$.

Similar arguments prove statement (ii). Whereas, statement (iii) follows from Proposition 6 and Proposition 7. □

3. Semitotal Roman domination in graphs under some operations

Here we investigate the semitotal Roman domination in the join, corona and complementary prism of graphs.

3.1. In the join of graphs

In what follows, by $f|_G$ we mean the restriction of f to G.

Theorem 1. *Let G and H be any graphs and $f = (V_0, V_1, V_2) : V(G+H) \to \{0, 1, 2\}$ be a function with $V_0 \neq \varnothing$. Then $f \in SRDF(G + H)$ if and only if one of the following holds:*

(i) *$V_2 \subseteq V(G)$, $f|_G \in RDF(G)$ and one of the following holds:*

- (a) *$V_1 \cap V(H) \neq \varnothing$;*
- (b) *$|(V_1 \cup V_2) \cap V(G)| \geq 2$;*

(ii) *$V_2 \subseteq V(H)$, $f|_H \in RDF(H)$, and one of the following holds:*

- (a) *$V_1 \cap V(G) \neq \varnothing$;*
- (b) *$|(V_1 \cup V_2) \cap V(H)| \geq 2$;*

(iii) *$V_2 \cap V(G) \neq \varnothing$ and $V_2 \cap V(H) \neq \varnothing$.*

Proof. Suppose that $f \in SRDF(G + H)$. Since $V_0 \neq \varnothing$, $V_2 \neq \varnothing$. Suppose that $V_2 \subseteq V(G)$, and let $v \in V_0 \cap V(G)$. Then there exists $u \in V_2$ for which $uv \in E(G + H)$. Since $V_2 \subseteq V(G)$, $uv \in E(G)$. Accordingly, $f|_G \in RDF(G)$. If $V_1 \cap V(H) \neq \varnothing$, then $(i)(a)$ holds. Suppose that $V_1 \cap V(H) = \varnothing$. Pick $v \in V_2$. Since $f \in SRDF(G + H)$, there exists $u \in V_1 \cup V_2$ for which $1 \leq d_{G+H}(v, w) \leq 2$. The assumptions imply that $w \in V(G)$. Thus, $(i)(b)$ holds. Similarly, if $V_2 \subseteq V(H)$, then (ii) holds. Clearly, if both (i) and (ii) do not hold, then (iii) holds.

Conversely, assume that (i) holds. Let $v \in V_0$. If $v \in V(H)$, then pick any $u \in V_2$. If $v \in V(G)$, then since $f|_G \in RDF(G)$, there exists $u \in V_2$ for which $uv \in E(G)$. In any case, $uv \in E(G+H)$. Let $v \in V_1 \cup V_2$. If $v \in V(H)$, then $v \in V_1$ and $uv \in E(G+H)$ for all $u \in V_2 \subseteq V(G)$. Suppose that $v \in V(G)$. If $(i)(a)$ holds, then $d_{G+H}(u,v) = 1$ for all $u \in V_1 \cap V(H)$. If $(i)(b)$ holds, then $d_{G+H}(u,v) \leq 2$ for all $u \in [(V_1 \cup V_2) \cap V(G)] \setminus \{v\}$. Accordingly, $f \in SRDF(G+H)$. Similarly, if (ii) holds, then $f \in SRDF(G+H)$. It is also clear that if (iii) holds, then $f \in SRDF(G+H)$.

Corollary 1. *Let G and H be any graphs. Then*

$$2 \leq \gamma_{tR}(G+H) \leq 4. \tag{2}$$

Proof. Since $G+H$ is at least P_2, $\gamma_{t2R}(G+H) \geq 2$ by Proposition 5. Now let $u \in V(G)$ and $v \in V(H)$. Define $V_0 = V(G+H) \setminus \{u,v\}$, $V_1 = \varnothing$ and $V_2 = \{u,v\}$. Then $V_2 \cap V(G) \neq \varnothing$ and $V_2 \cap V(H) \neq \varnothing$. By Theorem 1, $f = (V_0, V_1, V_2) \in TRD(G+H)$. Thus, $\gamma_{t2R}(G+H) \leq w_{G+H}(f) = 4$. □

