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Impulsive Protocols for Scaled Consensus in Edge-Dynamic Multi-Agent Systems

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Abstract. This study addresses the challenge of achieving scaled consensus in multi-agent systems with edge dynamics, a critical aspect for practical applications such as traffic networks and distributed energy systems. Utilizing graph theory, matrix analysis, and Lyapunov stability, we propose impulsive consensus protocols designed for both directed and undirected topologies. These protocols enable agents to converge proportionally to desired ratios among states, generalizing classical consensus approaches while significantly reducing communication and control costs. We establish sufficient conditions for achieving scaled consensus, demonstrating that it requires the line graph of the communication network to be connected and contain a spanning tree. Numerical simulations validate the robustness and efficiency of the proposed protocols, highlighting their scalability and adaptability in dynamic and resource-constrained environments. The findings provide a solid foundation for real-world applications, including autonomous vehicles, distributed energy systems, and robotic coordination, where proportional state alignment is essential.

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1. Introduction and preliminaries

The consensus problem in multi-agent systems has been a cornerstone of research in networked control systems, particularly over the last decade. Traditional investigations have predominantly focused on achieving agreement among node states, using nodal dynamics to model interactions. Seminal works, such as those by Olfati-Saber et al. [8, 9], developed integrator and linear models for consensus, which have since been extended to nonlinear [23], second order [21], and higher order systems [5]. These studies laid the groundwork for a variety of applications, from sensor networks to robotics.

However, practical scenarios often reveal the limitations of the nodal dynamics in capturing real-world complexities. For instance, in transportation systems or energy networks,

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edge dynamics representing interactions rather than individual states offer a more accurate framework. Edge-based modeling has been particularly relevant in scenarios such as traffic systems, where road conditions (edges) rather than junction states (nodes) dictate flow efficiency [6, 14, 18]. The burgeoning interest in edge dynamics has inspired a new wave of research aimed at solving edge consensus problems under dynamic topologies.

A pressing challenge in multi-agent systems is the coordination of agents in discrete time intervals, a scenario where continuous-time protocols falter. This has motivated the adoption of impulsive consensus protocols, which leverage impulsive control strategies to enable abrupt state adjustments at specified time instances. These protocols, as highlighted in studies such as Wang et al. [14] and Liu et al. [16, 17], significantly reduce communication overhead while maintaining robust system performance. Applications span autonomous systems [12], deep-space missions, and energy-efficient wireless sensor networks, where communication constraints are a critical consideration (see more examples in [1, 15, 19, 20, 22]).

A particularly intriguing domain within consensus research is the scaled consensus problem. Unlike traditional consensus, which focuses on convergence to a single shared state, scaled consensus aims to achieve predefined proportional relationships among agent states. This paradigm finds practical utility in biological systems, multi-robot coordination, and distributed energy systems [11]. Despite its potential, scaled consensus has remained relatively under explored, especially concerning edge dynamics and the impulsive protocols necessary to regulate them under complex topologies.

This paper aims to fill this gap by addressing the scaled consensus problem for edge-dynamic multi-agent systems under both directed and undirected topologies. Inspired by foundational works [7, 11, 14], we introduce novel impulsive control protocols, extending their applicability to scenarios characterized by heterogeneous agent dynamics. Specifically, our contributions include:

(1) A rigorous exploration of scaled consensus under impulsive protocols for edge dynamics, expanding on prior analyses of nodal dynamics [1, 2].

(2) The establishment of sufficient conditions for achieving scaled consensus, leveraging graph theory and Lyapunov-based stability criteria.

The rest of the paper is organized as follows. Section 2 reviews foundational concepts and frames the problem. Section 3 presents the main results, deriving conditions for scaled consensus in undirected and directed topologies. Numerical simulations, illustrating the efficacy of the proposed methods, are provided in Section 4. Finally, Section 5 concludes the paper with a discussion of future research directions.

2. Preliminaries and Problem formulations

2.1. Preliminaries

In this section, we introduce essential concepts from graph theory and matrix theory (for further details, see [3, 4]). Throughout the paper, interactions among n agents are represented as a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ denotes the set of nodes,$\mathcal{E} = \{e_{ij} = (v_i, v_j)\} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of edges, and $\mathcal{A} = [a_{ij}]_{n\times n}$ is a nonnegative matrix. The neighbors of agent i are indicated by $N_i = \{j : a_{ij} > 0\}$. The out-degree and in-degree of node v_i are represented as $deg_{out}(v_i)$ and $deg_{in}(v_i)$, corresponding to the number of edges $e_{ij} = (v_i, v_j)$ and $e_{ki} = (v_k, v_i)$, respectively. A graph is considered balanced if each node's out-degree equals its in-degree. A directed path in G consists of a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}), ...,$ and so on. A digraph $\mathcal{G}$ is termed strongly connected if there exists a directed path between any two nodes in $\mathcal{G}$. A directed tree is a digraph with a single root (no edges pointing to this node), where every node except the root has exactly one parent. A spanning tree of $\mathcal{G}$ is a directed tree that connects all nodes in $\mathcal{G}$. Additionally, we denote $\mathbb{R}$ as the set of real numbers, $\mathbb{N}$ as the set of positive integers, and $\mathbb{R}^n$ as the n-dimensional real vector space. For a vector or matrix B, B^T represents its transpose, and $\|X\|$ indicates the Euclidean norm of vector X. A vector is nonnegative if all its components are nonnegative, while the column vector with all entries equal to one or zero is denoted by 1_n and 0_n, respectively. In is the n-dimensional identity matrix, and a diagonal matrix with diagonal elements $a_1, a_2, ..., a_n$ is represented as $diaga_1, a_2, ..., a_n$. Furthermore, $[a_{ij}]_{nn}$ is an n by n matrix with a_{ij} as its (i, j)th entry. A matrix $B = [b_{ij}]_{nn}$ is classified as nonnegative, denoted by $B \geq 0$, if all its entries are nonnegative. For nonnegative matrices, we define an order such that if A and B are nonnegative, then $A \geq B$ implies $A - B$ is also a nonnegative matrix. A is termed a stochastic matrix if it is nonnegative and the sum of each row equals 1. A stochastic matrix A is called indecomposable and aperiodic (SIA) if there exists a column vector v such that $\lim_{n\to\infty} A^n = \mathbf{1}_n v^T$, where $\mathbf{1}_n = (1, 1, ..., 1)^T$ is a $n \times 1$ vector. Additionally, we provide some useful definitions, lemmas, and properties.

