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Ulam Stability of a Pexiderized Additive-quadratic Equation

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Abstract. Suppose that E is a normed space. In this work, using Brzdęk fixed point theorem, we prove the Hyers-Ulam stability of the Pexiderized additive-quadratic functional equation

$$f(x+y)+f(x-y)+h(x+y)=2f(x)+2f(y)+h(x)+h(y)$$

for all $x, y \in E$.

2020 Mathematics Subject Classifications: 39B72, 39B82, 47H10

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1. Introduction and preliminariess

The concept of stability of functional equations originated from a problem of Ulam [33]. In continue, Hyers gave a positive answer to the question of Ulam in the context of Banach spaces in the case of additive mappings, that was the first notable advance and a step toward more solutions in this field. Also, he answered the question of Ulam for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces (see [19]).

The method provided by Hyers [19] which produces the additive function will be called a direct method. This method is the most important and powerful tool to concerning the stability of system of different functional equations [31]. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the

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given approximate solution (see [14, 23, 32]). The other method is fixed point method, that is, the exact solution of the functional equation is explicitly constructed as a fixed point of some certain map [4, 11–13, 27].

Recently, a number of results concerning the stability have been obtained by different ways and been applied to a number of functional equations, functional inequalities and mappings (see [6, 7, 22, 29, 30]). Also, many mathematicians studied the stabilities additive-quadratic equation and the Drygas' equation (see [12, 18, 21]).

A mapping $f : E \to B$ is said to be additive if it satisfies

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in E$. A mapping $f : E \to B$ is called quadratic if f satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in E$.

In [1], Aczél and Dhombres showed that if E is a linear space over a field F of characteristic 0, then $q : E \to F$ is a solution of the quadratci functional equation if and only if there is a unique symmetric biadditive mapping $L : E^2 \to F$ such that $q(x) = L(x, x)$ for all $x \in E$.

Various Pexiderized versions of the quadratic functional equation have been studied in [5, 20]. Various works on stability of the quadratic functional equation can be found in [9, 16, 28].

In 2011, Brzdęk et al. [8] gave a simple fixed point theorem. Before stating Brzdęk fixed point theorem, let us introduce some hypothesis, which we will use in the sequel.

(A1) E is a nonempty set and B is a Banach space.

(A2) $\delta_1, \ldots, \delta_k : E \to E$ and $\lambda_1, \ldots, \lambda_k : E \to \mathbb{R}_+$ are given maps.

(A3) $\mathcal{H} : B^E \to B^E$ is an operator satisfying the inequality

$$\|\mathcal{H}g(x) - \mathcal{H}l(x)\| \le \sum_{i=1}^{k} \lambda_i(x) \|g\left(\delta_i(x)\right) - l\left(\delta_i(x)\right)\|$$

for all $g, h : E \to B$ and $x \in E$.

(A4) $\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ is a linear operator defined by

$$\Lambda\mathcal{F}(x) := \sum_{i=1}^{k} \lambda_i(x)\mathcal{F}\left(\delta_i(x)\right)$$

for $\mathcal{F} : E \to \mathbb{R}_+$ and $x \in E$.

Theorem 1. *[8] Suppose that the hypotheses* (A1)–(A4) *are satisfied. Assume that there are functions* $\mu : E \to \mathbb{R}_+$ *and* $\varphi : E \to B$ *such that, for all* $x \in E$*,*

$$\|\mathcal{H}\varphi(x) - \varphi(x)\| \le \mu(x)$$

and

$$\mu^*(x) := \sum_{n=0}^{\infty} \Lambda^n \mu(x) < \infty$$

hold. Then, for all $x \in E$ the limit

$$T(x) := \lim_{n\to\infty} \mathcal{H}^n \varphi(x)$$

exists and the mapping $T : E \to B$ is a unique fixed point of $\mathcal{H}$ with

$$\|\varphi(x) - T(x)\| \le \mu^*(x)$$

for all $x \in E$.

Theorem 2. *[2] Let $f, h : E \to B$ be mappings satisfying*

$$\|f(x+y) + f(x-y) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \le \epsilon$$

for some $\epsilon > 0$ and for all $x, y \in E$. Then there exist an additive mapping $A : E \to B$ and a unique quadratic mapping $Q : E \to B$ such that

$$\|h(x) - h(0) - A(x)\| \le 13\epsilon,$$
$$\|f(x) - f(0) - Q(x)\| \le 24\epsilon$$

for all $x \in E$.

