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Analytic Solutions of the Time-Fractional Date-Jimbo-Kashiwara-Miwa Equation via New Function Transformations

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Abstract. In this paper, a new analytical framework for solving the fundamental nonlinear model, the time-fractional Date-Jimbo-Kashiwara-Miwa equation, is proposed. The time-fractional Date-Jimbo-Kashiwara-Miwa equation is made simpler by reducing it to an ordinary differential equation through the Power Index Method's multiplication of variables x and t. Because it permits a change in variables, which can uncover a hidden pattern in the equation, multiplying x and t is significant. The original equation's terms may be removed or their complexity decreased with the aid of this transformation. The benefits of various variable transformations vary depending on the particular issue, but this transformation has the advantage of simplifying the equation, which makes it simpler to solve, analyze, and produce precise and explicit solutions. Both rational and logarithm functions are present in the solutions that were obtained. Through 3D visualizations of the general solutions, our method offers a deeper comprehension of the dynamics of the equation. The behavior of the solution to the time-fractional Date-Jimbo-Kashiwara-Miwa equation is depicted in the paper's 3D visualization. The solitons and nonlinear wave solutions are depicted in each plot. Our findings show the effectiveness of the Power Index Method in this situation and highlight its capacity to address challenging issues.

2020 Mathematics Subject Classifications: 35Qxx, 26A33

Key Words and Phrases: Analytic solution, Euler's second order ODE, Time-fractional derivative, Fractional calculus, Traveling wave transformation

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1. Introduction

In computational chemistry, particle physics, plasma physics, and other fields, nonlinear partial differential equations (NPDEs) have been used as models to explain nonlinear

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physical phenomena. The study of exact or analytical solutions has received a lot of attention because it is crucial to the examination of the model physical characteristics. Different wave types, such as solitary waves, optical solutions, singular solutions, periodic waves, breather waves, rogue waves, and rational wave solutions, can be created using analytical techniques. Investigating techniques for solving fractional nonlinear partial differential equations (PDEs) is necessary to better understand the dynamics of these real-world components. This investigation is necessary to learn more about and comprehend the intricate behaviors that are present in the systems indicated above. The greater richness and generality that fractional nonlinear PDE solutions offer, which surpasses classical solutions in terms of descriptive power, makes them of academic interest [25], [8], [9], [21], [19], [18]. Mathematicians have created and applied creative and trustworthy methods to explore novel discoveries. The generalized Khater method, generalized Kudryashov method, fractional sub-equation method, modified extended direct algebraic approach, (G'/G)-expansion method, and simple equation method are a few of these techniques. Numerous methods have been employed in recent research to look into solutions [17], [2], [20], [14], [22], [16].
In mathematics, analytic solutions often serve as a foundation for further analysis. However, when analytic solutions are not feasible. We discuss alternative approaches like: asymptotic expansions, perturbation theory, bifurcation analysis, numerical methods (e.g finite element, finite difference etc). These approaches can provide valuable insights into solution properties, such as: stability and instability, bifurcation and pattern formation, long-time behavior and attraction, symmetries and conservation law.

The time-fractional DJKM equation is a significant extension of the classical DJKM equation, incorporating fractional derivatives to model complex phenomena in physics, mathematics and engineering. Despite the existence of various fractional derivatives, our work focuses on the conformable fractional derivative due to its uniqueness. The motivation for this work is multifaceted:
(1) The increasing importance of fractional derivative in modeling real-world phenomena
(2) The need for analytic solutions to understand the underlying dynamics of the time-fractional DJKM equation
By addressing these motivations, our work provides a comprehensive analysis of the time-fractional DJKM equation and contributes to the growing field of fractional derivatives

DJKM equation was firstly presented by Kadomtsev and Petviashvili so as to study the stability of the KdV soliton [13]. Later, some important properties of DJKM have been investigated in [5], [24], [11] and [15]. In 2020, Wazwaz have observed the Painleve integrability and multiple soliton solutions by getting variable coefficient in [29]. In 2021, Khudija and Khalil applied Lie symmetry transformation on DJKM and reduced into linear PDEs [3].
In paper [10], the CDJKM equation has the following form [12], [7].

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \gamma u_{yyy} - 2\rho \frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = 0 \tag{1}$$

where $u = u(x, y, t)$ represents the physical quantity of wave amplitude and the subscripts indicate partial differentiation with regard to the specified variables whereas γ, ρ are constants, and $0 < \alpha \leq 1$ shows wave behavior of the solitions. This definition is widely used in different fields, including physics, engineering, and mathematics, and has been shown to be effective in modeling complex phenomena. This definition is chosen for its simplicity and applicability to many problem.