Definition 1. *[3] For an undirected graph $\mathcal{G}$ with the Laplacian matrix $\mathcal{L}$ the algebraic connectivity is defined as $\lambda_2(\mathcal{L})$, the second smallest eigenvalue of $\mathcal{L}$:*

$$\lambda_2(\mathcal{L}) = \min_{x\neq 0,\ 1^T x = 0} \frac{x^T \mathcal{L} x}{x^T x}.$$

Lemma 1. *[10] Let $\mathcal{G}$ be a digraph with adjacency matrix $\mathcal{A}$ and Laplacian $\mathcal{L}$. Then $\hat{\mathcal{L}} = Sym(\mathcal{L}) = (\mathcal{L} + \mathcal{L}^T)/2$ is a valid Laplacian matrix for $\hat{\mathcal{G}}$ if $\mathcal{G}$ is balanced.*

Lemma 2. *[13] A stochastic matrix has algebraic multiplicity equal to one for eigenvalue $\lambda = 1$ if and only if the graph associated with matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.*

Lemma 3. *[13] Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{k\to\infty} A^k = \mathbf{1}_n y^T$, where y is nonnegative and satisfies $A^T y = y$, $\mathbf{1}_n^T y = 1$.*

2.2. Problem formulation

Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with a set of N nodes and M edges, where $\mathcal{E} = \{(i, j) :$ if there is an edge between node i and node $j\}$ and $\mathcal{V} = \{1, 2, ..., N\}$.

The topology of the network is described by the adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$, where

$$a_{ji} = a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}; \\ 0, & \text{otherwise.} \end{cases}$$

Different from the undirected network, the directed network can be denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E} = \{(i,j); \text{if node } i \text{ can receive information from node } j\}$. The topology of the network is described by the adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}; \\ 0, & \text{otherwise.} \end{cases}$$

For any node i, its inbound edge (i,k) is adjacent to its outbound edge (j,i) while its outbound edge (j,i) is adjacent from its inbound edge (i,k). For better description and understanding, we say that the inbound edge (i,k) is the valid neighbor of the outbound edge (j,i). Let $x_{ij}(t)$ and $\beta_{ij} \neq 0$ be the state and scalar scale of edge (i,j) at time t, respectively. Thus, the edge dynamics of each edge can be designed as follows:

$$\beta_{ij}\dot{x}_{ij}(t) = u_{ij}(t), \quad \text{for } (i,j) \in \mathcal{E}, \tag{2.1}$$

where $u_{ij} \in \mathbb{R}$ is a consensus protocol. In general, one says that the protocol u_{ij} in (2.1) solves the edge consensus problems if the following definition is satisfied:

Definition 2. *The multi agent system (2.1) is said to reach scaled consensus in edge dynamics if for any initial conditions,*

$$\lim_{t_k\to\infty} \|\beta_{ij}x_{ij}(t_k) - \beta_{ks}x_{ks}(t_k)\| = 0, \quad \textit{for all } (i,j),(k,s) \in \mathcal{E}. \tag{2.2}$$

3. scaled consensus results on edge dynamics

3.1. Undirected communication networks

In this section, we solve scaled consensus problem of edge dynamics in MASs by designing the consensus protocol, denoted by u_{ij}, which depends on the states of the edge (i,j) and its neighboring edges. In addition, two edges are neighboring edges if they share exactly a common ending vertex.

Assuming that the multi-agent system (2.1) has been modelled as a connected digraph, where $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and $\mathcal{G}' = (\mathcal{V}, \mathcal{E}', \mathcal{A}')$, are the communication networks of system (2.1) at time t and at the impulsive time t_k, respectively. Given any scalar scales $\beta_{ij} \neq 0$ for all i,j, the scaled consensus protocol of multi-agent system (2.1) based on the state of edges in a communication network $\mathcal{G}\cup\mathcal{G}'$ and its line graph is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$u_{ij}(t) = |\beta_{ij}| \left[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t) - \beta_{ij}x_{ij}(t)] + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t) - \beta_{ij}x_{ij}(t)] \right]$$

$$+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \left[\sum_{s \in \mathcal{N}_i'} a_{is}' [\beta_{is} x_{is}(t) - \beta_{ij} x_{ij}(t)] \right.$$

$$\left. + \sum_{s \in \mathcal{N}_j'} a_{js}' [\beta_{js} x_{js}(t) - \beta_{ij} x_{ij}(t)] \right] \delta(t - t_k), \quad \text{for} \quad (i,j) \in \mathcal{E}, \tag{3.1}$$

where β_{ij} is the scalar scale of an edge (i,j); $h = t_k - t_{k-1}$ is a sampling period; $\mathcal{N}_i$ and $\mathcal{N}_i'$ are the neighboring sets of node i at time t and impulsive time t_k, respectively.

