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New Abundant Optical Solutions for The Stochastic Heisenberg Ferromagnetic Equation Used in a Ferromagnet via Two Different Methods

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Abstract. In this study, we look at the stochastic Heisenberg Ferromagnetic equation (SHFE) perturbed in the Itô sense by multiplicative Brownian motion. The SHFE is transformed into a different Heisenberg Ferromagnetic equation with random variable coefficients (HFE-RVCs) utilizing the proper transformation. To provide novel solutions for trigonometric functions, hyperbolic functions, and rational functions for HFE-RVCs, we employ the generalizing Riccati equation mapping technique (GREM-method) and Jacobi elliptic functions (JEF-method). The solutions of SHFE can then be obtained. For the first time, we postulate that the solution to the Heisenberg Ferromagnetic equation is stochastic, in contrast to earlier research that suggested it was deterministic. Additionally, we investigate the impact of multiplicative Brownian motion on the exact solutions of the SHFE by offering several graphical representations.

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1. Introduction

The Heisenberg ferromagnetic equation has important implications for the behavior of ferromagnetic materials. It predicts phenomena such as spontaneous magnetization, where the atoms' magnetic moments align parallel to each other even in the absence of an external magnetic field. It also explains the existence of magnetic domains and how they can change their alignment under the influence of an external field. Additionally, the Heisenberg equation helps scientists understand the temperature dependence of magnetism and phenomena such as the Curie temperature, which represents the transition from ferromagnetic to paramagnetic behavior [3, 4].

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On the other side, random fluctuations play a crucial role in the behavior of ferromagnetic materials as described by the Heisenberg ferromagnetic equation. These fluctuations arise from the probabilistic nature of quantum mechanics and lead to variations in the alignment of atomic magnetic moments, both in the absence and presence of an external magnetic field. This means that even when the system is in its lowest energy state, there will still be some degree of variation in the magnetic moments' orientation. These fluctuations can give rise to interesting phenomena, such as spontaneous magnetization and the formation of magnetic domains.

It seems more relevant here the stochastic Heisenberg Ferromagnetic equation (SHFE) is taken into consideration as follows:

$$id\mathcal{U} + [\rho_1\mathcal{U}_{xx} + \rho_2\mathcal{U}_{yy} + \rho_3\mathcal{U}_{xy} - \rho_4 |\mathcal{U}|^2 \mathcal{U}]dt = i\nu\mathcal{U}d\beta, \tag{1}$$

where $\mathcal{U}$ is a stochastic complex function of the variables t, x and y; ν is the amplitude of noise and $\beta(t)$ is Brownian motion in one variable t; $\rho_2,\ \rho_2,\ \rho_3$ and ρ_4 are real functions of the variable t and they are defined as

$$\rho_1 = \lambda^4(\theta + \theta_2),\ \rho_2 = \lambda^4(\theta_1 + \theta_2),\ \rho_3 = 2\lambda^4\theta_2,\quad \rho_4 = 2\lambda^4 K,$$

where λ is the lattice parameter with the interaction coefficients $\theta,\ \theta_1,\ \theta_2$, while K is the anisotropic parameter .

Because of the relevance of Heisenberg Ferromagnetic equation, many researchers have acquired the exact solutions for this equation without stochastic term by employing different methods such as F-expansion method [6], generalized Riccati mapping [9], modified exp-function expansion [2], Jacobi elliptic functions [12], auxiliary ordinary differential equation [13], sub-ODE [11], Hirota bilinear [5, 14], and sine-Gordon expansion and the modified $\exp(-\Phi(\xi))$-expansion function methods [10]. However, many authors have studied the SHFE (1) by utilizing different methods including (G'/G)-expansion method [1], Jacobi elliptic function method [7] and mapping method [8].

This work aims to obtain accurate solutions to the SHFE (1). In order to do this, we apply an appropriate transformation to change the SHFE into another HFE-RVCs. Following that, we use the JEF and GREM methods to obtain the solutions for HFE-RVCs. Finally, we may obtain the stochastic solutions of SHFE by applying the transformation that was applied. Because of the significance of the Heisenberg Ferromagnetic equation (1) in the behavior of ferromagnetic materials, these obtained solutions are essential for comprehending a number of challenging physical processes. Using MATLAB tools, we present some graphics to illustrate the impact of the stochastic term.

The rest of this paper is organized as follows: In Section 2, we derive HFE-RVCs from SHFE (1) and determine the solutions of HFE-RVCs. In Section 3, we obtain the solutions to SHFE (1). We discuss how Brownian motion affects the solutions in Section 4. Lastly, we present the conclusions.

2. HFE-RVCs and Its Solutions

In this section, we derive the HFE-RVCs. By applying the transformation

$$\mathcal{U}(t,x,y)=\mathcal{W}(t,x,y)\exp[i\varphi(t,x,y)+\nu\beta(t)-\frac{1}{2}\nu^2 t], \tag{2}$$

we get HFE-RVCs as follows

$$\begin{aligned} &i\mathcal{W}_t-\mathcal{W}\varphi_t+\rho_1[\mathcal{W}_{xx}+2i\varphi_x\mathcal{W}_x+i\varphi_{xx}\mathcal{W}-\varphi_x^2\mathcal{W}] \\ &+\rho_2[\mathcal{W}_{yy}+2i\varphi_y\mathcal{W}_y+i\varphi_{yy}\mathcal{W}-\varphi_y^2\mathcal{W}]+\rho_3[\mathcal{W}_{xy}+i\varphi_{xy}\mathcal{W} \\ &+i\varphi_x\mathcal{W}_y+i\varphi_y\mathcal{W}_x-\varphi_x\varphi_y\mathcal{W}]-A(t)\mathcal{W}^3=0, \end{aligned} \tag{3}$$

where we applied the Itô derivatives, $A(t)=\rho_4 e^{2\nu\beta(t)-\nu^2 t}$ and $\mathcal{W}$ is a stochastic real function.