Motivated by the above results on the Hyers-Ulam stability of additive functional equations, quadratic functional equations, cubic functional equations and quartic functional equations, in the current work, we try to examine the Hyers-Ulam stability of the following Pexiderized additive-quadratic functional equation

$$\phi(u+3v) - 5\phi(u+2v) - \phi(u-2v) + 10\phi(u+v) + 5\phi(u-v) - 10\phi(u) - 120\phi(v) = 0, \quad (1)$$

in Banach spaces by means of Brzdęk's fixed point approach.

A concept employed by Brzdęk, a lot of articles on hyperstability have been written on this topic and we refer to [24, 25].

Throughout the paper $\mathbb{N}_0$ denotes the set of all non-negative integers.

2. Some auxiliary results

In this section, we establish lemmas for the proof of Hyers-Ulam stability of the functional equation (1).

The next theorem is an example of a very classical result in Hyers-Ulam stability.

Theorem 3. *[17] Let $h : E \to B$ be a mapping satisfying*

$$\|h(x+y) - h(x) - h(y)\| \leq \eta(\|x\|^p + \|y\|^p)$$

for some $\eta \geq 0, p > 0, p \neq 1$ and for all $x, y \in E$. Then there is a unique additive mapping $A : E \to B$ such that

$$\|h(x) - A(x)\| \leq \frac{2\eta}{|2^p - 2|}\|x\|^p \tag{2}$$

for all $x \in E$.

Researchers obtained new results on Ulam stability of some functional equations using the Banach limit (see [3, 15]).

In the continue, we need the following lemma, whose proof is similar to the proof of Theorem 3, and so we will omit it.

Lemma 1. *Let $h : E \to B$ be a mapping satisfying*

$$\|h(x+y) - h(x) - h(y)\| \leq \eta(\|x\|^p + \|y\|^p) + \theta\|x-y\|^p$$

for some $\eta, \theta \geq 0, p > 0, p \neq 1$ and for all $x, y \in E$. Then there is a unique additive mapping $A : E \to B$ satisfying (2).

Lemma 2. *Let $f : E \to B$ be a mapping satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \eta(\|x\|^p + \|y\|^p) + \theta\|x-y\|^p \tag{3}$$

for some $\eta, \theta \geq 0, p > 0, p \neq 2$ and for all $x, y \in E$. Then there exists a unique quadratic mapping $Q : E \to B$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\eta}{|2^p - 4|}\|x\|^p$$

for all $x \in E$.

Proof. Putting $x = y = 0$ in (3), we obtain $f(0) = 0$.

Setting $y = x$ in (3) and dividing by 4, we obtain

$$\|f(x) - \frac{1}{4}f(2x)\| \leq \frac{\eta}{2}\|x\|^p \tag{4}$$

for all $x \in E$.

Let $\mathcal{H} : B^E \to B^E$ and $\mu : E \to \mathbb{R}_+$ be defined by

$$\mathcal{H}g(x) = \frac{1}{4}g(2x), \qquad g \in B^E$$

and

$$\mu(x) = \frac{\eta}{2}\|x\|^p$$

for all $x \in E$. Then

$$\|\mathcal{H}f(x) - f(x)\| \leq \mu(x)$$

for all $x \in E$. Hence

$$\|\mathcal{H}g(x) - \mathcal{H}l(x)\| \leq \frac{1}{4}\|g(2x) - l(2x)\|$$

For all $g, h \in B^E$ and $x \in E$, $\mathcal{H} : B^E \to B^E$ satisfies the condition (A3) with $\lambda_1(x) = \frac{1}{4}$ and $\delta_1(x) = 2x$. By (A4), the operator $\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ is defined by:

$$\Lambda\mathcal{F}(x) = \frac{1}{4}\mathcal{F}(2x), \qquad \mathcal{F} \in \mathbb{R}_+^E$$

for all $x \in E$. Hence

$$\Lambda\mu(x) = \frac{1}{4}\mu(2x) = 2^{p-2}\mu(x), \qquad \mu \in \mathbb{R}_+^E$$

for all $x \in E$. Since Λ is linear,

$$\Lambda^n\mu(x) = 2^{n(p-2)}\mu(x), \qquad n \in \mathbb{N}_0$$

for all $x \in E$.

