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Subordinate Average Structures on Random Walks

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Abstract. In a random walk $\{N, p(x, y)\}$ where N is an infinite graph and $\{p(x, y)\}$ is a set of transition probabilities, if $\{p^{'}(x, y)\}$ is another set subordinate to the set $\{p(x, y)\}$ such that $p^{'}(x, y) \leq p(x, y)$ for all pairs (x, y) and $p^{'}(x, y) < p(x, y)$ for atleast one pair (x, y), then $\{N, p^{'}(x, y)\}$ can be identified as a Schrödinger network. In general, we consider the random walk $\{N, P^{'}\}$ which is subordinate to $\{N, P\}$ and discuss the relation between the classes of super-average functions defined by the transition probabilities sets $\{p(x, y)\}$ and $\{p^{'}(x, y)\}$. Moreover, we define $\{N, P^{'}\}$ as parahyperbolic if 0 is the only bounded $P^{'}$-average function on N and study various potential-theoretic properties of parahyperbolic networks. We also give equivalent conditions for a random walk to be parahyperbolic. Finally, we discuss the relation between bounded $P^{'}$ and P-average functions.

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Key Words and Phrases: Superaverage functions, subordinate structure, parahyperbolic, $P^{'}$-Green's potential, bounded P and $P^{'}$ functions

1. Introduction

In the state space $N = \{0, 1, 2, ...\}$ with the set $P = \{p(x, y)\}$ of transition probabilities given by $p(n, n+1) = \alpha_n$, $p(n, n-1) = \beta_n$, for $n \geq 1$, $\alpha_n, \beta_n > 0$, $\alpha_n + \beta_n \leq 1$ and $0 \leq p(0, 1) \leq 1$, the transience, the recurrence, the hitting time etc. of the random walk $\{N, P\}$ depend on P. For example if $p(n, n+1) = p(n, n-1) = \frac{1}{2}$ for $n \geq 1$ and $p(0, 1) = 1$ then $\{N, P\}$ is recurrent and any function $u(x)$ on N such that $u(n) = \frac{1}{2}u(n+1) + \frac{1}{2}u(n-1)$ for $n \geq 1$ and $u(0) = u(1)$ is constant. This example is the motivation for the consideration of the following problem: Let $\{N, P\}$ be a random walk ([2] and [12]) where N is an infinite graph which is connected and $P = \{p(x, y)\}$ is a set of transition probabilities, $p(x, y) > 0$ if and only if x and y are neighbours; $p(x, y)$ and $p(y, x)$ may have different values. Suppose $P^{'} = \{p^{'}(x, y)\}$ is another set of transition probabilities on N such that $p^{'}(x, y) \leq p(x, y)$ for every pair x, y. The problem is to study how the properties of transience, recurrence and other probabilistic results in $\{N, P\}$ get transformed in the random walk $\{N, P^{'}\}$. We refer to $P^{'}$ as a transition probability structure on N subordinate to P.

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There is an analogy in the context of general infinite networks ([1] and [10]) $\{X, t(x,y)\}$. Here potential-theoretic properties of functions on X are studied using the Laplace operator $\Delta u(x) = \sum t(x,y)[u(y) - u(x)]$. If $q \geq 0$ is a function on X, there is another interesting Schrödinger operator ([4] and [7]) $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$. If we write $t^{'}(x,y) = \frac{t(x,y)}{q(x)+\sum t(x,y)}$ than $\{X, t^{'}(x,y)\}$ is another network where $t^{'}(x,y) \leq t(x,y)$ which provides a convenient base for the study of Schrödinger potentials with a comparable study of Laplace potentials.

In this article, we discuss the potential-theoretic aspects of functions on N determined by the original structure P and another structure $P^{'}$ that is subordinate to P. For related results, we have referred [6] and [11]. The classification of connected and locally finite infinite network into parabolic and hyperbolic of order p is investigated in [13], where as the same is done for non-locally finite networks in [3]. The concept of recurrent random walks on countable infinite state spaces is explored by V. R. Manivannan and M. Venkataraman [5]. In [8] M. Surya priya and N. Nathiya establishes the existence of Green's function on non-reversible random walks through the application of potential theoretic techniques. With reference to the aforementioned, we have looked into the concepts of $P^{'}$-Green's function as well as parahyperbolic and bounded hyperbolic subordinate structures.