A lot of studies have been presented to seek the solutions of equation (1). Wang and Hu [6] have derived the Grammian solutions of equation (1). Guo and Lin [5] have studied interaction solutions between lump and stripe soliton solutions via a quadratic function. Adem et al. [30] have used the extended transformed rational function that depends on the Hirota bilinear form to constructed Complexion solutions of the DJKM equation. Yuan et al. [27] have studied Wronskian and Grammian solutions to the DJKM equation. Singh and Gupta [1] have investigated the Painleve property of the suggested equation and have revealed some exact solutions to the studied equation by using the Pickering's algorithm. Sajid and Akram [24] have utilized $\exp(\phi(\xi))$-expansion method to seek some exact solutions to equation (1).

In recent years, the concept of fractional derivatives has been applied with great success to model various real-life phenomena in many scientific fields. Fractional order operators include the history of a physical phenomenon from the initial state to the current state. Therefore, fractional order operators are often applied to model systems that describe the influence of memory effects in [23], [26], [4] and [28].

Function transformation is a mathematical technique that uses a specific transformation to convert the time-fractional DJKM equation into a more tractable form, enabling us to derive analytical solutions using various methods, such as Power Index Method. We have derived closed-form analytical solutions for the time-fractional DJKM equation, which reveal the dynamics of wave propagation and dispersion in terms of fractional derivatives, providing new insights into the behavior of complex systems. Our results can be applied in various fields, such as signal processing, image processing, and optical communications, to model and analyze complex wave phenomena, leading to improved performance and enhanced understanding of these systems.

The purpose of the paper is to develop a novel analytical framework for solving nonlinear Partial differential equations using function transformation, focusing on the time-fractional DJKM equation. We address previously unsolved problems in nonlinear PDEs, are deriving exact solutions for nonlinear wave equations with fractional time derivatives. Overcoming limitations of existing methods in handling nonlinear wave dynamics. Our research has significant practical implications for accurate modeling and simulation of nonlinear wave phenomena in physics, engineering and optics. Enhanced understanding of nonlinear wave interactions and their role in complex systems. Informing the development of new numerical methods and analytical tools for nonlinear PDEs. There are many other definitions of fractional derivatives in different ways such as Caputo frac-

tional derivative, Riemann liouville fractional derivative and Grunwald-letnikov fractional derivative. Recently, Khalil et al. have introduced a new, straightforward definition of the fractional derivative called the conformable fractional derivative with the limit operator and he made the most significant contributions to Fractioanl derivative.
Remark 1 "We use of polynomial fractional derivatives as

$$D_t^{\alpha}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} t^{s-\alpha} \tag{2}$$

All the fractional derivatives applying on polynomial have same results".
This paper is divided as follows: In Section 2, Power Index Method (PIM) is introduced. In Section 3, new function transformations are applied to the DJKM equation. Section 4 are devoted to results and discussion. Finally conclusion is demonstrated in Section 5.

2. Power Index Method

Step:-1 Considering the PDE (1) and we want to show its exact solution, we introduce the variable ξ as;

$$\xi = x^m t^r$$

and transformation of general form is;

$$u(x,t) = x^n t^s f(\xi)$$

Since ξ and u depend only x and t, so all terms vanish except last term.
Step:-2 By using fractional derivatives of polynomial as

$$D_t^{\alpha}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} t^{s-\alpha}$$

we can find

$$D_t^{\alpha} u = D_t^{\alpha}(x^n t^s f(\xi)) = x^n \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} t^{s-\alpha} f(\xi) + x^n t^s D_t^{\alpha} f(\xi)$$

$$= x^n \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} t^{s-\alpha} f(\xi) + x^{m+n} t^{s+r-\alpha} \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} f'(\xi)$$

The indexes of x and t in every term may be defined the following form

$$p_x = a_1 m + a_2 n$$

$$p_t = b_1 r + b_2 s$$

where a_i, b_i, $i = 1, 2$ are constants that can be come by taking fractional derivatives. We observe coefficients of p_x and p_t so that the PDE may be transformed to ODE. The best optimal indexes of independent variable x and t are chosen in such away that only two indexes vary at time and others are fixed constants. We continue this process with different

indexes of x and t so that we find all well-defined transformations.
Step:-3 By using the analytic solution of ODE and transformation, we can easily find the relation between x and ξ in each term as;
The indexes of x, t in terms containing f and f' are;

$$p_x(f) = n, \qquad p_x(f') = m + n$$

$$p_t(f) = s - \alpha, \qquad p_t(f') = s + r - \alpha$$

Since $\xi = x^m t^n$, so the values of $p_x(f)$ and $p_x(f')$ must be multiplies of m and similarly, the values of $p_t(f)$ and $p_t(f')$ must be multiplies of r.
Step:-4 We select out a few members of the family of indexes of x and ξ. Further, we