Remark 1. *It can be seen that if $\beta_{ij} = 1$ for all i,j, the consensus protocol (3.1) will reduce to the following consensus protocol*

$$u_{ij}(t) = \left[\sum_{s \in \mathcal{N}_i} a_{is} [x_{is}(t) - x_{ij}(t)] + \sum_{s \in \mathcal{N}_j} a_{js} [x_{js}(t) - x_{ij}(t)] \right]$$

$$+ h \cdot \sum_{k=1}^{\infty} \left[\sum_{s \in \mathcal{N}_i'} a_{is}' [x_{is}(t) - x_{ij}(t)] \right.$$

$$\left. + \sum_{s \in \mathcal{N}_j'} a_{js}' [x_{js}(t) - x_{ij}(t)] \right] \delta(t - t_k), \quad \textit{for} \quad (i,j) \in \mathcal{E}. \tag{3.2}$$

Remark 2. *If there is no instantaneous contact or update at the impulsive time t_k and $\beta_{ij} = 1$ for all i,j, the consensus protocol (3.1) is described as in the following consensus protocol*

$$u_{ij}(t) = \left[\sum_{s \in \mathcal{N}_i} a_{is} [x_{is}(t) - x_{ij}(t)] + \sum_{s \in \mathcal{N}_j} a_{js} [x_{js}(t) - x_{ij}(t)] \right], \quad \textit{for} \quad (i,j) \in \mathcal{E}, \tag{3.3}$$

which was investigated by Wang et al., [14]. This shows the generalization of our protocol.

By the definition of the Dirac delta function, the system (2.1) and protocol (3.1) can be described as the impulsive system:

$$\begin{cases} \beta_{ij} \dot{x}_{ij}(t) & = |\beta_{ij}| \Bigg[\sum_{s \in \mathcal{N}_i} a_{is} [\beta_{is} x_{is}(t) - \beta_{ij} x_{ij}(t)] \\ & \quad + \sum_{s \in \mathcal{N}_j} a_{js} [x_{js}(t) - x_{ij}(t)] \Bigg], \quad t \in (t_{k-1}, t_k), \\ \Delta \beta_{ij} x_{ij}(t_k) & = h |\beta_i| \sum_{l \in \mathcal{N}_i'} a_{il}' \left[\sum_{s \in \mathcal{N}_i} a_{is} [\beta_{is} x_{is}(t_k) - \beta_{ij} x_{ij}(t_k)] + \sum_{s \in \mathcal{N}_j} a_{js} [x_{js}(t_k) - x_{ij}(t_k)] \right], \end{cases} \tag{3.4}$$

where $\Delta \beta_{ij} x_{ij}(t_k) = \beta_{ij} x_{ij}(t_k^+) - \beta_{ij} x_{ij}(t_k^-)$; $\beta_{ij} x_{ij}(t_k^+) = \lim_{h \to 0^+} \beta_{ij} x_{ij}(t_k + h)$ and $\beta_{ij} x_{ij}(t_k^-) = \lim_{h \to 0^+} \beta_{ij} x_{ij}(t_k - h)$.

With out loss of generality, we assume that the solution of system (3.4) is left continuous, that is, $\beta_{ij}x_{ij}(t_k^-) = \beta_{ij}x_{ij}(t_k)$ and let $Y(t) = (\beta_{ij}x_{ij}(t))^T \in \mathbb{R}^M,\ i = 1, 2, \ldots, N,\ i < j$; $Y_{ij} = \beta_{ij}x_{ij};\ \beta_{ij} \neq 0$, $\mathcal{B} = diag(\beta_{ij}) \in \mathbb{R}^{M\times M}$, $h \in \mathbb{R}^+$. Then, the system (3.4) can be written as the form:

$$\begin{cases} \dot{Y}(t) = h|\mathcal{B}|QY(t), & t \neq t_k, \\ Y(t_k^+) = [\mathbf{I}_M + h|\mathcal{B}|Q']Y(t_k), & k \in \mathbb{N}, \end{cases} \tag{3.5}$$

where h is a step size, I_M is an identity matrix, $|\mathcal{B}| = diag(|\beta_{ij}|)$, Q and Q' are the zero-row-sum symmetric matrices with nonnegative off-diagonal elements and the diagonal elements are $-\{\sum_{s\in\mathcal{N}_i,s\neq j} a_{is}+\sum_{s\in\mathcal{N}_j,s\neq i} a_{js}\}$ at time t and impulsive time t_k, respectively.

Let $\beta_{max} = \max\{|\beta_{ij}|\}$, for $i, j \in \mathcal{I}_N$ and $\Delta = \beta_{max}\cdot max\{\sum_{s\in\mathcal{N}_i,s\neq j} a_{is}+\sum_{s\in\mathcal{N}_j,s\neq i} a_{js}\}$. The following assumptions and lemma are provided in order to obtain our main results:

(A1) $0 < h < \dfrac{1}{\Delta}$,

(A2) there exists a constant $0 < \alpha \leq 1$ such that

$$(1-\alpha)\mathbf{I}_M + h|\mathcal{B}|Q' + h|\mathcal{B}|Q'^T + (h|\mathcal{B}|)^2Q'^TQ' \leq 0.$$

The conditions (A1) and (A2) are critical to the results as they ensure stability and convergence of the scaled consensus protocols. However, satisfying these conditions in real-world applications presents several practical challenges and limitations.

1. **Condition (A1)** imposes an upper bound on the step size h, which depends on the maximum scaling factor β_{ij} and the graph topology (e.g., adjacency matrix and degree distribution). In dense or highly connected networks, the maximum degree can be large, significantly reducing the permissible step size h. This can lead to slower convergence and may necessitate frequent impulsive updates, which increase communication and computational overhead.

2. **Condition (A2)** involves matrix inequalities that depend on the eigenvalues of the system's Laplacian matrix. Computing and verifying these conditions require precise knowledge of the network's topology and weights, which may not always be available or feasible in large-scale or dynamically changing networks. Furthermore, satisfying $(1-\alpha)I + h|B|Q' \leq 0$ can be restrictive in heterogeneous networks where β_{ij} varies significantly.