2.1. GREM-method

The GREM-method described in [15] is used here. To attain the solutions of the HFE-RVCs (3), we suppose the solutions of Eq. (3) in the special forms:

$$\mathcal{W}(t,x,y)=\sum_{k=0}^{N}\alpha_k(t)\mathcal{F}^k(\mu),\quad \mu=k_1x+k_2y+\int_0^t\lambda(\tau)d\tau, \tag{4}$$

and

$$\varphi(t,x,y)=\psi_0(t)+\psi_1(t)x+\psi_2(t)y. \tag{5}$$

where α_0, α_1,....., α_{m-1}, α_m, λ, ψ_0, ψ_1, and ψ_2, are functions of t and $\alpha_m\neq 0$, and $\mathcal{F}$ fulfills

$$\mathcal{F}'=s\mathcal{F}^2+r\mathcal{F}+p, \tag{6}$$

with s, r, and p are constants. First, let us balance $\mathcal{W}_{xx}$ with $\mathcal{W}^3$ in Eq. (3) in order to determine the value of N as follows

$$N+2=3N\Longrightarrow N=1.$$

Rewriting Eq. (4) as

$$\mathcal{W}(t,x,y)=\alpha_0(t)+\alpha_1(t)\mathcal{F}(\mu). \tag{7}$$

We differentiate Eqs. (7) and (5) with regards to t, x and y as follows

$$\begin{aligned} \mathcal{W}_t &= (\dot{\alpha}_0+p\alpha_1\lambda)+(\dot{\alpha}_1+\alpha_1 r\lambda)\mathcal{F}+s\lambda\alpha_1\mathcal{F}^2, \\ \mathcal{W}_x &= k_1\alpha_1[s\mathcal{F}^2+r\mathcal{F}+p],\ \mathcal{W}_y=k_2\alpha_1[s\mathcal{F}^2+r\mathcal{F}+p], \\ \mathcal{W}_{xx} &= k_1^2\alpha_1[2s^2\mathcal{F}^3+3sr\mathcal{F}^2+(2sp+r^2)\mathcal{F}+rp], \\ \mathcal{W}_{yy} &= k_2^2\alpha_1[2s^2\mathcal{F}^3+3sr\mathcal{F}^2+(2sp+r^2)\mathcal{F}+rp], \\ \mathcal{W}_{xy} &= k_1k_2\alpha_1[2s^2\mathcal{F}^3+3sr\mathcal{F}^2+(2sp+r^2)\mathcal{F}+rp], \end{aligned}$$

$$\mathcal{W}^3 = \alpha_1^3\mathcal{F}^3 + 3\alpha_0\alpha_1^2\mathcal{F}^2 + 3\alpha_0^2\alpha_1\mathcal{F} + \alpha_0^3, \tag{8}$$

and

$$\varphi_t = \dot{\psi}_0 + \dot{\psi}_1 x + \dot{\psi}_2 y, \quad \varphi_x = \psi_1, \quad \varphi_y = \psi_2, \quad \varphi_{xx} = \varphi_{yy} = \varphi_{xy} = 0. \tag{9}$$

Substituting Eqs. (5), (7), (8) and (9) into Eq. (3). Then, by setting each of $\mathcal{F}^k$'s coefficients to zero, we gain

$$\mathcal{F}^0: \quad \dot{\alpha}_0 + p\lambda\alpha_1 + 2pk_1\rho_1\psi_1\alpha_1 + 2pk_2\rho_1\psi_2\alpha_1 + pk_2\rho_3\psi_1\alpha_1 + pk_1\rho_3\psi_2\alpha_1 = 0,$$

$$\mathcal{F}^1: \quad \dot{\alpha}_1 + r\lambda\alpha_1 + 2rk_1\rho_1\psi_1\alpha_1 + 2rk_2\rho_1\psi_2\alpha_1 + rk_2\rho_3\psi_1\alpha_1 + rk_1\rho_3\psi_2\alpha_1 = 0,$$

$$\mathcal{F}^2: \quad s\lambda\alpha_1 + 2sk_1\rho_1\psi_1\alpha_1 + 2sk_2\rho_1\psi_2\alpha_1 + sk_2\rho_3\psi_1\alpha_1 + sk_1\rho_3\psi_2\alpha_1 = 0,$$

$$x\mathcal{F}^0 \quad : \quad \alpha_0\dot{\psi}_1 = 0,$$
$$y\mathcal{F}^0 \quad : \quad \alpha_0\dot{\psi}_2 = 0,$$

$$x\mathcal{F}^1 \quad : \quad \alpha_1\dot{\psi}_1 = 0,$$
$$y\mathcal{F}^1 \quad : \quad \alpha_1\dot{\psi}_2 = 0,$$