S. Sivan and M. Venkataraman [9], says that an infinite network is parahyperbolic if and only if constant 1 is a potential. They have given neccessary and sufficient condition for a network to be parahyperbolic. V. Anandam [1] has studied the Schrödinger operators and subordinate structures on infinite networks. Where as in the present article, we delve into the potential theory associated to a structure subordinate to a non-locally finite random walk. We define parahyperbolic random walk and its subordinate structure. Finally a section is devoted to investigate the relation between bounded P-average functions and bounded $P^{'}$-average functions.

2. Preliminaries

Definition 1. *Random walk: Let $\{N, P\}$ be a random walk with a countable infinite number of states N and $P = \{p(x,y)\}$ is the probability transition matrix, where $p(x,y)$ denotes the transition probability from state x to state y. We assume $\{N, P\}$ is connected (i.e, for any two distinct states there exists a path connecting them) and without self loops. As usual, we shall take N as an infinite graph by defining $[x,y]$ as an edge if and only if $p(x,y) > 0$. We say two states x and y are neighbours if there exists an edge between them and it is denoted by $x \sim y$ and $p(x) = \sum_{y \sim x} p(x,y) = 1$ for every $x \in N$.*

Note: We do not place the condition that the number of neighbours of any state is finite. Hence we consider only those real-valued functions s on N for which $\sum\limits_{y \sim x} p(x,y)|s(y)| < \infty$ for any $x \in N$. Write $As(x) = \sum\limits_{y} p(x,y)s(y)$.

Definition 2. *Interior and Boundary of a set: We say a state x is an interior state of* *a subset K if and only if x and all its neighbours are in a subset K of N. The set of all interior states of K is denoted by $\mathring{K}$ and the boundary of K by $\partial K = K \backslash \mathring{K}$.*

Definition 3. *Laplacian(Δ): Let $s(x)$ be a real valued function defined on N. For $x \in \mathring{K}$, $K \subset N$, the Laplacian (Δ) of s at x is defined as*

$$\Delta s(x) = \sum_{y \sim x} p(x,y)[s(y) - s(x)] = (A - I)s(x)$$

Definition 4. *A function u defined on a subset K is said to be P-superaverage (respectively, P-subaverage and P-average) on K if and only if $s(x) \geq As(x)$ (respectively $s(x) \leq As(x)$ and $s(x) = As(x)$) for every $x \in \mathring{K}$.*

Definition 5. *If $p \geq 0$ is a P-superaverage function such that any P-subaverage function majorized by p is non-positive, then p is called a P-potential.*

Definition 6. *If s is a P-superaverage function on N and if K is a subset of N such that $\Delta s(x) = 0$ for each x in $N \backslash K$, then K is said to be the P-average support of s in N.*

If there is a perturbation on Laplace operator indicated by the operator $\Delta_q u(x) = \Delta u(x) - q(x)u(x), q \geq 0$, (the operator Δ_q is commonly referred to as a Schrödinger operator on N) we have

$$\begin{aligned} \Delta_q u(x) =& \sum_y p(x,y)u(y) - [1 + q(x)]u(x) \\ =& [1 + q(x)][A^{'} - I]u(x) \end{aligned}$$

where $A^{'}u(x) = \sum p^{'}(x,y)u(y), p^{'}(x,y) = \frac{p(x,y)}{1+q(x)} \leq p(x,y)$.
Then the potential theory associated with the Schrödinger operator Δ_q depends on $A^{'}$, just as the potential theory associated with the Laplace operator Δ depends on the operator A.

Since $A^{'}\varphi(x) \leq A\varphi(x)$ for any functions $\varphi \geq 0$ on N, the relation between the Schrödinger potentials and the Laplace potentials is exhibited by the relation between $P^{'} = \{p^{'}(x,y)\}$ and $P = \{p(x,y)\}$. Here $p^{'}(x,y) \leq p(x,y)$ for any pair x, y and we say that $P^{'}$ is subordinate to P.
In the following sections we investigate this subordinate structure in an abstract setting.