In real-world applications, such as wireless sensor networks or distributed control systems, communication delays, packet losses, and resource constraints add complexity. For example:

- **Delays and Losses:** Even if h is chosen appropriately, delays in receiving updates can violate stability.

- **Dynamic Topologies:** Changes in the network structure (e.g., node or edge failures) can render the pre-computed conditions invalid.

To address these challenges, adaptive protocols that dynamically adjust h and β_{ij} based on real-time network conditions could be explored. Additionally, decentralized methods that rely on local information rather than global connectivity can improve scalability and robustness.

Lemma 4. *Let L be the Laplacian matrix of a communication network $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ with M nodes. Define $\beta_{max} = \max |\beta_{ij}|$, where β_{ij} be a nonzero scalar scale of edge (i,j). Assume the step size h satisfies **(A1)** and $|\mathcal{B}| = diag(|\beta_{ij}|)$, $(\beta_{ij}) \in \mathbb{R}^M$. Then, $\mathbf{I}_M + h|\mathcal{B}|Q$ is SIA, i.e., $\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M y^T$ if and only if $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ contains a spanning tree. Furthermore, $[\mathbf{I}_M + h|\mathcal{B}|Q]^T y = y$, $\mathbf{1}_M^T y = 1$, where each element of y is nonnegative.*

Proof. **(Sufficiency)** Since $0 < h < (\Delta)^{-1}$ and using the fact that $L = -Q$, one obtains $\mathbf{I}_M + h|\mathcal{B}|Q = \mathbf{I}_M - h|\mathcal{B}|L = (\mathbf{I}_M - h|\mathcal{B}|\mathcal{D}) + h|\mathcal{B}|\mathcal{A}$ is a stochastic matrix with positive diagonal entries, where $\mathcal{D} = diag(d_1, \ldots, d_M)$ and $\mathcal{A}$ are the degree matrix and adjacency matrix of $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$, respectively. Obviously, for all $r, s \in \mathcal{I}_M$; $r \neq s$, the (r,s)th entry of $\mathbf{I}_M - h|\mathcal{B}|L$ is positive if and only if $a_{rs} > 0$. Then, $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is the graph associated with $\mathbf{I}_M - h|\mathcal{B}|L$. Combining Lemma 2 and Lemma 3, gives $\lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|L]^k = \lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M y^T$, when $\mathcal{G}$ has a spanning tree, where y is nonnegative vector. Moreover, y satisfies $[\mathbf{I}_M + h|\mathcal{B}|Q]^T y = y$, $\mathbf{1}_M^T y = 1$.
(Necessary) From Lemma 2, if $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ does not contain a spanning tree, the algebraic multiplicity of eigenvalue $\lambda = 1$ of $\mathbf{I}_M - h|\mathcal{B}|L$ is $m > 1$. Then, the rank of $\lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|L]^k$ is not equal to 1. This implies that

$$\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|L]^k \neq \mathbf{1}_M y^T.$$

Therefore, $I_M + h|\mathcal{B}|Q$ is not SIA.

The following theorem is the scaled consensus results based on the undirected network.

Theorem 1. *Let $\mathcal{G}$ be a communication network of the multi-agent system (2.1) and $\beta_{ij} \neq 0$ be a scalar scale of edge (i,j). Assume that the assumptions **(A1)** and **(A2)** hold. Then, the multi-agent system (2.1) with the protocol (3.1) reaches scaled consensus on edge dynamics if and only if the line graph $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is connected.*

Proof. **(Sufficiency)** Let $Y = (\beta_{ij} x_{ij}) \in \mathbb{R}^M$, where $y_{ij} = \beta_{ij} x_{ij} \in \mathbb{R}$ for all $(i,j) \in \mathcal{E}$. Since $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is connected, one obtains that

$$\bar{y} = Ave(y) = \frac{1}{M} \sum_{(i,j)\in\mathbb{E}} y_{ij} \text{ is invariant quantity } i.e.,$$

$$\bar{y}(t) = \bar{y}(0) = \frac{1}{M} \sum_{(i,j)\in\mathbb{E}} y_{ij}(0).$$

The invariant of $\bar{y}$ allows decomposition of y_{ij} for $(i,j) \in \mathcal{E}$ as in the following equation:

$$\xi_{ij}(t) = y_{ij}(t) - \bar{y}, \quad t \in (t_{k-1}, t_k],$$

$\xi_{ij}(t_k^+) = y_{ij}(t_k^+) - \bar{y}$ and $\xi_{ij}(t_k^-) = \xi_{ij}(t_k)$, for all $(i,j) \in \mathcal{E}$, with initial conditions $y(t_0) = y(0) = (y_{ij}(0))^T$, where $\xi = (\xi_{ij})^T$, $(i,j) \in \mathcal{E}$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \dot{\xi}(t) = |\mathcal{B}|Q\xi(t), & t \neq t_k \\ \xi(t_k^+) = [\mathbf{I}_M + h|\mathcal{B}|Q']\xi(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{3.6}$$

Consider the Lyapunov function candidate as follows:

$$V(\xi) = \xi^T \xi.$$

Let $V(\xi) =: V(\xi(t))$. Since $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is connected, by Lemma 1 [10], we have $\hat{\mathcal{L}} = Sym(L) = (L + L^T)/2$, where $L = -Q$ is the Laplacian matrix of $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$. Hence, by the definition 1, the total derivation of $V(\xi)$ with respect to (3.6) is

$$\begin{aligned} \dot{V}(t) &= \dot{\xi}^T \xi + \xi^T \dot{\xi} \\ &= \xi^T ((|\mathcal{B}|Q)^T + |\mathcal{B}|Q)\xi \\ &= -2\xi^T (|\mathcal{B}|\hat{\mathcal{L}})\xi \\ &\leq -2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})V(t). \end{aligned}$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_{k-1})} V(t_{k-1}^+).$$