Proposition 12. *Let G and H be any graphs of orders n and m, respectively. Then*

(i) $\gamma_{t2R}(G+H) = 2$ *if and only if* $n = m = 1$;

(ii) $\gamma_{t2R}(G+H) = 3$ *if and only if* $n \neq 1$ *or* $m \neq 1$ *and one of the following holds:*

- (a) $\gamma(G) = 1$ *or* $\gamma(H) = 1$;
- (b) $\Delta(G) = n-2$ *or* $\Delta(H) = m-2$.

(iii) $\gamma_{t2R}(G+H) = 4$ *if and only if* $\Delta(G) \leq n-3$ *and* $\Delta(H) \leq m-3$.

Proof. By Proposition 5, $\gamma_{t2R}(G+H) = 2$ if and only if $G+H = P_2$. Thus, (i) holds.

Suppose that $\gamma_{t2R}(G+H) = 3$. By (i), $G \neq K_1$ or $H \neq K_1$. If $\gamma(G+H) = 1$, then $\gamma(G) = 1$ or $\gamma(H) = 1$ and $(ii)(a)$ holds. Otherwise, by Proposition 6, there exists $u, v \in V(G+H)$ for which $N_{G+H}[v] = V(G+H) \setminus \{u\}$ and $d_{G+H}(u,v) = 2$. Necessarily, either $u, v \in V(G)$ or $u, v \in V(H)$. This means that if $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$, then $(ii)(b)$ holds. Conversely, if $G \neq K_1$ or $H \neq K_1$, then $\gamma_{t2R}(G+H) \geq 3$ by (i). If $(ii)(a)$ holds, then $\gamma(G+H) = 1$ and $\gamma_{t2R}(G+H) = 3$ (by Proposition 6). If $(ii)(b)$ holds, then $\Delta(G) = m+n-2$. Thus, there exists $u, v \in V(G+H)$ such that $N_{G+H}[v] = V(G+H) \setminus \{u\}$ and $d_{G+H}(u,v) = 2$. By Proposition 6, $\gamma_{t2R}(G+H) = 3$. This proves (ii).

Statement (iii) follows immediately from statements (i) and (ii) and the Inequality 2. □

The join $G+H$, where $G = \overline{K_2}$, is a good example of Proposition 12$(ii)(b)$, where $\Delta(G) = n-2 = 0$. Proposition 12(iii) also implies that $\gamma_{t2R}(K_{m,n}) = 4$ for all integers $m, n \geq 3$.

3.2. In the corona of graphs

Let G and H be connected graphs. We adapt the notation H^v to denote the copy of H whose vertices is joined to $v \in V(G)$.

Theorem 2. *Let G be nontrivial connected graph and H be any graph, and let $f = (V_0, V_1, V_2) : V(G \circ H) \to \{0,1,2\}$. Then $f \in SRDF(G \circ H)$ if and only if each of the following holds:*

(i) For each $v \in V_0 \cap V(G)$, $f|_{H^v} \in RDF(H^v)$ and each of the following holds:

(a) $|(V_1 \cup V_2) \cap V(H^v)| \geq 2$ whenever $N_G(v) \subseteq V_0$;

(b) $V_2 \cap V(H^v) \neq \varnothing$ whenever $N_G(v) \cap V_2 = \varnothing$.

(ii) For each $v \in V_1 \cap V(G)$, $f|_{H^v} \in RDF(H^v)$.

(iii) For each $v \in V_2 \cap V(G)$, $(V_1 \cup V_2) \cap N_G(v,2) \neq \varnothing$ whenever $V(H^v) \subseteq V_0$.