If $p < 2$, then the series $\sum_{n=0}^{\infty} \Lambda^n\eta(x)$ is convergent for all $x \in E$ and

$$\mu^*(x) = \sum_{n=0}^{\infty} \Lambda^n\mu(x) = \sum_{n=0}^{\infty} 2^{n(p-2)}\mu(x) = \frac{2\eta}{4 - 2^p}\|x\|^p$$

for all $x \in E$. By Theorem 1, there exists a mapping $Q : E \to B$ such

$$Q(x) = \lim_{n\to\infty} \mathcal{H}^n f(x), \qquad Q(x) = \frac{1}{4}Q(2x)$$

and

$$\|f(x) - Q(x)\| \leq \frac{2\eta}{4 - 2^p}\|x\|^p$$

for all $x \in E$.

Next, by induction on n, one can see that

$$\begin{aligned}&\|\mathcal{H}^n f(x+y) + \mathcal{H}^n f(x-y) - 2\mathcal{H}^n f(x) - 2\mathcal{H}^n f(y)\|\\ &\leq 2^{n(p-2)}\left[\eta\left(\|x\|^p + \|y\|^p\right) + \theta\|x-y\|^p\right]\end{aligned}$$

for all $x, y \in E$ and $n \in \mathbb{N}_0$. By letting $n \to \infty$ we conclude that Q is a quadratic mapping.

Now, we consider the case $p > 2$. Replacing x and y by $\frac{x}{2}$ in (3), we have

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \frac{2\eta}{2^p}\|x\|^p$$

for all $x \in E$. Consider

$$\mathcal{H}g(x) = 4g\left(\frac{x}{2}\right), \qquad g \in B^E,$$
$$\Lambda\mathcal{F}(x) = 4\mathcal{F}\left(\frac{x}{2}\right), \qquad \mathcal{F} \in \mathbb{R}_+^E$$

and $\mu(x) = \frac{2\eta}{2^p}\|x\|^p$ for all $x \in E$. Also,

$$\Lambda\mu(x) = 2^{2-p}\mu(x)$$

for all $x \in E$. Since $p > 2$, the serie $\sum_{n=0}^{\infty} \Lambda^n \mu(x)$ is convergent for all $x \in E$ and

$$\mu^*(x) = \sum_{n=0}^{\infty} \Lambda^n \mu(x) = \frac{2\eta}{2^p - 4}\|x\|^p$$

for all $x \in E$. So, by Theorem 1 there is $Q : E \to B$ such that

$$Q(x) = \lim_{n\to\infty} \mathcal{H}^n f(x), \qquad Q(x) = 4Q\left(\frac{x}{2}\right)$$

and

$$\|f(x) - Q(x)\| \le \frac{2\eta}{2^p - 4}\|x\|^p$$

for all $x \in E$. It follows from (3) and by induction $n \in \mathbb{N}_0$ that

$$\begin{aligned}&\|\mathcal{H}^n f(x+y) + \mathcal{H}^n f(x-y) - 2\mathcal{H}^n f(x) - 2\mathcal{H}^n f(y)\| \\ &\le 2^{n(2-p)}\left[\eta\left(\|x\|^p + \|y\|^p\right) + \theta\|x-y\|^p\right]\end{aligned}$$

for all $x, y \in E$. Therefore, Q satisfies the quadratic functional equation.

To prove the uniqueness of Q for the case $p < 2$, assume that $Q_1, Q_2 : E \to B$ satisfy the quadratic functional equation on E and

$$\|f(x) - Q_1(x)\| \le \eta_1\|x\|^p, \qquad \|f(x) - Q_2(x)\| \le \eta_2\|x\|^p$$

for some $\eta_1, \eta_2 \ge 0$ and for all $x \in E$. Then

$$\|Q_1(x) - Q_2(x)\| \le (\eta_1 + \eta_2)\|x\|^p$$

for all $x \in E$. Hence

$$Q_1(x) = \frac{1}{4}Q_1(2x), \qquad Q_2(x) = \frac{1}{4}Q_2(2x)$$

for all $x \in E$. Thus

$$\|Q_1(x) - Q_2(x)\| \le \frac{1}{4}\|Q_1(2x) - Q_2(2x)\| \le \frac{2^p}{4}(\eta_1 + \eta_2)\|x\|^p$$

for all $x \in E$. By induction on $n \in \mathbb{N}_0$ we see that

$$\|Q_1(x) - Q_2(x)\| \leq \left(\frac{2^p}{4}\right)^n (\eta_1 + \eta_2)\|x\|^p$$

which tends to 0 as $n \to \infty$ for all $x \in E$. This implies

$$Q_1(x) = Q_2(x)$$

for all $x \in E$. The proofs of the cases $p > 2$ runs as before.