On the other hand, when $t = t_{k-1}$, using $(1-\alpha)\mathbf{I}_M + h|\mathcal{B}|Q' + h|\mathcal{B}|Q'^T + (h|\mathcal{B}|)^2 Q'^T Q' \leq 0$, for $0 < \alpha \leq 1$, gives

$$\begin{aligned} V(t_{k-1}^+) &= \xi^T(t_{k-1})(\mathbf{I}_M + h|\mathcal{B}|Q')^T(\mathbf{I}_M + h|\mathcal{B}|Q')\xi(t_{k-1}) \\ &= \xi^T(t_{k-1})[\mathbf{I}_M + h|\mathcal{B}|Q'^T + h|\mathcal{B}|Q' + (h|\mathcal{B}|)^2 Q'^T Q' - \alpha\mathbf{I}_M + \alpha\mathbf{I}_M]\xi(t_{k-1}) \\ &= \xi^T(t_{k-1})[(1-\alpha)\mathbf{I}_M + h|\mathcal{B}|Q'^T + h|\mathcal{B}|Q' + (h|\mathcal{B}|)^2 Q'^T Q']\xi(t_{k-1}) \\ &\quad + \alpha\xi^T(t_{k-1})\xi(t_{k-1}) \\ &\leq \alpha\xi^T(t_{k-1})\xi(t_{k-1}) \\ &= \alpha V(t_{k-1}). \end{aligned}$$

In general, for $t \in (t_{k-1}, t_k]$, we have

$$V(t) \leq \alpha^{k-1} e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0^+)} V(t_0^+).$$

Hence,

$$|\xi(t)| \leq \alpha^{(k-1)/2} e^{-\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)} |\xi(t_0^+)|, t \in (t_{k-1}, t_k].$$

Therefore,

$$\|Y_{ij}(t) - \bar{y}\| \to 0 \;\text{ as }\; t \to \infty \;\text{ or }\; \lim_{t\to\infty} \beta_{ij} x_{ij}(t) = \bar{y}, \quad \forall\ (i,j) \in \mathcal{E}.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|\beta_{ij} x_{ij}(t) - \beta_{kl} x_{kl}(t)\| = 0 \qquad \text{for all } (i,j), (k,l) \in \mathcal{E}.$$

(Necessity) Suppose that $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is not connected. Thus, it does not contain a spanning tree. Then, by Lemma 4, we have $\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q']^k \neq \mathbf{1}_n y^T$. Hence,

$$\lim_{t_k\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{kl}x_{kl}(t)\| \neq 0, \text{ for some } (i,j),(k,l) \in \mathcal{E}.$$

This implies that the multi-agent system (2.1) under the protocol (3.1) cannot achieve scaled consensus.

If the scalar scales of all edge are equal to one, the consensus problem can be solved as in following corollary.

Corollary 1. *Let $\mathcal{G}$ be a communication network of the multi-agent system (2.1). Assume that the following assumptions are satisfied:*

(I) $0 < h < \dfrac{1}{max\{\sum_{s\in\mathcal{N}_i, s\neq j} a_{is} + \sum_{s\in\mathcal{N}_j, s\neq i} a_{js}}$,

(II) *there exists a constant $0 < \alpha \leq 1$ such that*

$$(1-\alpha)\mathbf{I}_M + hQ' + hQ'^T + h^2 Q'^T Q' \leq 0.$$

Then, the multi-agent system (2.1) with the protocol (3.2) reaches consensus on edge dynamics if and only if the line graph $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is connected.

3.2. Directed topology

In this section, scaled consensus problems of edge dynamics via directed networks are investigated. Similar to undirected network, we transform the original nodal graph into its corresponding line graph. Obviously, the line graph $\mathbb{L}(\mathcal{G})$ of a digraph $\mathcal{G}$ is a directed graph, and there are $\sum_{i=1}^{N} d(in)_i$ vertices and $\sum_{i=1}^{N} d(in)_i d(out)_i$ edges in $\mathbb{L}(\mathcal{G})$, where $d(in)_i$ and $d(out)_i$ is the in-degree and out-degree of node i, respectively. In addition, the order pair (i,j) refers to the directed edge from node j to node i, and if the direct edge (i_1, j_i) is the valid neighbor of the directed edge (i_2, j_2) in the original nodal graph, add a new directed edge between the generated nodes i_1, j_i and i_2, j_2, and (i_1, j_i) is the initial node of the new directed edge while (i_2, j_2) is its terminal node. The detailed evolution of a digraph to its line digraph can be seen in [18].

The edge consensus protocol for directed topology can be designed as

$$\begin{aligned} u_{ij}(t) = |\beta_{ij}| &\left[\sum_{s\in\mathcal{N}_j} a_{js}[\beta_{is}x_{is}(t) - \beta_{ij}x_{ij}(t)] \right] \\ &+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{(i,s)\in\mathcal{E}'} a'_{is}[\beta_{is}x_{is}(t) - \beta_{ij}x_{ij}(t)]\delta(t-t_k), \forall (i,j) \in \mathcal{E}, \end{aligned} \tag{3.7}$$

where all variables are defined as in the previous section.

Remark 3. *If $\beta_{ij} = 1$ for all edge (i,j), the protocol (3.7) can be written as*

$$u_{ij}(t) = \sum_{s\in\mathcal{N}_j} a_{js}[x_{is}(t) - x_{ij}(t)] + h \cdot \sum_{k=1}^{\infty} \sum_{(i,s)\in\mathcal{E}'} a'_{is}[x_{is}(t) - x_{ij}(t)]\delta(t - t_k), \forall (i,j) \in \mathcal{E}, \tag{3.8}$$

which can be called as an impulsive protocol and if $\mathcal{E}' = \emptyset$ or there is no instantaneous contact at the sampling time t_k and $\beta_{ij} = 1$ for all edge (i,j), one obtains

$$u_{ij}(t) = \sum_{s\in\mathcal{N}_j} a_{js}[x_{is}(t_k) - x_{ij}(t_k)], \quad \forall (i,j) \in \mathcal{E}, \tag{3.9}$$

which were studied in [18].