3. Subordinate structure

Definition 7. *Let $\{p^{'}(x,y)\}$ be a set of transition indices on N such that $p(x,y) \geq p^{'}(x,y) \geq 0$ for any pair of states x and y and $p^{'}(x,y) < p(x,y)$ for atleast one pair of x and y. Then we say that $P^{'} = \{p^{'}(x,y)\}$ defines a submarkov average structure on N that is subordinate to the average structure defined by $P = \{p(x,y)\}$; or simply that $P^{'}$ is subordinate to P on N.*

Remark 1. *A Schrödinger operator Δ_q defines a subordinate structure on N, when $q > 0$ and $p^{'}(x,y) = \frac{p(x,y)}{1+q(x)}$. But for a subordinate structure $\{N, P^{'}\}$ to define a Schrödinger operator on N, it is necessary that for every $x \in N$, the quantity $\frac{p(x,y)}{p^{'}(x,y)}$ is independent of y, for any $y \sim x$.*

Definition 8. *A real valued function u on a subset K of N is said to be $P^{'}$-superaverage (respectively $P^{'}$-subaverage) on K if and only if $\Delta^{'}u(x) \leq 0$ ($\Delta^{'}u(x) \geq 0$ respectively) or $u(x) \geq \sum_y p^{'}(x,y)u(y)$ for every x in $\mathring{K}$. (Here $\Delta^{'}u(x) = [\sum_y p^{'}(x,y)u(y)] - u(x) = (A^{'} - I)u(x)$).*

Definition 9. *A real valued function u on a subset K of N is said to be $P^{'}$-average on K if and only if $\Delta^{'}u(x) = 0$ for every x in $\mathring{K}$.*

Proposition 1. *If $P^{'}$ is subordinate to P then for any $u \geq 0$ on a subset K of N, $\sum p(x,y)u(y) \geq \sum P^{'}(x,y)u(y)$, for every $x \in \mathring{K}$. Hence*

(i) If $u \geq 0$ is P-superaverage on K then u is $P^{'}$ -superaverage on K.

(ii) If $u \geq 0$ is $P^{'}$-subaverage on K then u is P-subaverage on K.

(iii) If $u = 0$ is $P^{'}$-average on K then u is $P^{'}$ -superaverage on K.

Proof. For a P-superaverage function u on $K \subset N$, we have $\sum p(x,y)u(y) \leq u(x)$. Since $P^{'}$ is subordinate to P,

$$\sum p(x,y)u(y) \geq \sum p^{'}(x,y)u(y)$$
$$u(x) \geq \sum p(x,y)u(y) \geq \sum p^{'}(x,y)u(y)$$
$$u(x) \geq \sum p(x,y)^{'}u(y)$$

Thus, if u is P-superaverage on a subset K of N then u is $P^{'}$-superaverage on $K \subset N$. Similarly the proof follows for (ii) and (iii).

3.1. Properties of $P^{'}$-Superaverage Functions

(i) If s_1 and s_2 are $P^{'}$-superaverage on a subset K and if α_1, α_2 are two non-negative numbers, then $\alpha_1 s_1 + \alpha_2 s_2$ and $inf(s_1, s_2)$ are $P^{'}$-superaverage on K.

(ii) If $\{s_i\}$ is a lower directed family of $P^{'}$-superaverage functions on K, then $s(x) = \inf_i u_i(x)$ and then s is $P^{'}$-superaverage on K. (A lower directed family $\mathcal{F}$ of functions means that if $f, g \in \mathcal{F}$ then $inf(f,g)$ is also in $\mathcal{F}$)

(iii) **Greatest $P^{'}$-average minorant (g.$P^{'}$-a.m):** Suppose $u(x) \geq v(x)$ on N where $u(x)$ is $P^{'}$-superaverage and $v(x)$ is $P^{'}$-subaverage on N. Then there exists a $P^{'}$-average function $h(x)$ on N, $u(x) \geq h(x) \geq v(x)$ and if h_1 is any other $P^{'}$-average function on N between $u(x)$ and $v(x)$, then $h(x) \geq h_1(x)$ on N (Section 3.1, [1]).