3. Main results

In this section, we investigate the Hyers-Ulam stability of the Pexiderized additive-quadratic functional equation (1) in Banach spaces.

Theorem 4. *Let $f, h : E \to B$ be mapings satisfying*

$$\|f(x+y) + f(x-y) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \quad (5)$$

for some $\epsilon \geq 0, p > 0, p \neq 1, 2$ and for all $x, y \in E$. Then there exist an additive mapping $A : E \to B$ and a unique quadratic mapping $Q : E \to B$ such that

$$\|h(x) - h(0) - A(x)\| \leq \frac{8 + 2^{4-p}}{|2^p - 2|}\epsilon\|x\|^p,$$
$$\|f(x) - f(0) - Q(x)\| \leq \frac{10 + 2^{4-p}}{|2^p - 4|}\epsilon\|x\|^p$$

for all $x \in E$.

Proof. Interchanging x with y in (5), we obtain

$$\|f(x+y) + f(y-x) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \quad (6)$$

for all $x, y \in E$. From (5) and (6) it follows that

$$\|f(x-y) - f(y-x)\| \leq 2\epsilon\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in E$. Putting $y = 0$ in the above inequality, we get

$$\|f(x) - f(-x)\| \leq 2\epsilon\|x\|^p \quad (7)$$

for all $x \in E$.

Substituting x and then $-x$ in the place of y in (5), we have

$$\|f(2x) + f(0) + h(2x) - 4f(x) - 2h(x)\| \leq 2\epsilon\|x\|^p, \quad (8)$$
$$\|f(0) + f(2x) + h(0) - 2f(x) - 2f(-x) - h(x) - h(-x)\| \leq 2\epsilon\|x\|^p \quad (9)$$

for all $x \in E$. From (7), (8) and (9), we obtain

$$\begin{aligned} &\|h(2x) - h(x) + h(-x) - h(0)\| \\ &\leq \|f(2x) + f(0) + h(2x) - 4f(x) - 2h(x)\| \\ &\quad + \|f(0) + f(2x) + h(0) - 2f(x) - 2f(-x) - h(x) - h(-x)\| + 2\|f(x) - f(-x)\| \\ &\leq 2\epsilon\|x\|^p + 2\epsilon\|x\|^p + 4\epsilon\|x\|^p = 8\epsilon\|x\|^p \end{aligned} \tag{10}$$

for all $x \in E$.

Replacing x by $-x$ in the last inequality, we obtain

$$\|h(-2x) + h(x) - h(-x) - h(0)\| \leq 8\epsilon\|x\|^p \tag{11}$$

for all $x \in E$.

It follows from (10) and (11) that

$$\|h(2x) + h(-2x) - 2h(0)\| \leq 16\epsilon\|x\|^p,$$

which implies

$$\|h(x) + h(-x) - 2h(0)\| \leq \frac{16}{2^p}\epsilon\|x\|^p \tag{12}$$

for all $x \in E$. Replacing y by $-y$ in (5), we have

$$\|f(x+y) + f(x-y) + h(x-y) - 2f(x) - 2f(-y) - h(x) - h(-y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{13}$$

for all $x, y \in E$. By (5), (7), (12) and (13), we have

$$\begin{aligned} &\|h(x+y) - h(x-y) - 2h(y) + 2h(0)\| \\ &\leq \|f(x+y) + f(x-y) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \\ &\quad + \|f(x+y) + f(x-y) + h(x-y) - 2f(x) - 2f(-y) - h(x) - h(-y)\| \\ &\quad + 2\|f(y) - f(-y)\| + \|h(y) + h(-y) - 2h(0)\| \\ &\leq \epsilon\left(\|x\|^p + \|y\|^p\right) + \epsilon\left(\|x\|^p + \|y\|^p\right) + 4\epsilon\|y\|^p + \frac{16}{2^p}\epsilon\|y\|^p \\ &= \epsilon\left(2\|x\|^p + \left(6 + 2^{4-p}\right)\|y\|^p\right) \end{aligned} \tag{14}$$