Theorem 2. *Consider a directed communication network $\mathcal{G}$ with N nodes and M edges, where the edge dynamics described as in (2.1). Then, the consensus protocol (3.7) solves scaled consensus of edge dynamics if and only if, for any initial state, the following conditions are satisfied:*

(I) the step size h is satisfies

$$0 < h < \frac{1}{max_{i,j}\{\sum_{j\in\mathcal{N}_j, s\neq i} a_{js}\}\beta_{max}}; \tag{3.10}$$

(II) there exists a constant $0 < \alpha \leq 1$ such that

$$(1-\alpha)\mathbf{I}_M + h|\mathcal{B}|W' + h|\mathcal{B}|W'^T + (h|\mathcal{B}|)^2 W'^T W' \leq 0;$$

(III) the line digraph $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is balanced and contains a spanning tree.

Proof. **(Sufficiency)** Assume that the conditions $(I), (II)$ and (III) are satisfied. According to the definition of the Dirac delta function, the system (2.1) under protocol (3.7) can be written as

$$\begin{cases} \beta_{ij}\dot{x}_{ij}(t) = & |\beta_{ij}| \sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t) - \beta_{ij}x_{ij}(t)], \quad t \in (t_{k-1}, t_k), \\ \Delta\beta_{ij}x_{ij}(t_k) & = h|\beta_i| \sum\limits_{s\in\mathcal{N}'_i} a'_{js}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)], \end{cases} \tag{3.11}$$

where $\Delta\beta_{ij}x_{ij}(t_k) = \beta_{ij}x_{ij}(t_k^+) - \beta_{ij}x_{ij}(t_k^-)$; $\beta_{ij}x_{ij}(t_k^+) = \lim\limits_{h\to 0^+} \beta_{ij}x_{ij}(t_k + h)$ and $\beta_{ij}x_{ij}(t_k^-) = \lim\limits_{h\to 0^+} \beta_{ij}x_{ij}(t_k - h)$.

With out loss of generality, we assume that the solution of system (3.11) is left continuous, that is, $\beta_{ij}x_{ij}(t_k^-) = \beta_{ij}x_{ij}(t_k)$ and let $Y(t) = (\beta_{ij}x_{ij}(t))^T \in \mathbb{R}^M$, $i,j = 1,2,\ldots,N$, $i < j$; $y_{ij} = \beta_{ij}x_{ij}$; $\beta_{ij} \neq 0$, $\mathcal{B} = diag(\beta_{ij}) \in \mathbb{R}^{M\times M}$, $h \in \mathbb{R}^+$. Then, the system (3.11) can be written as the form:

$$\begin{cases} \dot{Y}(t) = h|\mathcal{B}|WY(t), & t \neq t_k, \\ Y(t_k^+) = [\mathbf{I}_M + h|\mathcal{B}|W']Y(t_k), & k \in \mathbb{N}, \end{cases} \tag{3.12}$$

where h is a step size, I_M is an identity matrix, $|\mathcal{B}| = diag(|\beta_{ij}|)$, W and W' are the zero-row-sum symmetric matrices with nonnegative off-diagonal elements and the diagonal elements are $-\sum_{s\in\mathcal{N}_j,s\neq i} a_{js}$ at time t and impulsive time t_k, respectively. Since $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$ is balanced and contains a spanning tree, one obtains that $\bar{\mu} = \dfrac{1}{M}\displaystyle\sum_{(i,j)\in\mathbb{E}} y_{ij}$ is invariant quantity.

Therefore,

$$\eta_{ij}(t) = y_{ij}(t) - \bar{\mu}, \quad t \in (t_{k-1}, t_k],$$

$\eta_{ij}(t_k^+) = y_{ij}(t_k^+) - \bar{\mu}$ and $\eta_{ij}(t_k^-) = \eta_{ij}(t_k)$, for all $(i,j) \in \mathcal{E}$, with initial conditions $y(t_0) = y(0) = (y_{ij}(0))^T$, where $\eta = (\eta_{ij})^T$, $(i,j) \in \mathcal{E}$ is an error vector. Thus,

$$\begin{cases} \dot{\eta}(t) = |\mathcal{B}|W\eta(t), & t \neq t_k \\ \eta(t_k^+) = [\mathbf{I}_M + h|\mathcal{B}|W']\eta(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{3.13}$$

Consider the Lyapunov function candidate as follows:

$$V(\eta) = \eta^T\eta.$$

Similar to the proof of Theorem 1, by using the fact that $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$ is balanced and contains a spanning tree together with condition $(I), (II)$ and Lemmas, we can prove that

$$\|Y_{ij}(t) - \bar{\mu}\| \to 0 \;\text{ as }\; t \to \infty \;\text{ or }\; \lim_{t\to\infty} \beta_{ij}x_{ij}(t) = \bar{\mu}, \quad \forall\,(i,j) \in \mathcal{E}.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{kl}x_{kl}(t)\| = 0 \qquad \text{for all } (i,j),(k,l) \in \mathcal{E}.$$

(Necessity) Suppose that $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$ does not contain a spanning tree. Then, by Lemma 4, we have $\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|W']^k \neq \mathbf{1}_n y^T$. Hence,

$$\lim_{t_k\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{kl}x_{kl}(t)\| \neq 0, \text{ for some } (i,j),(k,l) \in \mathbb{E}.$$

This implies that the multi-agent system (2.1) under the protocol (3.7) cannot achieve scaled consensus.

If the scalar scales of all edges are equal to one, then the consensus problem is solved as in following corollary.