Proof. Let $\{K_n\}$ be an exhaustion of N by finite sets; that is $K_n \subset \mathring{K}_{n+1} \subset K_{n+1}$ and $X = \cup K_n$. let $D_n u$ denote the $P^{'}$-superaverage function on N, equal to the Dirichlet solution on K_n with boundary values u and extended by u outside K_n. Then $\{D_n u\}$ is a decreasing sequence of $P^{'}$-superaverage functions, each $D_n u \geq v$ on N. Hence $D[u] = lim_n D_n u$ is $P^{'}$-superaverage function.
Now for any z in N, $z \in \mathring{K}_m$ for some m. Hence $D_n(x)$ is $P^{'}$-average at $x = z$ for all $n \geq m$. Consequently, $D[u](x)$ is $P^{'}$-average at $x = z$. This shows that $D[u]$ is $P^{'}$-average on N.
Thus $u \geq D[u] \geq v$. Moreover if h_1 is P-average, $u \geq h_1 \geq v$, then $D_n u \geq h_1$ for any n so that $D[u] \geq h_1$. we term $D[u]$ as the greatest $P^{'}$-average minorant of u on N.

(iv) **Riesz representation theorem:** Any non-negative $P^{'}$-superaverage function s on a subset K can be written as the sum of a $P^{'}$-potential and a non-negative $P^{'}$-average function on K and this representation is unique.

Proposition 2. *If s is a P-potential then it is a $P^{'}$-potential.*

Proof. First note that s is $P^{'}$-superaverage on N. Let $u \geq 0$ be a $P^{'}$-subaverage function such that $u \leq s$ on N. Note that u is P-subaverage function on N and s is a P-potential, therefore u = 0. Consequently s is a $P^{'}$-potential on N.

3.2. $P^{'}$-Green's Potential

Though positive $P^{'}$-potentials always exist on N, positive P-potentials may exist (hyperbolic random walk) or may not exist (parabolic random walk) on N. Thus on a hyperbolic random walk N, for a fixed state e, we have the P-Green's potential $G_e(x)$, $-\Delta[G_e(x)] = \delta_e(x)$ and $P^{'}$-Green's potential $G^{'}_e(x) = -\Delta^{'}[G^{'}_e(x)] = \delta_e(x)$. The following theorem indicates a relation between them.

Lemma 1. *Let $s \geq 0$ be a $P^{'}$-superaverage function and p be a $P^{'}$-potential on N. If $(-\Delta^{'})s \geq (-\Delta^{'})p$, then $s \geq p$ on N.*

Proof. By hypothesis, $s = p + u$ where u is $P^{'}$-superaverage on N. Since $s \geq 0, -u \leq p$ on N. Hence $-u \leq 0$ so that $s \geq p$ on N.

Theorem 1. *Let (N, P) be a hyperbolic network. Let $G_e(x)$ be the P-Green's function on N with average support $\{e\}$. Then ${G_e}^{'}(x) \leq G_e(x)$ for every $x \in N$.*

Proof. Since any positive P-superaverage function on N is a $P^{'}$-superaverage function on N, $G_e(x)$ is a $P^{'}$-superaverage function on N.
If $x \neq e$,
$(-\Delta^{'})G_e(x) \geq 0$ and $(-\Delta^{'})G^{'}_e(x) = 0$
when $x = e$,

$$(-\Delta^{'})G_e(e) = G_e(e) - \sum_y p^{'}(e,y)G_e(y)$$

$$
\begin{aligned}
&\geq G_e(e) - \sum_y p(e,y)G_e(y)\\
&=(-\Delta)G_e(e)\\
&=1\\
&=(-\Delta^{'})G^{'}_e(e)
\end{aligned}
$$

Since for all $x \in N$, $(-\Delta^{'})G_e(x) \geq (-\Delta^{'})G^{'}_e(x)$
By the above Lemma 1, ${G_e}^{'}(x) \leq G_e(x)$ for every $x \in N$.

Lemma 2. *If p_n is a sequence of $P^{'}$-potentials and if $p(x) = \sum_n p_n(x)$ is finite at one state, then p is a $P^{'}$-potential.*

Proof. The $P^{'}$-superaverage function (actually a $P^{'}$-potential) $s_m = \sum_1^m p_n$ introduces a sequence $\{s_m\}$ of increasing $P^{'}$-superaverage functions so that $s = \lim_m s_m$ is a $P^{'}$-superaverage function if s is finite at one state. Hence $p(x) = \sum_n p_n(x)$ is a $P^{'}$-superaverage function.
To show $p(x)$ is a $P^{'}$-potential: Let $h(x)$ be a non-negative $P^{'}$-average and $h \leq p$. Then $h - \sum\limits_2^\infty p_n \leq p_1$. Here the left side is $P^{'}$-subaverage and the right side a $P^{'}$-potential, so that $h - \sum\limits_2^\infty p_n \leq 0$. Continuing this process we find $h(x) \leq \sum\limits_m^\infty p_n(x)$ for any m. For any z in N, since $\sum\limits_1^\infty p_n(z)$ is convergent $h(z) \leq \sum\limits_m^\infty p_n(z) \leq \epsilon$ for sufficiently large m. This leads to $h(z) = 0$, hence $h = 0$ and consequently $p = \sum\limits_1^\infty p_n$ is a $P^{'}$-potential on N.