for all $x, y \in E$. Interchanging x with y in (14), we get

$$\|h(x+y) - h(y-x) - 2h(x) + 2h(0)\| \leq \epsilon\left(2\|y\|^p + \left(6 + 2^{4-p}\right)\|x\|^p\right) \tag{15}$$

for all $x, y \in E$. By (12), (14) and (15), we have

$$\begin{aligned} &\|2h(x+y) - 2h(x) - 2h(y) + 2h(0)\| \\ &\leq \|h(x+y) - h(x-y) - 2h(y) + 2h(0)\| + \|h(x+y) - h(y-x) - 2h(x) + 2h(0)\| \\ &\quad + \|h(x-y) + h(y-x) - 2h(0)\| \\ &\leq \left(8 + 2^{4-p}\right)\epsilon\left(\|x\|^p + \|y\|^p\right) + 2^{4-p}\epsilon\|x-y\|^p, \end{aligned}$$

which implies

$$\|h(x+y)-h(x)-h(y)+h(0)\| \leq \left(4+2^{3-p}\right)\epsilon\left(\|x\|^p+\|y\|^p\right)+2^{3-p}\epsilon\|x-y\|^p \tag{16}$$

for all $x, y \in E$. Define $\widehat{h}: E \rightarrow B$ by $\widehat{h}(x) := h(x)-h(0)$ for all $x \in E$. Then we can rewrite the inequality (16) in the form

$$\|\widehat{h}(x+y)-\widehat{h}(x)-\widehat{h}(y)\| \leq \left(4+2^{3-p}\right)\epsilon\left(\|x\|^p+\|y\|^p\right)+2^{3-p}\epsilon\|x-y\|^p$$

for all $x, y \in E$. Applying Lemma 1 to $\widehat{h}$, we get a unique additive mapping $A_1 : E \rightarrow B$ such that

$$\|\widehat{h}(x)-A_1(x)\| \leq \frac{8+2^{4-p}}{|2^p-2|}\epsilon\|x\|^p,$$

which implies

$$\|h(x)-h(0)-A_1(x)\| \leq \frac{8+2^{4-p}}{|2^p-2|}\epsilon\|x\|^p$$

for all $x \in E$. Letting $x=y=0$ in (5), we have

$$2f(0)+h(0)=0. \tag{17}$$

Next, using (5), (16) and (17), we get

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)+2f(0)\| \\
&\leq \|f(x+y)+f(x-y)+h(x+y)-2f(x)-2f(y)-h(x)-h(y)\| \\
&\quad +\|h(x+y)-h(x)-h(y)+h(0)\| \\
&\leq \epsilon\left(\|x\|^p+\|y\|^p\right)+\left(4+2^{3-p}\right)\epsilon\left(\|x\|^p+\|y\|^p\right)+2^{3-p}\epsilon\|x-y\|^p \\
&= \left(5+2^{3-p}\right)\epsilon\left(\|x\|^p+\|y\|^p\right)+2^{3-p}\epsilon\|x-y\|^p
\end{aligned}$$

for all $x, y \in E$.

Similarly, we define $\widehat{f}: E \rightarrow B$ by $\widehat{f}(x) := f(x)-f(0)$ for all $x \in E$. Then we obtain

$$\|\widehat{f}(x+y)+\widehat{f}(x-y)-2\widehat{f}(x)-2\widehat{f}(y)\| \leq \left(5+2^{3-p}\right)\epsilon\left(\|x\|^p+\|y\|^p\right)+2^{3-p}\epsilon\|x-y\|^p$$

for all $x, y \in E$. By Lemma 2, there exists a unique quadratic mapping $Q : E \rightarrow B$ such that

$$\|\widehat{f}(x)-Q(x)\| \leq \frac{10+2^{4-p}}{|2^p-4|}\epsilon\|x\|^p,$$

which implies

$$\|f(x)-f(0)-Q(x)\| \leq \frac{10+2^{4-p}}{|2^p-4|}\epsilon\|x\|^p$$

for all $x \in E$, which ends our proof.