Corollary 2. *Consider a directed communication network $\mathcal{G}$ with N nodes and M edges, where the edge dynamics described as in (2.1). Then, the consensus protocol (3.8) solves consensus of edge dynamics if and only if, for any initial state, the following conditions are satisfied:*

(I) *the step size h satisfies*

$$0 < h < \frac{1}{max_{i,j}\{\sum_{j\in\mathcal{N}_j,s\neq i} a_{js}\}}; \tag{3.10}$$

(II) *there exists a constant $0 < \alpha \leq 1$ such that*

$$(1-\alpha)\mathbf{I}_M + hW' + hW'^T + h^2W'^TW' \leq 0;$$

(III) *the line digraph $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$ is balanced and contains a spanning tree.*

4. Simulations and discussion

In order to demonstrate the effectiveness of the theoretical results in this work, the following example is provided.

Example 1. *Consider a communication network of* 8 *agents denoted by* $1-8$*, where the dashed line refers to a communication at the impulsive time* t_k*. Then, the communications* $\mathcal{G}\cup\mathcal{G}'$ *can be described as in Figure 1.*

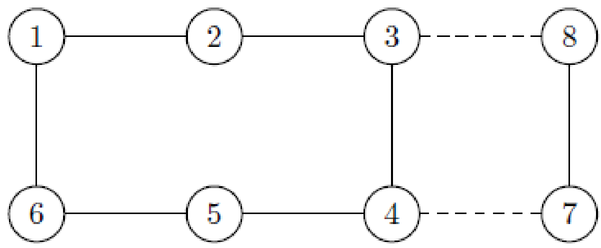

Figure 1: The communication net work $\mathcal{G}\cup\mathcal{G}'$.

In order to solve edge consensus problems, we first transform the graph $\mathcal{G}\cup\mathcal{G}'$ to its line graph, using the same methodology as [14], denoted by $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$, as shown in Figure 2. It can be seen that $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$ has 9 nodes, which are $x_{12}, x_{16}, x_{23}, x_{34}, x_{38}, x_{45}, x_{47}, x_{56}, x_{78}$ and 12 edges.

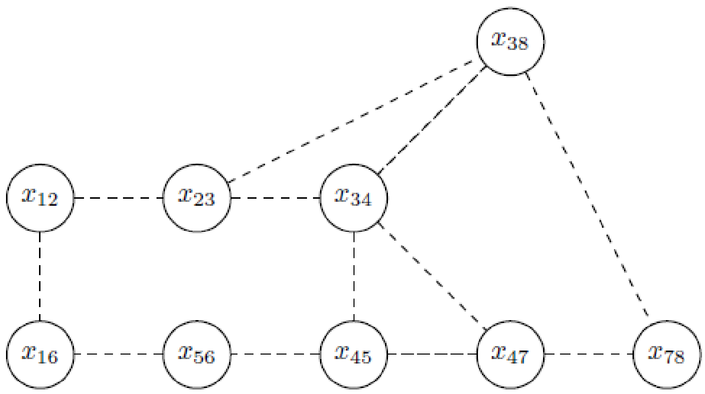

Figure 2: The communication network of $\mathbb{L}(\mathcal{G}\cup\mathcal{G}')$.

According to the communication network $\mathcal{G} \cup \mathcal{G}'$, the adjacency matrix of $\mathcal{G} \cup \mathcal{G}'$ is denoted by $\mathcal{A}$ and the Laplacian matrix of its line graph is $-Q$, where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix}.$$

Let $(\beta_{ij}) = (0.75, 2, -2, 1, -1, 1.5, -1.5, 0.5, -0.5)^T \in \mathbb{R}^9$ be the scalar scales of edges and $(x_{ij}(t_0)) = (2.5, 1, -1, 0.5, -0.5, 1.5, -1.5, -2, 2)^T \in \mathbb{R}^9$ be the initial values of edges. By choosing $h = 0.125$, one obtains that the condition **(A1)** holds. Moreover, by using MATLAB, it can be seen that the condition **(A2)** is satisfied. Since the line graph $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is connected, theorem 1 can guarantee reaching scaled consensus of edge dynamics under protocol (3.1) and the state trajectories of all edge can be described as in Figure 3.

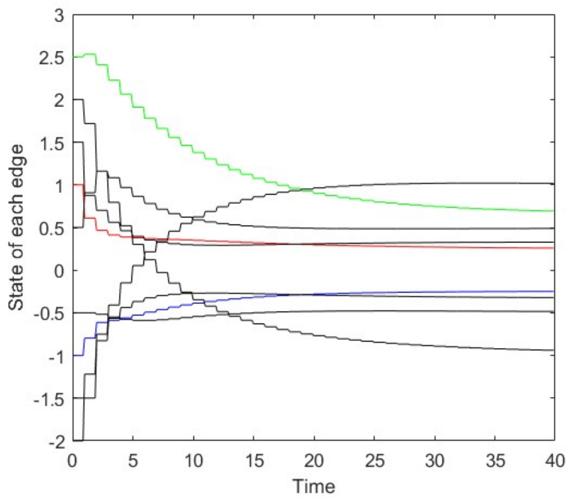

Figure 3: The state trajectory of each edge using protocol(3.1) with $h = 0.125$ and $(\beta_{ij}) = (0.75, 2, -2, 1, -1, 1.5, -1.5, 0.5, -0.5)^T$.

In addition, when $(\beta_{ij}) = (1,1,1,1,1,1,1,1,1)^T$ the scaled consensus problems are the usual consensus problems (see Figure 4).