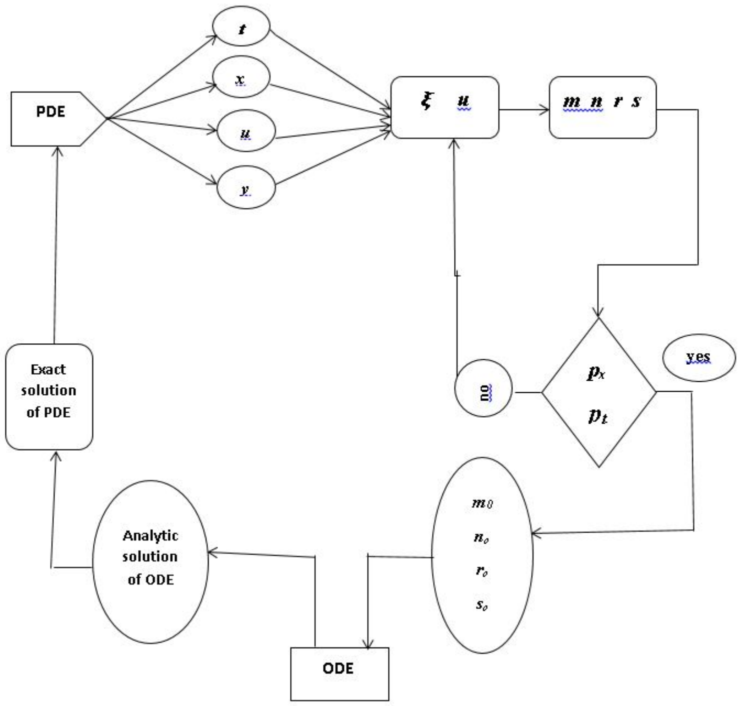

Figure 1: "Flow Chart of Power Index Method which shows how we transition from a PDE to an ODE, and subsequently obtain analytical solutions. Furthermore, when we substitute the soultion back into the original solution PDE, it satisfies the equation, validating our approach".

get different members of Power Indexes in order that we will select unique values of Power Indexed. We retain the same technique for other variables. Now, we have got selected a new variable and transformation with changeable indexes.
Next each term is changed by using a new variable and transformation. Our objective is concentrated PDE into ODE

$$R(\xi, f', f'', ...) = 0 \tag{3}$$

where R is polynomial of $f(\xi)$ and its derivatives.
By using Power Index Method, we can gain a deeper understanding of power and influence within various contexts, and develop more effective strategies for promoting positive

change.
Step:-5 Solve the ODE (3) by using computerized symbolic package like Maple. If we get exact solution of the ODE then expressing this solution by using new variable and transformation.

3. Analytic solutions of Date-Jimbo-Kashiwara-Miwa equation by using new function transformations

Case 1:-
We choose function transformation

$$\xi = x^{2r}t^{2n} \tag{4}$$

$$u = x^{\beta} f(\xi) \tag{5}$$

Using time-fractional derivative, we have

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)} x^{\beta+2r} t^{2n-\alpha} f'(\xi) \tag{6}$$

$$\frac{\partial}{\partial x}(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}) = \frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)} t^{2n-\alpha}\left(2rt^{2n}x^{\beta+4r-1}f'' + x^{\beta+2r-1}(\beta+2r)f'\right) \tag{7}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}) &= \frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)} t^{2n-\alpha}\frac{\partial}{\partial x}\left(2rt^{2n}x^{\beta+4r-1}f'' + x^{\beta+2r-1}(\beta+2r)f'\right)\\ &= \frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)} t^{2n-\alpha}\Bigg(4r^2t^{4n}x^{\beta+6r-2}f''' + 2rt^{2n}x^{\beta+4r-2}(2\beta+6r-1)f''\\ &\qquad +(\beta+2r)(\beta+2r-1)x^{\beta+2r-2}f'\Bigg)\end{aligned} \tag{8}$$