Proof. If $f \in SRDF(G \circ H)$, then the statements (i)-(iii) are clear. Conversely, assume that all conditions (i), (ii) and (iii) hold for f. First, let $x \in V_0$ and let $v \in V(G)$ for which $x \in V(v + H^v)$. Suppose that $x = v$. If $N_G(v) \cap V_2 \neq \varnothing$, say $u \in N_G(v) \cap V_2$, then $u \in V_2 \cap N_{G\circ H}(x)$. If, on the other hand, $N_G(v) \cap V_2 = \varnothing$, then by $(i)(b)$, $V_2 \cap V(H^v) \neq \varnothing$. Pick $u \in V_2 \cap V(H^v)$. Then $u \in V_2 \cap N_{G\circ H}(x)$. Suppose that $x \neq v$. If $v \in V_0 \cup V_1$, then since $f|_{H^v} \in RDF(H^v)$, there exists $u \in V_2 \cap N_{H^v}(x)$. This means that $u \in V_2 \cap N_{G\circ H}(x)$. And, if $v \in V_2$, then $v \in V_2 \cap N_{G\circ H}(x)$.

Next, let $x \in V_1 \cup V_2$ and let $v \in V(G)$ for which $x \in V(v + H^v)$. If $x = v \in V_1$, then since $f|_{H^v} \in RDF(H^v)$, $(V_1 \cup V_2) \cap V(H^v) \neq \varnothing$. Then $1 \leq d_{G\circ H}(x,u) \leq 2$ for all $u \in (V_1 \cup V_2) \cap V(H^v)$. Suppose $x = v \in V_2$. If $V(H^v) \setminus V_0 \neq \varnothing$, then $1 \leq d_{G\circ H}(u,x) \leq 2$ for all $u \in V(H^v) \setminus V_0$. If $V(H^v) \subseteq V_0$, then by (iii), there exists $u \in (V_1 \cup V_2) \cap N_G(x,2)$. Finally, suppose that $x \neq v$. If $v \in V_1 \cup V_2$, then we are done. Suppose that $v \in V_0$. If $N_G(v) \setminus V_0 \neq \varnothing$, then $d_{G\circ H}(x,u) \leq 2$ for all $u \in N_G(v) \setminus V_0 \subseteq (V_1 \cup V_2)$. If $N_G(v) \subseteq V_0$, then by $(i)(a)$, there exists $u \in (V_1 \cup V_2) \cap V(H^v)$ with $u \neq x$. Here, $d_{G\circ H}(x,u) \leq 2$.

The above results imply that $f \in SRDF(G \circ H)$. □

It is worth noting that Theorem 2$(i)(a)$ does not happen if $H = K_1$.

Corollary 2. *Let G be a nontrivial connected graph of order n. Then*

(i) $\gamma_{t2R}(G \circ K_1) = n + \gamma(G)$;

(ii) $\gamma_{t2R}(G \circ H) = 2n$ for all nontrivial graphs H.

Proof. In the case where $H = K_1 = \{u\}$, we define for each $v \in V(G)$, $V(H^v) = \{u^v\}$. Let $S \subseteq V(G)$ be a γ-set of G. Define $V_0 = (V(G) \setminus S) \cup (\cup_{v\in S}\{u^v\})$, $V_1 = \cup_{v\in V(G)\setminus S}\{u^v\}$ and $V_2 = S$. It is easy to see that $f = (V_0, V_1, V_2) \in SRDF(G \circ H)$. Thus, $\gamma_{t2R}(G \circ H) \leq \omega_{G\circ H}(f) = 2|S| + (n - |S|) = n + \gamma(G)$. Now, let $f = (V_0, V_1, V_2)$ be a γ_{t2R}-function of

$G \circ H$. Since f is a γ_{t2R}-function, $f(u^v) = 1$ for all $v \in V_1 \cap V(G)$ and $f(u^v) = 0$ for all $v \in V_2 \cap V(G)$. Let

$$X = \{v \in V_0 \cap V(G) : f(u^v) = 2\},\ Y = \{u^v : v \in X\} \text{ and } W = \{u^v : v \in V_1 \cap V(G)\},$$

and define

$$V_0^* = (V_0 \setminus X) \cup Y \cup W,\ V_1^* = V_1 \setminus [(V_1 \cap V(G)) \cup W] \text{ and } V_2^* = ((V_1 \cup V_2) \cap V(G)) \cup X.$$