The following example shows that for $p=1$ the Pexider additive-quadratic functional equation (1) is not stable (see [26]).

Example 1. *Define* $\psi : \mathbb{R} \to \mathbb{R}$ *by*

$$\psi(x) = \begin{cases} -b & x \leq -1 \\ bx & -1 < x < 1 \\ b & x \geq 1 \end{cases}$$

where $a, b > 0$ *and assume that* $f, h : \mathbb{R} \to \mathbb{R}$ *are defined by*

$$f(x) = ax^2 \qquad \text{and} \qquad h(x) = \sum_{n=0}^{\infty} \frac{\varphi\left(2^n x\right)}{2^n}$$

for all $x \in \mathbb{R}$. *We show that*

$$\|f(x+y) + f(x-y) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \leq 8b\left(|x| + |y|\right)$$

for all $x, y \in \mathbb{R}$, *but there are no constant* $k \geq 0$ *and no mapping* $A : \mathbb{R} \to \mathbb{R}$ *satisfying* (1) *and*

$$|h(x) - h(0) - A(x)| \leq k|x|$$

for all $x \in \mathbb{R}$.

In the following we show that for $p = 2$ the equation (1) is not stable (see [10]).

Example 2. *Define* $\varphi : \mathbb{R} \to \mathbb{R}$ *by*

$$\varphi(x) = \begin{cases} ax^2 & -1 < x < 1 \\ a & |x| \geq 1 \end{cases}$$

where $a, b > 0$ *and assume that* $f, h : \mathbb{R} \to \mathbb{R}$ *are defined by*

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi\left(2^n x\right)}{4^n} \qquad \text{and} \qquad h(x) = bx$$

for all $x \in \mathbb{R}$. *We show that*

$$\|f(x+y) + f(x-y) + h(x+y) - 2f(x) - 2f(y) - h(x) - h(y)\| \leq 32a\left(|x|^2 + |y|^2\right)$$

for all $x, y \in \mathbb{R}$, *but there are no constant* $k \geq 0$ *and no mapping* $Q : \mathbb{R} \to \mathbb{R}$ *satisfying* (1) *and*

$$|f(x) - f(0) - Q(x)| \leq k|x|^2$$

for all $x \in \mathbb{R}$.

4. Conclusion and Future Works

In this work, using Brzdęk fixed point theorem, we proved the Hyers-Ulam stability of the Pexiderized additive-quadratic functional equation (1) in Banach spaces. We can apply the method to study the Hyers-Ulam stability problems of the Pexiderized additive-quadratic functional equation (1) in fuzzy Banach spaces, matrix Banach spaces, Hilbert C^*-modules and fuzzy Hilbert C^*-modules, in future work.

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Conflict of interest

The authors declare that they have no competing interests.

References

[1] J Aczél and J Dhombre. *Functional Equastions in Several Variables.* Cambridge University Press., Cambridge, 1989.

[2] M Adam. Alienation of the quadratic and additive functional equations. *Anal. Math.*, 45(3):449–460, 2019.

[3] R Badora, J Brzdęk, and K Ciepliński. Applications of banach limit in ulam stability. *Symmetry*, 13(5(841)):1–18, 2021.

[4] A Batool, S Nawaz, O Ege, and M de cla Sen. Hyers-ulam stability of functional inequalities: A fixed point approach. *J. Inequal. Appl.*, 2020(1(251)):1–18, 2020.

[5] B Bouikhalene and E Elqorachi. Ulam-gǎvriţa-rassias stability of the pexider functional equation. *Int. J. Appl. Math. Stat.*, 7(FE07):27–39, 2007.

[6] S Bowmiya, G Balasubramanian, V Govindan, M Donganont, and H Byeon. Generalized linear differential equation using hyers-ulam stability approach. *Eur. J. Pure Appl. Math.*, 17(4):3415–3435, 2024.

[7] S Bowmiya, G Balasubramanian, V Govindan, M Donganont, and H Byeon. Hyers-ulam stability of fifth order linear differential equations. *Eur. J. Pure Appl. Math.*, 17(4):3585–3609, 2024.

[8] J Brzdęk, J Chudziak, and Z Páles. A fixed point approach to stability of functional equations. *Nonlinear Anal.*, 74:6728–6732, 2011.