Using (6), (7) and (8) the PDE (1) can be expressed in simplified as;

$$4r^2t^{4n}x^{4r}f''' + 2rt^{2n}x^{2r}(3+2r)f'' + 2f' = 0 \tag{9}$$

Since $\xi = x^{2r}t^{2n}$, so (9) reduces to ODE. For simplification, we take $V = f'$ then (9) becomes

$$\xi^2 V'' + \xi(\frac{2\beta+6r-1}{2r})V' + \frac{\beta^2+4\beta r+4r^2-\beta-2r}{4r^2}V = 0 \tag{10}$$

It is easy to see that (10) is Euler second order linear ODE. If we choose $V = \xi^k, V' = k\xi^{k-1}$ and $V'' = k(k-1)\xi^{k-2}$ then the characteristic equation of (10) is

$$k(k-1) + k(\frac{2\beta+6r-1}{2r}) + \frac{\beta^2+4\beta r+4r^2-\beta-2r}{4r^2} = 0 \tag{11}$$

The roots of equation (11) are

$$k_1 = \frac{1-2\beta-4r}{2r} + \frac{\sqrt{(2\beta+4r-1)^2 - 4\beta(\beta+4r-1) + 2r(2r-1)}}{2r}$$

$$k_2 = \frac{1-2\beta-4r}{2r} - \frac{\sqrt{(2\beta+4r-1)^2 - 4\beta(\beta+4r-1) + 2r(2r-1)}}{2r} \tag{12}$$

Hence the analytic solution of ODE (10) is

$$f' = V = c_1\xi^{k_1} + c_2\xi^{k_2}$$

By replacing $V = f'$ and taking integral on both sides, we get

$$f(\xi) = c_1\frac{\xi^{k_1+1}}{k_1+1} + c_2\frac{\xi^{k_2+1}}{k_2+1} + c_3 \tag{13}$$

Using transformations (4), (5) and analytic solution of ODE (13), we get exact solution of PDE (1) which is

$$u = c_1x^{\beta}\frac{(x^{2r}t^{2n})^{k_1+1}}{k_1+1} + c_2x^{\beta}\frac{(x^{2r}t^{2n})^{k_2+1}}{k_2+1} + c_3x^{\beta}$$

$$u = c_1\frac{x^{\beta+2rk_1+2r}t^{2nk_1+2n}}{k_1+1} + c_2\frac{x^{\beta+2rk_2+2r}t^{2nk_2+2n}}{k_2+1} + c_3x^{\beta} \tag{14}$$

where k_1 and k_2 are given in (12). When $\beta = 0.5$ it become rapidly change in the wave amplitude.

Case 2:-

$$\xi = xt^m \tag{15}$$

$$u = x^{\beta}t^n f(\xi) \tag{16}$$

Using time-fractional derivative, we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{\beta+1}t^{n+m-\alpha}f'(\xi) + \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{\beta}t^{n-\alpha}f(\xi) \tag{17}$$

$$\frac{\partial}{\partial x}(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}) = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}t^{n+m-\alpha}\left(t^m x^{\beta+1}f'' + x^{\beta}(\beta+1)f'\right)$$
$$+\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}\left(\beta x^{\beta-1}t^m f' + \beta(\beta-1)f + x^{\beta}t^{2m}f'' + \beta x^{\beta-1}t^m f'\right) \tag{18}$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}) = m_1t^{3m}x^3f''' + (2m_1(\beta+1)+n_1)t^{2m}x^{\beta}(2)f'' +$$
$$\beta(m_1(\beta+1)+2n_1)xt^m f' + n_1\beta(\beta-1)f = 0, \tag{19}$$

where $m_1 = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}$ and $n_1 = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}$. Using (17), (19) and $\xi = xt^m$, the PDE (1) can be expressed as;

$$m_1\xi^3 f''' + (2m_1(\beta+1)+n_1)\xi^2 f'' + \beta(m_1(\beta+1)+2n_1)\xi f' + n_1\beta(\beta-1)f = 0 \tag{20}$$

It is easy to see that (20) is Euler second order linear ODE. If we choose $f = \xi^k, f' = k\xi^{k-1}, f'' = k(k-1)\xi^{k-2}$ and $f''' = k(k-1)(k-2)\xi^{k-3}$ then the characteristic equation is

$$m_1k(k-1)(k-2)+k(k-1)(2m_1(\beta+1)+n_1)+\beta(m_1(\beta+1)+2n_1)k+n_1\beta(\beta-1) = 0 \tag{21}$$

The roots of equation (21) are $k = -\beta$, $k = 1-\beta$ and $k = -\frac{n_1}{m_1}$.
Hence the analytic solution of ODE (20) is

$$f(\xi) = c_1\xi^{-\beta} + c_2\xi^{1-\beta} + c_3\xi^{-\frac{n_1}{m_1}} \tag{22}$$