$$\mathcal{F}^0: \quad -\alpha_0\dot{\psi}_0 + pr\alpha_1(k_1^2\rho_1 + k_2^2\rho_2 + k_1k_2\rho_3) - \psi_1^2\rho_1\alpha_0 - \psi_2^2\rho_2\alpha_0 - \psi_1\psi_2\rho_3\alpha_0 - A\alpha_0^3 = 0,$$

$$\mathcal{F}^1: \quad \alpha_1[\dot{\psi}_0 + (2sp + r^2)(k_1^2\rho_1 + k_2^2\rho_2 + k_1k_2\rho_3) - \psi_1^2\rho_1 - \psi_2^2\rho_2 - \psi_1\psi_2\rho_3 - 3A\alpha_0^2] = 0,$$

$$\mathcal{F}^2: \quad 3sr\alpha_1(k_1^2\rho_1 + k_2^2\rho_2 + k_1k_2\rho_3) - 3A\alpha_0\alpha_1^2 = 0,$$

and

$$\mathcal{F}^3: \quad 2s^2\alpha_1(k_1^2\rho_1 + k_2^2\rho_2 + k_1k_2\rho_3) - A\alpha_1^3 = 0.$$

By solving these equations, we have

$$\alpha_0(t) = r = 0, \quad \alpha_1 = \ell, \ k_1^2\rho_1 + k_2^2\rho_2 + k_1k_2\rho_3 = \frac{\ell^2 A(t)}{2s^2}, \tag{10}$$

$$\lambda(t) = -2k_1\rho_1\psi_1 - 2k_2\rho_2\psi_2 - k_2\rho_3\psi_1 - k_1\rho_3\psi_2, \tag{11}$$

and

$$\psi_1 = \hbar_1, \ \psi_2 = \hbar_2, \ \psi_0(t) = -\frac{p\ell^2}{s}\int_0^t A(\tau)d\tau + \int_0^t (\hbar_1^2\rho_1 + \hbar_2^2\rho_2 + \hbar_1\hbar_2\rho_3)d\tau, \tag{12}$$

where $\hbar_1$, $\hbar_2$ and ℓ are constants. For simplicity, let $\hbar_1 = k_1$ and $\hbar_2 = k_2$ in order to obtain

$$\lambda(t) = -\frac{\ell^2 A(t)}{s^2}, \ \psi_0(t) = (\frac{\ell^2}{2s^2} - \frac{p\ell^2}{s})\int_0^t A(\tau)d\tau,$$

and

$$\alpha_0(t) = 0, \quad \alpha_1 = \ell, \ \psi_1 = k_1, \ \psi_2 = k_2, \ r = 0.$$

Hence, by utilizing Eqs (7) and (10), the solution of the HFE-RVCs (3) is

$$\mathcal{W}(t, x, y) = \ell\mathcal{F}(\mu), \ \mu = k_1 x + k_2 y - \frac{\ell^2 \rho_4}{s^2} \int_0^t e^{2\nu\beta(\tau)-\nu^2\tau} d\tau. \tag{13}$$

Depending on p and s, there exist many sets to find the solution $\mathcal{F}$ of Eq. (6) as follows:

Set I: If $ps < 0$, hence Eq. (6) possess the solutions:

$$\mathcal{F}_1(\mu) = -\sqrt{\frac{-p}{s}} \tanh(\sqrt{-ps}\mu),$$

$$\mathcal{F}_2(\mu) = -\sqrt{\frac{-p}{s}} \coth(\sqrt{-ps}\mu),$$

$$\mathcal{F}_3(\mu) = -\sqrt{\frac{-p}{s}} \Big(\coth(\sqrt{-4ps}\mu) \pm \operatorname{csch}(\sqrt{-4ps}\mu) \Big),$$

$$\mathcal{F}_4(\mu) = \frac{-1}{2}\sqrt{\frac{-p}{s}} \Big(\tanh(\frac{1}{2}\sqrt{-ps}\mu) + \coth(\frac{1}{2}\sqrt{-ps}\mu) \Big).$$

Consequently, HFE-RVCs (3) has the hyperbolic function solutions:

$$\mathcal{W}_1(t, x, y) = -\ell\sqrt{\frac{-p}{s}} \tanh(\sqrt{-ps}\mu), \tag{14}$$

$$\mathcal{W}_2(t, x, y) = -\ell\sqrt{\frac{-p}{s}} \coth(\sqrt{-ps}\mu), \tag{15}$$

$$\mathcal{W}_3(t, x, y) = -\ell\sqrt{\frac{-p}{s}} \Big(\coth(\sqrt{-4ps}\mu) \pm \operatorname{csch}(\sqrt{-4ps}\mu) \Big), \tag{16}$$

$$\mathcal{W}_4(t, x, y) = -\frac{1}{2}\ell\sqrt{\frac{-p}{s}} \Big(\tanh(\frac{1}{2}\sqrt{-ps}\mu) + \coth(\frac{1}{2}\sqrt{-ps}\mu) \Big). \tag{17}$$