First, we claim that $g = (V_0^*, V_1^*, V_2^*) \in SRDF(G \circ H)$. Let $x \in V_0^*$. We consider the following cases:

Case 1: Suppose $x \in V_0 \setminus X$. If $x = u^v$ for some v, then $v \in V_2 \cap V(G)$. Hence, $v \in V_2^* \cap N_{G\circ H}(x)$. If $x \in V(G)$, then $f(u^x) \neq 2$ and thus, there exists $y \in V_2 \cap V(G)$ such that $xy \in E(G \circ H)$. This means that $y \in V_2^* \cap N_{G\circ H}(x)$.

Case 2: Suppose $x \in Y \cup W$. Then $x = u^v$ where $v \in X$ or $v \in V_1 \cap V(G)$. In any case, $v \in V_2^* \cap N_{G\circ H}(x)$.

Now, let $x \in V_1^*$. Then $x = u^v \in V_1$ with $v \notin V_1 \cap V(G)$. Thus, $v \in V_0 \cap V(G)$. Since $v \notin X$, there exists $y \in V_2 \cap N_G(v)$. This means that $d_{G\circ H}(x, y) = 2$.

Finally, let $x \in V_2^*$. If $x \in X$, then $x \in V_0 \cap V(G)$ for which $u^x \in V_2$, and there exists $y \in V_2 \cap V(G)$ so that $d_{G\circ H}(u^x, y) = 2$. This means that $d_{G\circ H}(x, y) = 1$. Suppose $x \in (V_1 \cup V_2) \cap V(G)$. Take $y \in V(G)$ such that $xy \in E(G)$. If $y \in V_0$, then either $y \in X \subseteq V_2^*$ or $u^y \in V_1 \setminus W \subseteq V_1^*$. In any case, there exists $u \in V_1^* \cup V_2^*$ such that $1 \leq d_{G\circ H}(x, u) \leq 2$. then since $u^x \in V_0$, there exists $y \in V_2$ such that $1 \leq d_{G\circ H}(x, y) \leq 2$. On the other hand, if $y \in V_1 \cup V_2$, then $y \in V_2^*$ and $d_{G\circ H}(x, y) = 1$.

The above arguments show that $g \in SRDF(G \circ H)$. And since $V(G) \subseteq V_0^*$, V_2^* is a dominating set of G. Moreover, since $X = V_2 \setminus (V_2 \cap V(G)$ and $|W| = |V_1 \cap V(G)|$,

$$\begin{aligned}
\omega_{G\circ H}(g) &= |V_1^*| + 2|V_2^*| \\
&= |V_1| - 2|V_1 \cap V(G)| + 2[|V_2| + |V_1 \cap V(G)|] \\
&= |V_1| + 2|V_2| \\
&= \gamma_{t2R}(G \circ H).
\end{aligned}$$

Thus, $n + \gamma(G) \leq n + |V_2^*| = |V_1^*| + 2|V_2^*| = \gamma_{t2R}(G \circ H)$. This proves (i).

To prove (ii), since $f = (\cup_{v\in V(G)} V(H^v), \varnothing, V(G)) \in SRDF(G \circ H)$, $\gamma_{t2R}(G \circ H) \leq 2n$. To get the other inequality, let $f = (V_0, V_1, V_2) \in SRDF(G \circ H)$. For each $v \in V_2 \cap V(G)$, clearly $f(V(H^v + v) \geq 2$. For each $v \in (V_0 \cup V_1) \cap V(G)$, since $f|_{H^v} \in RDF(H^v)$, $f(V(H^v + v)) \geq 2$. Thus, $\omega_{G\circ H}(f) \geq \sum_{v\in V(G)} f(V(H^v + v)) \geq 2n$. □

3.3. In the complementary prism of graphs

Proposition 13. *Let G be a graph of order n. Then* (*i*) $\gamma_{t2R}(G\overline{G}) = 2$ *if and only if* $G = K_1$.