Using transformations (15), (16) and analytic solution of ODE (22), we get exact solution of PDE (1) which is

$$u = x^\beta t^n(c_1x^{-\beta}t^{-\beta m} + c_2x^{-\beta+1}t^{(1-\beta)m} + c_3x^{\frac{-\beta n_1}{m_1}}t^{\frac{-n_1 m}{m_1}}) \tag{23}$$

When β becomes more decrease then wave amplitude become frequent.
Case 3:-
If we choose $2n = 1$and $2r = \alpha$ in Case 1

$$\xi = x^\alpha t \tag{24}$$

$$u = x^\beta f(\xi) \tag{25}$$

Using time-fractional derivative, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{\alpha+\beta}t^{1-\alpha}f' \tag{26}$$

$$\frac{\partial}{\partial x}(\frac{\partial^\alpha u}{\partial t^\alpha}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\left(\alpha t x^{2\alpha+\beta-1}f'' + (\alpha+\beta)x^{\alpha+\beta-1}f'\right) \tag{27}$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^\alpha u}{\partial t^\alpha}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\frac{\partial}{\partial x}\left(\alpha t x^{2\alpha+\beta-1}f'' + (\alpha+\beta)x^{\alpha+\beta-1}f'\right)$$

$$= \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\Big(\alpha^2 t x^{3\alpha+\beta-2}f''' + \alpha(2\alpha+\beta-1)tx^{2\alpha+\beta-2}f'' + \alpha(\alpha+\beta)tx^{2\alpha+\beta-2}f''$$

$$+(\alpha+\beta)(\alpha+\beta-1)x^{\alpha+\beta-2}f'\Big) \tag{28}$$

Using (26), (27) and (28) the PDE (1) can be expressed as;

$$\alpha^2(x^\alpha t)^2 f''' + \alpha(x^\alpha t)(2\alpha\beta + 3\alpha^2 - \alpha)f'' + (\beta(\beta-1) + 2\alpha\beta + \alpha^2 - \alpha)f' = 0 \tag{29}$$

Since $\xi = x^{\alpha}t$, so (29) reduces to ODE. For simplification, we take $V = f'$ then (29) becomes

$$\xi^2 V'' + \xi\bigg(\frac{2\beta + 3\alpha - 1}{\alpha}\bigg)V' + \bigg(\frac{\beta(\beta-1) + 2\alpha\beta + \alpha(\alpha-1)}{\alpha^2}\bigg)V = 0 \tag{30}$$

It is easy to see that (30) is Euler second order linear ODE.
If we choose $V = \xi^k, V' = k\xi^{k-1}$ and $V'' = k(k-1)\xi^{k-2}$
then the characteristic equation is

$$k(k-1) + k\bigg(\frac{2\beta + 3\alpha - 1}{\alpha}\bigg) + \frac{\beta(\beta-1) + 2\alpha\beta + \alpha(\alpha-1)}{\alpha^2} = 0 \tag{31}$$

The roots of equation (31) are $k_1 = \frac{-\beta+\alpha-1}{\alpha}$ and $k_2 = -(\frac{\beta+\alpha}{\alpha})$
Hence the analytic solution of ODE (30) is

$$f' = V = c_1\xi^{\frac{-\beta+\alpha-1}{\alpha}} + c_2\xi^{-(\frac{\beta+\alpha}{\alpha})}$$

By replacing $V = f'$ and taking integral on both sides, we get

$$f(\xi) = c_1\frac{\xi^{k_1+1}}{k_1+1} + c_2\frac{\xi^{k_1+1}}{k_2+1} + c_3 \tag{32}$$

Using transformations (24), (25) and analytic solution of ODE (32), we get exact solution of PDE (1) which is

$$u = x^{\beta}\bigg(c_1\frac{(x^{\alpha}t)^{k_1+1}}{k_1+1} + c_2\frac{(x^{\alpha}t)^{k_2+1}}{k_2+1} + c_3\bigg) \tag{33}$$