Set II: If $ps > 0$, hence Eq. (6) possess the solutions:

$$\mathcal{F}_5(\mu) = \sqrt{\frac{p}{s}} \tan(\sqrt{ps}\mu),$$

$$\mathcal{F}_6(\mu) = -\sqrt{\frac{p}{s}} \cot(\sqrt{ps}\mu),$$

$$\mathcal{F}_7(\mu) = \sqrt{\frac{p}{s}} \Big(\tan(\sqrt{4ps}\mu) \pm \sec(\sqrt{4ps}\mu) \Big),$$

$$\mathcal{F}_8(\mu) = -\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}\mu) \pm \csc(\sqrt{4ps}\mu)\Big),$$

$$\mathcal{F}_9(\mu) = \frac{1}{2}\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\mu) - \cot(\frac{1}{2}\sqrt{ps}\mu)\Big),$$

Consequently, HFE-RVCs (3) has the trigonometric function solutions:

$$\mathcal{W}_5(t,x,y) = \ell\sqrt{\frac{p}{s}}\tan(\sqrt{ps}\mu), \tag{18}$$

$$\mathcal{W}_6(t,x,y) = -\ell\sqrt{\frac{p}{s}}\cot(\sqrt{ps}\mu), \tag{19}$$

$$\mathcal{W}_7(t,x,y) = \ell\sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}\mu) \pm \sec(\sqrt{4ps}\mu)\Big), \tag{20}$$

$$\mathcal{W}_8(t,x,y) = \ell_1\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}\mu) \pm \csc(\sqrt{4ps}\mu)\Big), \tag{21}$$

$$\mathcal{W}_9(t,x,y) = \frac{1}{2}\ell_1\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\mu) - \cot(\frac{1}{2}\sqrt{ps}\mu)\Big). \tag{22}$$

Set III: If $p = 0$ and $s \neq 0$, hence Eq. (6) possess the the rational function solution

$$\mathcal{F}_{10}(\mu) = \frac{-1}{s\mu}.$$

Consequently, the HFE-RVCs (3) has the solution

$$\mathcal{W}_{10}(t,x,y) = -\frac{\ell}{s\mu}, \tag{23}$$

where $\mu = k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau$.

2.2. JEF-method

Assuming that the solutions to Eq. (3) with $N = 1$ have the following type

$$\mathcal{W}(t,x,y) = a_0(t) + a_1(t)J(\mu), \tag{24}$$

where $J(\mu)$ can be any of the elliptic functions $cn(\omega\mu, \check{n})$, $sn(\omega\mu, \check{n})$ or $dn(\omega\mu, \check{n})$. We differentiate Eq. (24) with regards to t, x and y to get

$$\begin{aligned}
\mathcal{W}_t &= \dot{a}_0 + \dot{\alpha}_1 J + \omega\lambda a_1 J', \quad \mathcal{W}_x = \omega k_1 a_1 J', \quad \mathcal{W}_y = \omega k_2 a_1 J', \\
\mathcal{W}_{xx} &= a_1 k_1^2 \omega^2 J'' = k_1^2 a_1 B_1 J + k_1^2 a_1 B_2 J^3, \\
\mathcal{W}_{yy} &= k_2^2 a_1 B_1 J + k_2^2 a_1 B_2 J^3, \\
\mathcal{W}_{xy} &= k_1 k_2 a_1 B_1 J + k_1 k_2 a_1 B_2 J^3,
\end{aligned}$$

$$\mathcal{W}^3 = a_1^3 J^3 + 3a_0 a_1^2 J^2 + 3a_0^2 a_1 J + a_0^3, \tag{25}$$

where B_1 and B_2 are constants that will be determined later and rely on ω, and $\check{n}$. Plugging Eqs. (25) and (9) into Eq. (3). Then, by setting each of J^k's coefficients to zero, we gain

$$J^0: \quad \dot{a}_0 = 0,$$

$$J^1: \quad \dot{a}_1 = 0,$$

$$J': \quad \omega a_1[\lambda + 2\rho_1\psi_1 k_1 + 2\rho_2\psi_2 k_2 + \rho_3\psi_1 k_2 + \rho_3\psi_2 k_1] = 0,$$

$$xJ^0: \quad a_0\dot{\varphi}_1 = 0,$$

$$yJ^0: \quad a_0\dot{\varphi}_2 = 0,$$

$$xJ^1: \quad a_1\dot{\varphi}_1 = 0,$$

$$yJ^1: \quad a_1\dot{\varphi}_2 = 0,$$

$$J^0: \quad -a_0\dot{\varphi}_0 - \varphi_1^2\rho_1 a_0 - \varphi_2^2\rho_2 a_0 - \varphi_1\varphi_2\rho_3 a_0 - Aa_0^3 = 0,$$

$$J^1: \quad -a_1\dot{\varphi}_0 + \rho_1 a_1 k_1^2 B_1 + \rho_2 a_1 k_2^2 B_1 + \rho_3 a_1 k_1 k_2 B_1 - \rho_1 a_1 \varphi_1^2 - \rho_2 a_1 \varphi_2^2 - \rho_3 a_1 \varphi_1\varphi_2 - 3Aa_0^2 a_1 = 0,$$

$$J^2: \quad 3Aa_0 a_1^2 = 0,$$

and

$$J^3: \quad \rho_1 a_1 k_1^2 B_2 + \rho_2 a_1 k_2^2 B_2 + \rho_3 a_1 k_1 k_2 B_2 - Aa_1^3 = 0.$$