(*ii*) $\gamma_{t2R}(G\overline{G}) = 3$ *if and only if* $G \in \{K_2, \overline{K_2}\}$.

(*iii*) $\gamma_{t2R}(G\overline{G}) = 4$ *if and only if either*

(*a*) $G \in \{K_3, \overline{K_3}\}$; *or*

(*b*) $G \neq K_2$ *and* G *(resp.* $\overline{G} \neq K_2$ *and* $\overline{G}$*) has vertices* u *and* v *for which* $N_G[v] = V(G)$ *and* $N_G[u] = \{u, v\}$ *(resp.* $N_{\overline{G}}[v] = V(\overline{G})$ *and* $N_{\overline{G}}[u] = \{u, v\}$*).*

Proof. By Proposition 5, $\gamma_{t2R}(G\overline{G}) = 2$ if and only if $G\overline{G} = K_2$. Thus, (*i*) holds.

Suppose that $\gamma_{t2R}(G\overline{G}) = 3$. Since $G\overline{G} \neq K_2$, $\gamma(G\overline{G}) \geq 2$ by (*i*). By Proposition 6, $G\overline{G}$ has vertices u and v for which $N_{G\overline{G}}[v] = V(G\overline{G}) \setminus \{u\}$ and $d_{G\overline{G}}(u, v) = 2$. This is possible only if $G\overline{G} = P_4$ or equivalently, $G \in \{K_2, \overline{K_2}\}$. The converse of (*ii*) is immediate.

Suppose that $\gamma_{t2R}(G\overline{G}) = 4$. In view of Proposition 7, we consider two cases:

Case 1: Suppose that $\gamma_{t2}(G\overline{G}) = 2$, and let $\{u, v\}$ be a γ_{t2}-set of $G\overline{G}$. Since $G\overline{G} \neq K_2$, if $d_{G\overline{G}}(u, v) = 1$, then $G\overline{G} = P_4$, a contradiction. Thus, $d_{G\overline{G}}(u, v) = 2$. Assume $v \in V(G)$. Necessarily, $u \in V(\overline{G})$ and $u \neq \overline{v}$. Since $\{u, v\}$ is a dominating set of $G\overline{G}$, $N_G[v] = V(G)$ and $N_{\overline{G}}[u] = V(\overline{G}) \setminus \{\overline{v}\}$ or equivalently, $N_G[u] = \{u, v\}$.

Case 2: Suppose that $\gamma_{t2}(G\overline{G}) = 3$ and u, v and w are vertices of $G\overline{G}$ for which the following hold:

(*a*) $V(G\overline{G}) \setminus \{u, w\} = N_{G\overline{G}}[v]$;

(*b*) $d_{G\overline{G}}(u, v) = 2$ and $d_{G\overline{G}}(w, v) = 2$.

WLOG assume that $v \in V(G)$. If $u \in V(G)$, then by (*b*), $d_G(u, v) = 2$, say $[u, z, v]$ is a u-v geodesic in G. Then by (*a*), $\overline{u} \in N_{G\overline{G}}[v]$ or $\overline{z} \in N_{G\overline{G}}[v]$, which is impossible. Thus $u \in V(\overline{G})$. Similarly, $w \in V(\overline{G})$. Moreover, (*a*) implies that $V(G) = \{v, \overline{u}, \overline{w}\}$. Thus, either $G = P_3$ or $G = K_3$. But if $G = P_3$, then G is among the graphs described in Case 1. Thus, $G = K_3$. The converse of (*iii*) is also immediate. □

Corollary 3. *If* G *is a graph with isolated vertex* v *such that* $\gamma(G - v) = 1$*, then* $\gamma_{t2R}(G\overline{G}) = 4$.