When $\beta = 1.5$ then small oscillation in wave amplitude are observed.
Case 4:-
We choose function transformation

$$\xi = x^m t \tag{34}$$

$$u = x^n f(\xi) - x^n sint \tag{35}$$

Using time-fractional derivative, we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{m+n}t^{1-\alpha}f' - x^n sin\big(t + \frac{\alpha\pi}{2}\big) \tag{36}$$

$$\frac{\partial}{\partial x}(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\bigg(mtx^{2m+n-1}f'' + (m+n)x^{m+n-1}f' - nx^{n-1}sin\big(t + \frac{\alpha\pi}{2}\big)\bigg) \tag{37}$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\frac{\partial}{\partial x}\bigg(mt^{2m+n-1}f'' + (m+n)x^{m+n-1}f' - nx^{n-1}sin\big(t + \frac{\alpha\pi}{2}\big)\bigg)$$

$$= \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \Big(m^2 t^2 x^{3m+n-2} f''' + m(2m+n-1) t x^{2m+n-2} f'' + m(m+n) t x^{2m+n-2} f''$$
$$+ (m+n)(m+n-1) x^{m+n-2} f' - n(n-1) x^{n-2} sin\big(t + \frac{\alpha\pi}{2}\big) \Big) \tag{38}$$

Using (36), (37) and (38) into the PDE (1). To convert our equation (38) into ODE we have to take the only value $n = 1$, otherwise we are unable to convert ODE.

$$m^2 x^{2m} t^2 f''' + x^m t(3m^2 + m) f'' + m(m+1) f' = 0 \tag{39}$$

Since $\xi = x^m t$, so (39) reduces to ODE. For simplification, we take $V = f'$ then (39) becomes

$$\xi^2 V'' + \xi \left(\frac{3m+1}{m} \right) V' + \left(\frac{m+1}{m} \right) V = 0 \tag{40}$$

It is easy to see that (40) is Euler second order linear ODE.
If we choose $V = \xi^k, V' = k\xi^{k-1}$ and $V'' = k(k-1)\xi^{k-2}$
then the characteristic equation is

$$k(k-1) + k\left(\frac{3m+1}{m} \right) + \frac{m+1}{m} = 0 \tag{41}$$

The roots of equation (41) are $k = -1$ and $k = -1 - \frac{1}{m}$
Hence the analytic solution of ODE (40) is

$$f' = V = c_1 \xi^{-1} + c_2 \xi^{-1-\frac{1}{m}}$$

By replacing $V = f'$ and taking integral on both sides, we get

$$f(\xi) = c_1 ln(\xi) - m c_2 \xi^{-\frac{1}{m}} + c_3 \tag{42}$$

Using transformations (34), (35) and analytic solution of ODE (42), we get exact solution of PDE (1) which is

$$u = x^n \left(c_1 ln(x^m t) - \frac{m c_2 t^{-\frac{1}{m}}}{x} + c_3 \right) - x^n sint \tag{43}$$

When $\beta = 2.5$ then it become rapidly changing in the wave amplitude.
Case 5:-
We choose function transformation

$$\xi = e^{ax} t \tag{44}$$

$$u = e^{-bx} f(\xi) \tag{45}$$

Using time-fractional derivative, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} e^{(a-b)x} f' \tag{46}$$

$$\frac{\partial}{\partial x}(\frac{\partial^\alpha u}{\partial t^\alpha}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\left(ate^{(2a-b)x}f'' + (a-b)e^{(a-b)x}f'\right) \tag{47}$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^\alpha u}{\partial t^\alpha}) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\frac{\partial}{\partial x}\left(ate^{(2a-b)x}f'' + (a-b)e^{(a-b)x}f'\right)$$

$$= \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\left(a^2t^2e^{(3a-b)x}f''' + (3a^2-2ab)te^{(2a-b)x}f'' + (a-b)^2e^{(a-b)x}f'\right) \tag{48}$$

Using (46), (47) and (48) into the PDE (1). To convert our equation (48) into ODE we divide a^2e^{ax}

$$a^2e^{2ax}t^2f''' + (3a^2-2ab)e^{ax}tf'' + (a-b)^2f' = 0 \tag{49}$$

Since $\xi = x^m t$, so (49) reduces to ODE. For simplification, we take $V = f'$ then (49) becomes

$$\xi^2V'' + \left(\frac{3a-2b}{a}\right)\xi V' + \frac{(a-b)^2}{a^2}V = 0 \tag{50}$$