Solving these equations, yields

$$a_0(t) = 0, \quad a_1 = \hbar, \ \rho_1 k_1^2 + \rho_2 k_2^2 + \rho_3 k_1 k_2 = \frac{\hbar^2 A(t)}{B_2}, \ \varphi_1 = k_1, \ \varphi_2 = k_2, \tag{26}$$

and

$$\varphi_0 = \frac{\hbar^2(B_1 - 1)}{B_2}\int_0^t A(\tau)d\tau, \quad \lambda(t) = \frac{-2\hbar^2}{B_2}A(t),$$

where $\hbar$ is a constant. Therefore, the solutions of the HFE-RVCs (3), using Eqs (24) and (26), are

$$\mathcal{W}(t,x,y) = \hbar J(\mu), \ \mu = k_1 x + k_2 y - \frac{2\hbar^2\rho_4}{B_2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau. \tag{27}$$

In the following, we define $J(\mu)$ as:

Set 1: When $J(\mu) = sn(\omega\mu, \check{n})$, Eq. (27) has the form

$$\mathcal{W}(t,x,y) = \hbar[sn(k_1\omega x + \omega k_2 y - \frac{2\omega\hbar^2\rho_4}{B_2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})], \tag{28}$$

where

$$B_1 = -\omega^2(1 + \check{n}^2) \text{ and } B_2 = 2\omega^2\check{n}^2.$$

Set 2: When $J(\mu) = cn(\omega\mu, \check{n})$, Eq. (27) has the form

$$\mathcal{W}(t,x,y) = \hbar[cn(k_1\omega x + \omega k_2 y - \frac{2\omega\hbar^2\rho_4}{B_2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})], \quad (29)$$

where

$$B_1 = \omega^2(1-2\check{n}^2) \text{ and } B_2 = -2\omega^2\check{n}^2.$$

Set 3: When $J(\mu) = dn(\omega\mu, \check{n})$, Eq. (27) has the form

$$\mathcal{W}(t,x,y) = \hbar[dn(k_1\omega x + \omega k_2 y - \frac{2\omega\hbar^2\rho_4}{B_2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})], \quad (30)$$

where

$$B_1 = \omega^2(2-\check{n}^2) \text{ and } B_2 = -2\omega^2.$$

3. Exact Solutions of SHFE

The solution of SHFE (1) is obtained by putting Eq. (13) into Eq. (2) as

$$\mathcal{U}(t,x,y) = \mathcal{W}(t,x,y)\exp[i\varphi(t,x,y) + \nu\beta(t) - \frac{1}{2}\nu^2 t], \quad (31)$$

where

$$\varphi(t,x,y) = k_1 x + k_2 y + (\frac{\rho_4\ell^2}{2s^2} - \frac{p\rho_4\ell^2}{s})\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau.$$

3.1. GREM-Method

Substituting Eqs (18)-(23) into (31), we get the solutions of the SHFE (1) as:

$$\mathcal{U}_1(t,x,y) = \ell\sqrt{\frac{-p}{s}}\Big(\tanh[\sqrt{-ps}(k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (32)$$

$$\mathcal{U}_2(t,x,y) = \ell\sqrt{\frac{-p}{s}}\Big(\coth[\sqrt{-ps}(k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (33)$$

$$\begin{aligned}\mathcal{U}_3(t,x,y) = \ell\sqrt{\tfrac{-p}{s}}\Big(&\coth[\sqrt{-4ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\\ &+\text{csch}[\sqrt{-4ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]},\end{aligned} \quad (34)$$

$$\begin{aligned}\mathcal{U}_4(t,x,y) = -\tfrac{1}{2}\ell\sqrt{\tfrac{-p}{s}}\Big(&\tanh[\tfrac{1}{2}\sqrt{-ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\\ &+\coth[\tfrac{1}{2}\sqrt{-ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]},\end{aligned} \quad (35)$$

for $ps < 0$,

$$\mathcal{U}_5(t,x,y) = \ell\sqrt{\frac{p}{s}}\tan[\sqrt{ps}(k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (36)$$

$$\mathcal{U}_6(t,x,y) = -\ell\sqrt{\frac{p}{s}}\cot[\sqrt{ps}(k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (37)$$

$$\begin{aligned}\mathcal{U}_7(t,x,y) = \ell\sqrt{\tfrac{p}{s}}\Big(&\tan[\sqrt{4ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)] \\ &- \sec[\sqrt{ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]},\end{aligned} \quad (38)$$

$$\begin{aligned}\mathcal{U}_8(t,x,y) = \ell\sqrt{\tfrac{p}{s}}\Big(&\cot[\sqrt{4ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)] \\ &+ \csc[\sqrt{4ps}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}\end{aligned} \quad (39)$$

$$\begin{aligned}\mathcal{U}_9(t,x,y) = \tfrac{1}{2}\ell\sqrt{\tfrac{p}{s}}\Big(&\tan[\tfrac{\sqrt{ps}}{2}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)] \\ &- \cot[\tfrac{\sqrt{ps}}{2}(k_1x + k_2y - \tfrac{\ell^2\rho_4}{s^2}\textstyle\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)]\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]},\end{aligned} \quad (40)$$

for $ps > 0$, and

$$\mathcal{U}_{10}(t,x,y) = \Big(\frac{-1}{s(k_1x + k_2y - \frac{\ell^2\rho_4}{s^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)}\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]} \text{ for } s \neq 0. \quad (41)$$