Proposition 14. *Suppose that both* G *and* $\overline{G}$ *contain no isolated vertices. Then*

$$\max\{\gamma_{t2R}(G), \gamma_{t2R}(\overline{G})\} \leq \gamma_{t2R}(G\overline{G}) \leq \gamma_{t2R}(G) + \gamma_{t2R}(\overline{G}).$$

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ and $g = (V_0^g, V_1^g, V_2^g)$ be a γ_{t2R}-functions of G and $\overline{G}$, respectively. It is straightforward to show that $F = (V_0^f \cup V_0^g, V_1^f \cup V_1^g, V_2^f \cup V_2^g) \in SRDF(G\overline{G})$. Therefore, $\gamma_{t2R}(G\overline{G}) \leq \omega_{G\overline{G}}(F) = \gamma_{t2R}(G) + \gamma_{t2R}(\overline{G})$.

To get the other inequality, let $p = (V_0, V_1, V_2)$ be a γ_{t2R}-function of $G\overline{G}$. Let

$$A = \{v \in V_0 \cap V(G) \mid V_2 \cap N_{G\overline{G}}(v) = \{\overline{v}\} \text{ and } N_G(v) \subseteq V_0\},$$

$$B = \{v \in V_0 \cap V(G) \mid V_2 \cap N_{G\overline{G}}(v) = \{\overline{v}\} \text{ and } N_G(v) \cap V_1 \neq \varnothing\}, \text{ and}$$
$$C = \{v \in V(G) \cap (V_1 \cup V_2) \mid (V_1 \cup V_2) \cap N_{G\overline{G}}(v, 2) \subseteq V(\overline{G})\}.$$

For each $v \in A \cup C$, choose exactly one $u_v \in N_G(v)$ and let $D = \{u_v \mid v \in A\}$ and $E = \{u_v \mid v \in C\}$.

Define $q = (V_0^*, V_1^*, V_2^*)$ by

$$V_0^* = (V_0 \cap V(G)) \setminus (A \cup B \cup D \cup E),$$
$$V_1^* = (V_1 \cap V(G)) \cup A \cup B \cup D \cup E, \text{ and}$$
$$V_2^* = V_2 \cap V(G).$$

We claim that $q \in SRDF(G)$. Let $u \in V_0^*$. Since $u \in V_0$, there exists $v \in V_2$ for which $uv \in E(G\overline{G})$. Beacause $u \notin (A \cup B)$, $v \in V(G)$. Thus, $v \in V_2^*$ and $uv \in E(G)$. Let $x \in V_1^* \cup V_2^*$. We consider the following cases:

Case 1: $x \in V_1^*$.
Suppose that $x \in V_1 \cap V(G)$. Then there exists $y \in V_1 \cup V_2$ for which $d_{G\overline{G}}(x, y) \leq 2$. If $x \notin C$, then $y \in (V_1 \cup V_2) \cap V(G) \subseteq V_1^* \cup V_2^*$. If $x \in C$, then there exists $u_x \in N_G(x) \cap E \subseteq V_1^*$. If $x \in A$, then there exists $u_x \in N_G(x) \cap D \subseteq V_1^*$. If $x \in D$, then $x = u_v$ for some $v \in A \subseteq V_1^*$ and $v \in N_G(x)$. Suppose that $x \in B$. Since $N_G(x) \cap V_1 \neq \varnothing$, say $y \in N_G(x) \cap V_1$, we have $y \in V_1^*$ and $xy \in E(G)$. Finally, suppose $x \in E$. Then $x = u_v$ for some $v \in C$. This means that $v \in V_1^* \cup V_2^*$ and $xv \in E(G)$.

Case 2: $x \in V_2^*$.
Then there exists $y \in V_1 \cup V_2$ for which $d_{G\overline{G}}(x, y) \leq 2$. If $x \notin C$, then $y \in V(G)$ so that $y \in V_1^* \cup V_2^*$ and $d_G(x, y) \leq 2$. Suppose that $x \in C$. Then $u_x \in E$ and $xu_x \in E(G)$.

Accordingly, $q = (V_0^*, V_1^*, V_2^*) \in SRDF(G)$, showing that $\gamma_{t2R}(G\overline{G}) = \omega_{G\overline{G}}(p) \geq \omega_G(q) \geq \gamma_{t2R}(G)$.

Similarly, $\gamma_{t2R}(G\overline{G}) \geq \gamma_{t2R}(\overline{G})$. □

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