It is easy to see that (50) is Euler second order linear ODE.
If we choose $V = \xi^k, V' = k\xi^{k-1}$ and $V'' = k(k-1)\xi^{k-2}$
then the characteristic equation is

$$k(k-1) + \left(\frac{3a-2b}{a}\right)k + \frac{(a-b)^2}{a^2} = 0 \tag{51}$$

The roots of equation (51) are real and repeated i-e $k = -1 + \frac{b}{a}$ and $k = -1 + \frac{b}{a}$
Hence the analytic solution of ODE (50) is

$$f' = V = (c_1 + c_2 ln\xi)\xi^{-1+\frac{b}{a}}$$

By replacing $V = f'$ and taking integral on both sides, we get

$$f(\xi) = (c_1 + c_2 ln\xi)\xi^{\frac{b}{a}} + c_3 \tag{52}$$

Using transformations (44), (45) and analytic soltuion of ODE (52), we get exact solution of PDE (1) which is

$$u = (c_1 + c_2(ax + ln(t)))t^{\frac{b}{a}} + c_3e^{bx} \tag{53}$$

When $a = 1.0$ and $b = 1.4$ amplitude of the wave ocillates.
Case 6:-
We choose function transformation

$$\xi = \frac{x^m}{1+t} \tag{54}$$

$$u = x^n f(\xi) \tag{55}$$

Using time-fractional derivative, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\Gamma(3/2)}{(1+t)^{3/2}} x^{n+m} f' \tag{56}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = \frac{\Gamma(3/2)}{(1+t)^{3/2}}\left(m\frac{x^{n+2m-1}}{(1+t)} f'' + (n+m)x^{n+m-1} f'\right) \tag{57}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) &= \frac{\Gamma(3/2)}{(1+t)^{3/2}}\frac{\partial}{\partial x}\left(m\frac{x^{n+2m-1}}{(1+t)} f'' + (n+m)x^{n+m-1} f'\right) \\ &= \frac{\Gamma(3/2)}{(1+t)^{3/2}}\left(m^2\frac{x^{n+3m-2}}{(1+t)^2} f''' + m(2m+n-1)\frac{x^{2m+n-2}}{(1+t)} f'' \right. \\ &\left. + m(m+n)\frac{x^{2m+n-2}}{(1+t)} f'' + (m+n)(n+m-1)x^{n+m-2} f'\right)\end{aligned} \tag{58}$$

Using (56), (57) and (58) the PDE (1) can be expressed as;

$$m^2\xi^2 f''' + (m^2+3m)\xi f'' + 2f' = 0 \tag{59}$$

Since $\xi = \frac{x^m}{(1+t)}$, so (59) reduces to ODE. The ODE is same as in case 3. It's solution will also be same. For simplification, we take $V = f'$ then (59) becomes

$$\xi^2 V'' + (\frac{m^2+3m}{m^2})\xi V' + \frac{2}{m^2} V = 0 \tag{60}$$

It is easy to see that (60) is Euler second order linear ODE. If we choose $V = \xi^k, V' = k\xi^{k-1}$ and $V'' = k(k-1)\xi^{k-2}$.
then the characteristic equation of (60) is;

$$k(k-1) + (\frac{m+3}{m})k + \frac{2}{m^2} = 0 \tag{61}$$

The roots of equation (61) are $k = -1$ and $k = -1$
Hence the analytic solution of ODE (60) is

$$f' = V = c_1\xi^{-\frac{2}{m}} + c_2\xi^{-\frac{1}{m}}$$

By replacing $V = f'$ and taking integral on both sides, we get

$$f(\xi) = c_1\frac{\xi^{-\frac{2}{m}+1}}{-\frac{2}{m}+1} + c_2\frac{\xi^{-\frac{1}{m}+1}}{-\frac{1}{m}+1} + c_3 \tag{62}$$

Using transformation (54), (55) and analytic solution of ODE (62), we get exact solution of PDE (1) which is

$$u = x^n c_1\frac{\left(\frac{x^m}{1+t}\right)^{-\frac{2}{m}+1}}{-\frac{2}{m}+1} + x^n c_2\frac{\left(\frac{x^m}{1+t}\right)^{-\frac{1}{m}+1}}{-\frac{1}{m}+1} + x^n c_3$$

$$u = mc_1 \frac{x^{m+n-2}}{(m-2)(1+t)^{-\frac{2}{m}+1}} + mc_2 \frac{x^{m+n-1}}{(m-1)(1+t)^{-\frac{1}{m}+1}} + x^n c_3 \quad (63)$$

When $m = -0.5$ and $n = 3$ there is sudden change in wave amplitude.