Recall that for any $P^{'}$-superaverage function $s \geq 0$ we write by Riesz representation, $s = p + D[s]$, where $D[s]$ is the greatest $P^{'}$-verage minorant of s.

Theorem 2. *Any $P^{'}$-superaverage function $s \geq 0$ has a unique representation $s(x) = \sum\limits_y [-\Delta^{'} s(y)]G^{'}_y(x) + D[s](x)$.*

Proof. Let K be a finite set and $u_k(x) = s(x) - \sum\limits_{y\in k} [-\Delta^{'} s(y)]G^{'}_y(x)$. Then $-\Delta^{'}[u_k(x)] = 0$ if $x \in k$ and $-\Delta^{'}[u_k(x)] \geq 0$ if $x \in N\backslash K$. Then $u_k(x)$ is a $P^{'}$-superaverage function on N and $-u_k(x) \leq \sum\limits_{y\in K} [-\Delta^{'} s(y)]G^{'}_y(x)$. Since the left side is $P^{'}$-subaverage and the right side is a $P^{'}$-potential; $-u_k(x) \leq 0$ on N. That is $\sum\limits_{y\in K} [-\Delta^{'} s(y)]G^{'}_y(x) \leq s(x)$. Allowing K to grow into N, $s(x) \geq \sum\limits_{y\in N} [-\Delta^{'} s(y)]G^{'}_y(x)$. Note that the right side is a $P^{'}$-potential by Lemma 2.
Write $h(x) = s(x) - \sum\limits_{y\in N} [-\Delta^{'} s(y)]G^{'}_y(x)$. Note $-\Delta^{'} h = 0$ so that h is a $P^{'}$-average function on N. By the uniqueness of Riesz representation, $h(x) = D[s](x)$.

In the particular case, when $s = 1$ is the constant function then $-\Delta' s(x) = s(x) - \sum_{y} p'(x,y)s(y) = 1 - p'(x)$ when $p'(x) = \sum_y p'(x,y)$. Hence obtain the following result.

Corollary 1. *For any $x \in N$, $1 = \sum_y [1 - p'(y)]G'_y(x) + D[1](x)$.*

4. Parahyperbolic subordinate structures

Since $A'u(x) \leq Au(x)$ for any non-negative functions $u(x)$, then any non-negative P-superaverage functions is a P'-superaverage function. In particular, the constant function 1 is a P'-superaverage function so that $1 = s + h$ where $s > 0$ is P'-superaverage and $h \geq 0$ is a P'-average function. Since $h > 0$ or $h \equiv 0$, the constant 1 is a P'-potential or just a positive P'-superaverage function that is not a P'-potential. This opens up two possibilities in the study of P'-superaverage functions on N, as shown in this section.
In a random walk (N, P) the constant 1 is P-average. It is possible that any positive P-superaverage function is constant, hence there may not be any positive P-potential on N. On the other hand, the constant 1 is P'-superaverage but not P'-average. Hence there are always P'-potentials on N.
Let P' be a subordinate structure to P. Then the constant 1 is a P'-superaverage function, write $1 = v + h$ where v is a P'-potential and $h \geq 0$ is a P'-average function.

(i) It is possible that $h \neq 0$. It means that there are bounded positive P'-average functions N.

(ii) If $h = 0$, then 1 is a P'-potential, hence there is no bounded positive P'-average functions on N.

Definition 10. *If the constant 1 is a P'-potential, then (N, P') is referred to as parahyperbolic. Otherwise (N, P') is termed bounded hyperbolic.*

Proposition 3. *(Maximum Principle:) The following are equivalent(Theorem 4.3.7, [1]):*

(i) (N, P') is parahyperbolic.

(ii) In an arbitrary subset F of N, if u is an upper bounded subaverage function such that $u \leq 0$ on ∂F, then $u \leq 0$ on F.