[9] P W Cholewa. Remarks on the stability of functional equations. *Aequationes Math.*, 27:76–86, 1984.

[10] S Czerwik. On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg*, 62:59–64, 1992.

[11] M Dehghanian, C Park, and Y Sayyari. Stability of ternary antiderivation in ternary banach algebras via fixed point theorem. *Cubo*, 25(2):273–288, 2023.

[12] M Dehghanian and Y Sayyari. The application of brzdęk's fixed point theorem in the stability problem of the drygas functional equation. *Turk. J. Math.*, 47(6):1778–1790, 2023.

[13] M Dehghanian and Y Sayyari. A fixed point technique to the stability of hadamard $\mathfrak{D}$-hom-der in banach algebras. *Math. Slovaca*, 74(1):151–158, 2023.

[14] M Dehghanian, Y Sayyari, and C Park. Hadamard homomorphisms and hadamard derivations on banach algebras. *Misk. Math. Notes*, 24(1):129–137, 2023.

[15] E El-hady and J Brzdęk. Banach limit and ulam stability of nonhomogeneous cauchy equation. *Math.*, 10(1695):1–15, 2022.

[16] E El-hady, Y Sayyari, M Dehghanian, and Y Alruwaily. Stability results for some classes of cubic functional equations. *Axioms*, 13(7(480)):1–12, 2024.

[17] Z Gajda. On stability of additive mappings. *Int. J. Math. Math. Sci.*, 14:431–434, 1991.

[18] V Govindan, C Park, S Pinelas, and T M Rassias. Hyers-ulam stability of an additive-quadratic functional equation. *Cubo*, 22(2):233–255, 2020.

[19] D H Hyers. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. U.S.A.*, 27:222–224, 1941.

[20] S M Jung and P K Sahoo. Hyers-ulam stability of the quadratic equation of pexider type. *J. Korean Math. Soc.*, 38(3):645–656, 2001.

[21] S M Jung and P K Sahoo. Stability of a functional of drygas. *Aequationes Math.*, 64:263–273, 2002.

[22] B V Senthil Kumar, H Dutta, and S Sabarinathan. Modular stabilities of a reciprocal second power functional equation. *Eur. J. Pure Appl. Math.*, 13(5):1162–1175, 2020.

[23] S Paokanta, M Dehghanian, C Park, and Y Sayyari. A system of additive functional equations in complex banach algebras. *Demonstr. Math.*, 56(1(20220165)):1–10, 2023.

[24] P Phochai and S Saejung. The hyperstability of general linear equation via that of cauchy equation. *Aequationes Math.*, 93(4):781–789, 2019.

[25] P Phochai and S Saejung. Hyperstability of generalized linear functional equations in several variables. *Bull. Aust. Math. Soc.*, 102(2):293–302, 2020.

[26] M Piszczek and J Szczawinska. Stability of the drygas functional equation on restricted domain. *Results Math.*, 68:11–24, 2015.

[27] A Rani, S Devi, and M K Antil. Stability results of the additive-quadratic functional equations in random normed spaces by using direct and fixed-point method. *Commun. Math. Appl.*, 14(2):827–843, 2023.

[28] T M Rassias. On the stability of the quadratic functional equation and its applications. *Studia Univ. Babes-Bolyai Math.*, 43(3):89–124, 1998.

[29] K Ravi and B V Senthil Kumar. Generalized hyers-ulam-rassias stability of a system of bi-reciprocal functional equations. *Eur. J. Pure Appl. Math.*, 8(2):283–293, 2015.

[30] Y Sayyari, M Dehghanian, and S Nasiri. Solution of some irregular functional equa-

tions and their stability. *J. Linear Topol. Algebra*, 11(4):271–277, 2022.

[31] Y Sayyari, M Dehghanian, and C Park. Some stabilities of system of differential equations using laplace transform. *J. Appl. Math. Comput.*, 69(4):3113–3129, 2023.

[32] Y Sayyari, M Dehghanian, and C Park. A system of biadditive functional equations in banach algebras. *Appl. Math. Sci. Eng.*, 31(1(2176851)):1–11, 2023.

[33] S M Ulam. *Problems in Modern Mathematics.* John Wiley & Sons, Inc., New York, 1964.