4. Figures

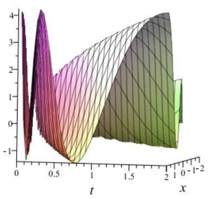

Figure 2: 3D plot of solution (14) of PDE (1) with $r = -1$; $n = 1$ and $\beta = 0.5$ shows more rapid decay in the wave amplitude

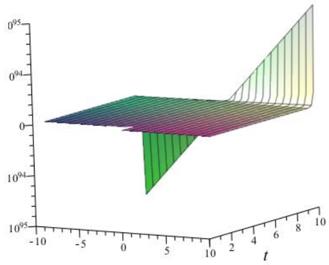

Figure 3: 3D plot of solution (23) of PDE (1) with $m = 8$, $\beta = -10$ and $n = 6$ wave amplitude become more frequent

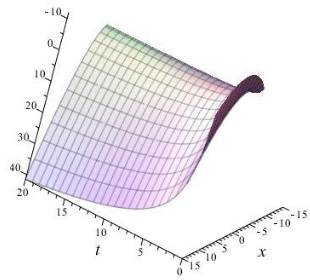

Figure 4: 3D plot of solution (33) of PDE (1) with $\alpha = 1.5$; $\beta = 1.5$ wave amplitude exhibits oscillatory behavior

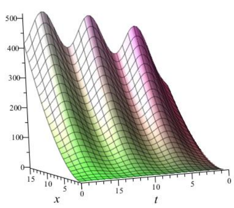

Figure 5: 3D plot of solution (43) of PDE (1) with $\alpha = 1.5$; $\beta = 1.5$ shows more rapid decay in the wave amplitude

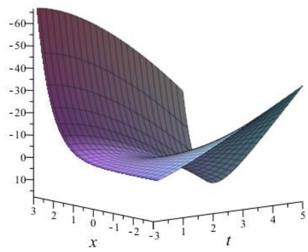

Figure 6: 3D plot of solution (53) of PDE (1) with $a = 1.0$ and $b = 1.4$ wave amplitude exhibits oscillatory behavior

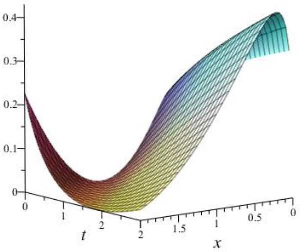

Figure 7: 3D plot of solution (63) of PDE (1) with $m = -0.5$ and $n = 3$ sudden change in wave amplitude

5. Discussion of the Results

By developing an analytical approach to solve the time-fractional DJKM equation, this paper significantly advances the field of nonlinear science. The study's successful application of the Power Index Method reveals a variety of solutions with unique wave structures, demonstrating the method effectiveness in solving challenging nonlinear problems. The

paper is a useful tool for scientists and researchers because of its thorough analysis and graphical representations, which provide a deeper understanding of the equation's dynamics and physical phenomena. Additionally, the study opens up new research directions in a number of scientific fields by showcasing the Power Index Method capacity to solve fractional differential equations. All things considered, the paper's distinctive methodology and conclusions make it a noteworthy and influential addition to the field.
In this study, we presented novel solutions known as a rational and logarithm functions. As a result of the above thorough discussion and compression, we are able to produce new, intriguing, and thorough results that have not been addressed by other approaches in the prior literature. To solve the nonlinear equation, we employed a potent technique. Other complex equations can be studied using this method.

6. Conclusions

This work represents a major advancement in the effort to solve the time-fractional Date-Jimbo-Kashiwara-Miwa equation, a basic nonlinear model with broad applications in many scientific fields. We have effectively derived a wide range of solutions by utilizing the Power Index Method, demonstrating the method's potential for modeling intricate wave structures. Various solutions for the current model are derived in the form of rational and logarithm functions. Every result found in this paper is fresh and original. The 3D under some suitable values of parameters are also plotted. These solutions provide important insights into the complex dynamics of nonlinear phenomena and have important applications in contemporary science and engineering. It provides a versatile and flexible method for solving challenging equations. It can be used with a variety of physical models and systems.
Further investigation into the use of fractional calculus in nonlinear dynamics is made possible by the efficacy of the Power Index Method as shown in this work. Determining precise solutions for such intricate equations creates new opportunities to investigate the complex behavior of nonlinear systems, which is crucial for improving our comprehension of a range of physical and natural phenomena. Furthermore, the solutions graphical representations offer a visual depiction of the underlying dynamics, promoting a better understanding of the complex interactions among the different parameters. The obtained solutions are verified using Maple software by substituting the solutions back into the equation.

7. Future Recommendations

Future studies could focus on applying this methodology to more complex issues, like coupled nonlinear time-fractional Date-Jimbo-Kashiwara-Miwa equations and systems of nonlinear PDEs, which are essential for simulating intricate physics and engineering phenomena. Additionally, this study will be expanded upon to create a new technique for exact solutions with a variety of behaviors, especially in the fields of neuroscience, machine learning for flow, and power electronics, where nonlinear oscillators are crucial. The world will transform in a new way as a result of our research.

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