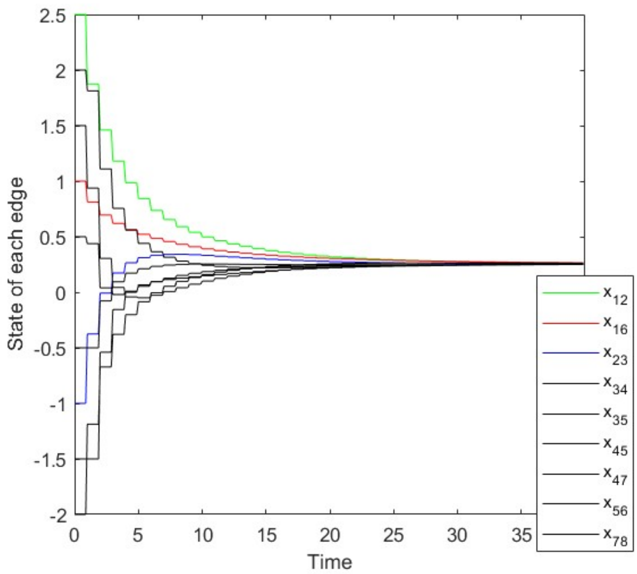

Figure 4: The state of each edge under protocol (3.1) with $(\beta_{ij}) = (1,1,1,1,1,1,1,1,1)^T$ and $h = 0.125$.

However, if h is not satisfied the condition **(A1)**, the protocol (3.1) cannot guarantee reaching scaled consensus problem as shown in Figure 5.

It can be seen that protocols (3.1) and (3.7) can solve scaled consensus problems if the step size is small enough satisfying **(A1)**. In addition, our results demonstrate that the scaled consensus problems on edge dynamics are solved if the communication network $\mathbb{L}(\mathcal{G} \cup \mathcal{G}')$ is balanced and contains a spanning tree (see Figure 3). On the other hand, if the sampling period h is not satisfied the condition **(A1)**, edge consensus cannot guarantee (see Figure 5). Moreover, if the scalar scale $\beta_{ij} = 1$, the simulations results (see Figure 4) show the effectiveness and generalization of our theorems compared to the results of [14, 18].

The paper's theoretical contributions offer practical advancements in domains such as autonomous vehicles and distributed energy systems, where scaled consensus protocols can enhance both efficiency and reliability. By dynamically adjusting proportional metrics and reducing communication demands, these protocols provide robust solutions for complex, dynamic, and heterogeneous environments.

In autonomous vehicles, scaled consensus enables dynamic inter-vehicle distance adjustments based on individual capabilities like braking power or acceleration limits, improving safety and traffic efficiency. The impulsive protocol's ability to minimize communication frequency proves valuable in areas with intermittent connectivity, such as dense urban networks or rural highways. Similarly, in distributed energy systems, scaled consen-

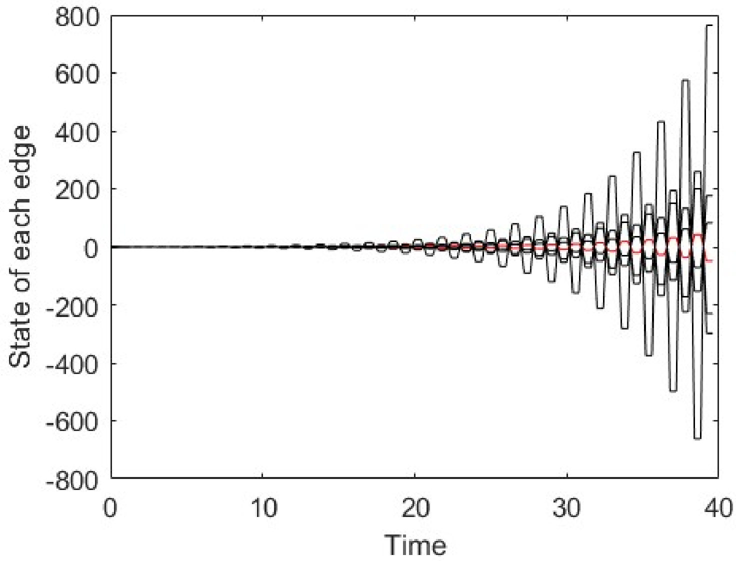

Figure 5: The state trajectories of all edges using the consensus protocol (3.1) with $h = 0.3$ and $(\beta_{ij}) = (0.75, 2, -2, 1, -1, 1.5, -1.5, 0.5, -0.5)^T$.

sus facilitates proportional power distribution among resources like solar panels or wind turbines, optimizing load balancing and efficiency. By reducing communication demands, the approach ensures stable operation even during peak loads or in resource-constrained networks. These methods also extend to applications in multi-robot systems and sensor networks, where their scalability and adaptability support efficient task allocation and data aggregation.

5. Conclusion

This paper presented impulsive protocols for scaled consensus in edge-dynamic multi-agent systems. Using graph theory, matrix analysis, and Lyapunov stability, we derived sufficient conditions for achieving scaled consensus under directed and undirected topologies, demonstrating robustness and efficiency in handling edge dynamics while ensuring convergence to desired proportional ratios.

Numerical simulations validated the theoretical results, showcasing the protocols' effectiveness in reducing communication overhead and accelerating convergence compared to continuous-time methods. These findings underline the advantages of impulsive protocols in dynamic and heterogeneous network environments.

The proposed protocols have significant real-world applications. In autonomous vehicles, they ensure proportional inter-vehicle distances for safer and more efficient traffic flow. In distributed energy systems, they balance power generation and consumption across resources proportionally, optimizing energy utilization in smart grids. Additional applications include robotics, sensor networks, and traffic management, where their scalability and adaptability provide practical advantages.

Future research will explore extensions to weakly connected graphs, highly dynamic topologies, and adaptive mechanisms for real-time parameter tuning. Real-world implementations in autonomous systems and smart grids will further validate their practicality and impact, establishing a foundation for broader adoption in complex multi-agent systems.

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Declarations

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Conflict of interest

The authors declare that they have no competing interests.

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