Remark 1. *Eqs (32) and (33), with $\nu = 0$ coincide with the solutions (45) and (46) that reported in [6].*

3.2. JEF-method

The solutions of the SHFE (1) are obtained by substituting Eqs (28)-(30) into (31) as:

$$\mathcal{U}(t,x,y) = \hbar\Big(sn(k_1\omega x + \omega k_2 y - \frac{\hbar^2\rho_4}{\omega\check{n}^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (42)$$

$$\mathcal{U}(t,x,y) = \hbar\Big(cn(k_1\omega x + \omega k_2 y + \frac{\hbar^2\rho_4}{\omega\check{n}^2}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (43)$$

and

$$\mathcal{U}(t,x,y) = \hbar\Big(dn(k_1\omega x + \omega k_2 y + \frac{\hbar^2\rho_4}{\omega}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau, \check{n})\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (44)$$

where

$$\varphi_0 = k_1x + k_2y + \frac{\hbar^2(B_1 - 1)}{B_2}\int_0^t A(\tau)d\tau.$$

If $\check{n} \to 1$, then the Eqs (42)-(44) become

$$\mathcal{U}(t,x,y) = \hbar\Big(\tanh(k_1\omega x + \omega k_2 y - \frac{\hbar^2\rho_4}{\omega}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}, \quad (45)$$

and

$$\mathcal{U}(t,x,y) = \hbar\Big(\text{sech}(k_1\omega x + \omega k_2 y + \frac{\hbar^2\rho_4}{\omega}\int_0^t e^{2\nu\beta(\tau)-\nu^2\tau}d\tau)\Big)e^{[i\varphi+\nu\beta(t)-\frac{1}{2}\nu^2 t]}. \quad (46)$$

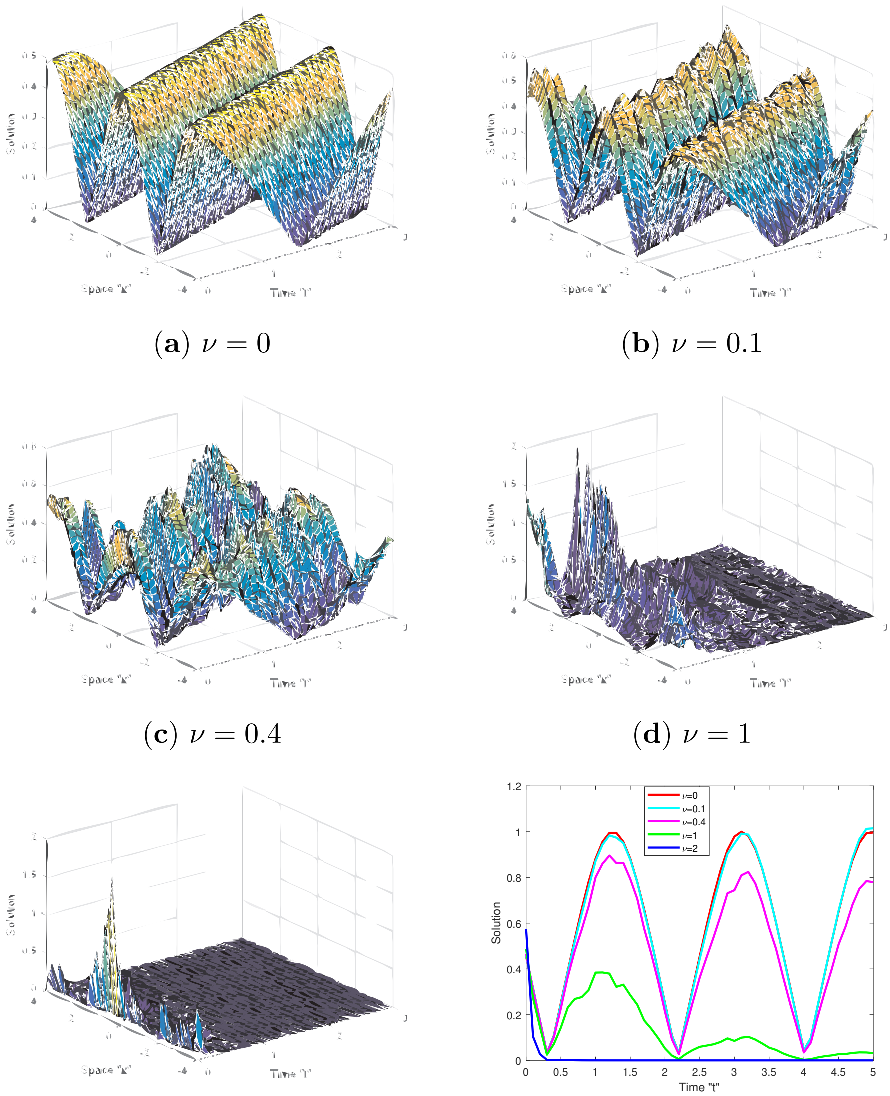
Remark 2. *If we put $\nu = 0$ in Eqs (42)-(46), we have the same solutions that stated in [12].*

4. Discussion and impacts of noise

Discussion: In this paper, the exact solutions of the SHFE (1) were acquired. We applied two methods such as the GREM-method and JEF-method which they provided many types of solutions such as optical kink solution (32), optical singular solution (33), optical singular periodic (36) and (37), elliptic solutions (42)-(44) and etc. One of the key features of singular solitons in the Heisenberg ferromagnet equation is their ability to maintain their shape and energy despite interacting with other solitons or external perturbations. This stability arises from the non-linear nature of the equation, which allows solitons to balance the competing effects of dispersion and nonlinearity. Solitons in the Heisenberg ferromagnet equation often exhibit complex and interesting dynamics, such as the formation of bound states or the reflection and transmission of solitons at interfaces between different regions of the material.

Impacts of Brownian motion: the impacts of noise on the solutions to the Heisenberg ferromagnet equation can have significant implications for the design and performance of magnetic devices. By understanding the impacts of noise on the solutions to the Heisenberg ferromagnet equation, scientists and engineers can work towards developing better strategies for mitigating the effects of noise and improving the performance of magnetic devices based on these solutions. Ultimately, addressing the challenges posed by noise will be crucial in unlocking the full potential of the Heisenberg ferromagnet equation and advancing the field of magnetic materials and technology.

Now, the effect of noise on the exact solutions of the SHFE (1) will be examined. We provide some figures for different solutions with varying noise intensity ν. Graphs 1, 2 and 3 show the solutions $\mathcal{U}(t,x,y)$ presented in Eqs (43), (45) and (46) for distinct value of ν as follows:

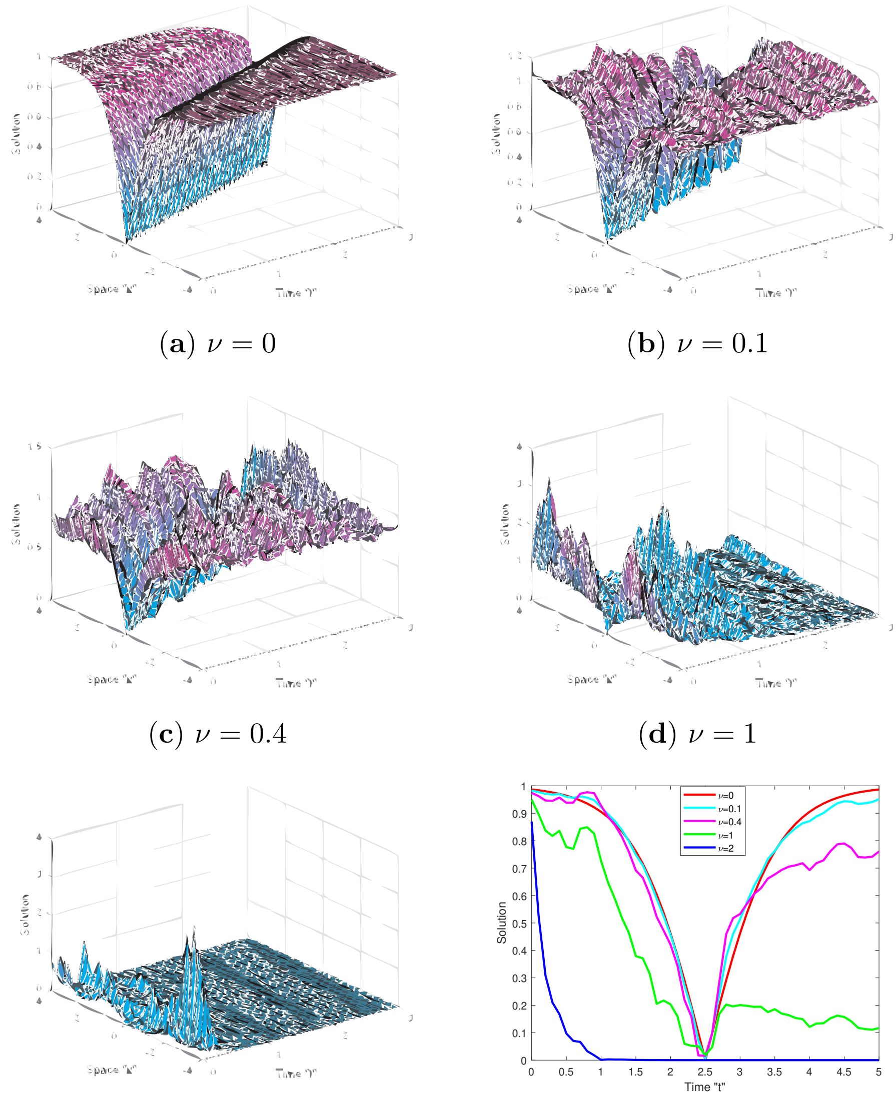


(**a**) $\nu = 0$ (**b**) $\nu = 0.1$

(**c**) $\nu = 0.4$ (**d**) $\nu = 1$

(**e**) $\nu = 2$ (**f**) $\nu = 0, 0.1, 0.4,\ 1,\ 2$

Figure 1: (a-e) display 3D-shape of $|\mathcal{U}(x, y, t)|$ described in Eq. (43) with $k_1 = k_2 = 1$, $p = s = 0.5$, $\ell = 1$, $\rho_4 = 1$, $x \in [-4, 4]$, $y = 0$ and $t \in [0, 3]$ (f) exhibits 2D-shape of Eq. (43) with distinct ν

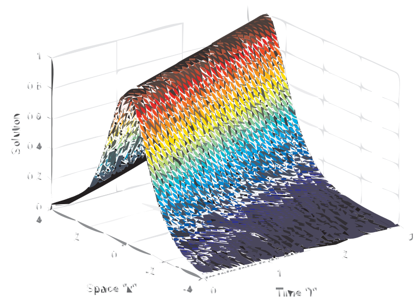


(**a**) $\nu = 0$ (**b**) $\nu = 0.1$

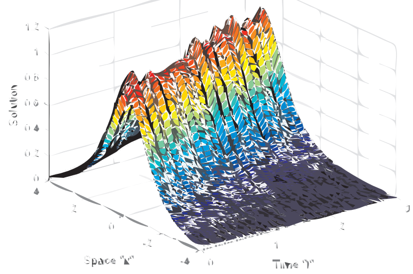
(**c**) $\nu = 0.4$ (**d**) $\nu = 1$

(**e**) $\nu = 2$ (**f**) $\nu = 0, 0.1, 0.4,\ 1,\ 2$

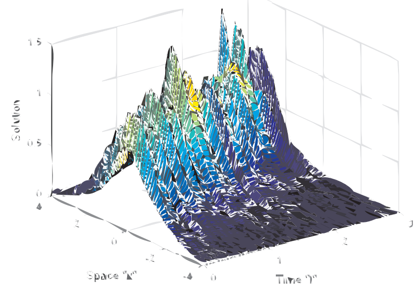
Figure 2: (a-e) display 3D-shape of $|\mathcal{U}(x, y, t)|$ presented in Eq. (43) with $\check{n} = \hbar = 0.5$, $k_1 = k_2 = 1$, $\rho_4 = 2$, $\omega = 1$, $x \in [-4, 4]$, $y = 0$, and $t \in [0, 4]$ (f) exhibits 2D-shape of Eq. (43) with different ν



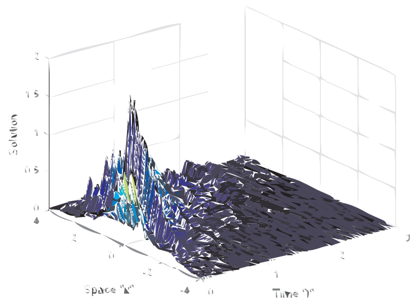
(**a**) $\nu = 0$



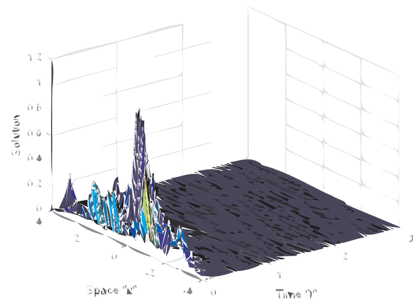
(**b**) $\nu = 0.1$



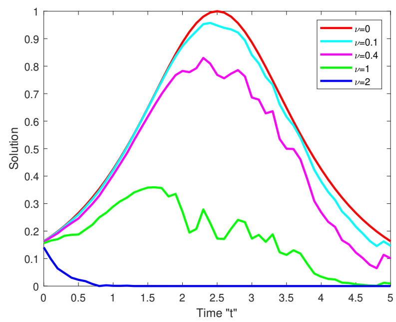
(**c**) $\nu = 0.4$



(**d**) $\nu = 1$



(**e**) $\nu = 2$



(**f**) $\nu = 0, 0.1, 0.4,\ 1,\ 2$

Figure 3: (a-e) display 3D-shape of $|\mathcal{U}(x, y, t)|$ presented in Eq (46) with $k_1 = k_2 = 1,\ \omega = 2,\ \rho_4 = 1,\ x \in [-4, 4]$, $y = 0$ and $t \in [0, 3]$ (f) shows 2D-shape of Eq. (46)

As seen in Figures 1-3, a wide variety of solutions, such as optical kink solutions, optical singular solutions, optical periodic solutions, and others, arise when noise is disregarded (i.e., $\nu = 0$). After a few transit patterns, the surface flattens when noise is added at $\nu = 0.1,\ 0.4,\ 1,\ 2$. This finding demonstrates the stabilization of the SHFE Eq. (1) solutions around zero due to multiplicative Brownian motion.

5. Conclusions

The stochastic Heisenberg ferromagnet equation (SHFE) (1) driven by multiplicative noise in the Itô sense was examined in this study. We transformed the SHFE into another Heisenberg ferromagnet equation with random variable coefficients (3) (HFE-RVCs) by applying the proper transformation. We found novel stochastic exact solutions for HFE-RVCs in the form of hyperbolic, trigonometric, rational, and elliptic functions utilizing two various methods including the JEF-method and GREM-method, and then we acquired the solutions of SHFE (1). Additionally, we generated a few earlier solutions, such the solutions documented in [6, 12]. The obtained solutions are essential for comprehending a number of challenging physical processes due to the significance of the Heisenberg ferromagnet equation in the behavior of ferromagnetic materials. The impact of the stochastic term on the stochastic exact solutions of SHFE was finally illustrated using a few graphics. In future work, we can study SHFE (1) with additive noise.

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