Definition 11. *(Perron family:) Let $\mathbb{F}$ be the family of all P'-subaverage functions u on N such that for a P'-superaverage function v on N, $u \leq v$ on N. If $u_1, u_2 \in \mathbb{F}$, then $sup(u_1, u_2) \in \mathbb{F}$, hence is an upper directed family of P'-subaverage functions .*
Fix a state z and choose any $u \in \mathbb{F}$. Then the function

$$u_z(x) = \begin{cases} u(x), & \text{if } x \neq z \\ \sum p'(z,y)u(y), & \text{if } x = z \end{cases}$$

(Known as the Poisson modification of $u(x)$ at $x = z$) also is in $\mathbb{F}$. Note $u_z \geq u$ and $u_z(x)$ is P'-average at $x = z$. Consequently, $h(x) = \sup_{u \in \mathbb{F}} u(x)$ is P'-subaverage on

N and at $x = z$, $h(z) = \sup_{u \in \mathbb{F}} u_z(z)$ is $P^{'}$-average. Since z is arbitrary, we conclude that $h(x) = \sup_{u \in \mathbb{F}} u(x)$ is $P^{'}$-harmonic on N. We refer to $\mathbb{F}$ as the Perron family of $P^{'}$-subaverage functions.

Theorem 3. *The following are equivalent:*

(i) Any bounded $P^{'}$-superaverage function u defined outside a finite set is of the form $u = p - q$ where p and q are bounded $P^{'}$-potentials on N.

(ii) Any bounded $P^{'}$-superaverage function in N is a $P^{'}$-potential.

(iii) 0 is the only bounded $P^{'}$-average function in N.

(iv) The constant function 1 is a $P^{'}$-potential on N, that is N is parahyperbolic.

Proof. (i) implies (ii). Let s be a bounded $P^{'}$-superaverage function in N. Then by (i), $s = p - q$ outside a finite set A. Hence $|s| \leq p + q$ on N/A. Since A is a finite set, s is bounded on A and we select a large constant $\alpha > 1$ such that $|s| \leq \alpha(p + q)$ on A. Consequently, $|s| \leq \alpha(p + q)$ on N. Since $-s \leq \alpha(p + q)$, we see that $-s \leq 0$, then $0 \leq s \leq \alpha(p + q)$ so that s is a $P^{'}$-potential on N.
(ii) implies (iii) if $h \neq 0$ is a bounded $P^{'}$-average function on N, then by (ii) it is a $P^{'}$-potential.
(iii) implies (iv) Since 1 is $P^{'}$-superaverage on N, the greatest $P^{'}$-average minorant of 1 is 0. Hence 1 is a $P^{'}$-potential, thus $\{N, P^{'}\}$ is parahyperbolic.
(iv) implies $i)$ Let $u = p - q$ outside a finite set in N. Since u is bounded by hypothesis and q is bounded, it is clear that p is bounded on N. Since 1 is a $P^{'}$-potential by (iv) the bounded $P^{'}$-superaverage function p is a $P^{'}$-potential.

Theorem 4. *If (N, P) is parabolic, then $(N, P^{'})$ is parahyperbolic.*

Proof. For let $h^{'}$ be a $P^{'}$-average function on N such that $|h^{'}| \leq M$, where M is a constant. Then, $|h^{'}|$ is $P^{'}$-subaverage on N and hence P-subaverage. Since, by assumption there is no positive P-potential on N, $|h|$ must be a constant thus $|h| = c$. If $c \neq 0$, in $|h| = c$, $|h|$ is $P^{'}$-subaverage and c is $P^{'}$-superaverage which is a contradiction. Hence $c = 0$ that is $h = 0$. Thus 0 is the only bounded $P^{'}$-average function on N. Hence the constant function 1 is a $P^{'}$-potential on N by the Theorem 3.

Theorem 5. *If $(N, P^{'})$ is parahyperbolic, then any lower bounded $P^{'}$-superaverage function is non-negative. Conversely, if any lower bounded $P^{'}$-average function is non-negative, then $(N, P^{'})$ is parahyperbolic.*

Proof. Let $(N, P^{'})$ be parahyperbolic. Suppose s is a $P^{'}$-superaverage function on N such that $s \geq -M$ for some $M > 0$. Since M is $P^{'}$-potential by assumption , $-s \leq M$ implies that $s \geq 0$. Conversely, suppose any lower bounded $P^{'}$-average function on N is non-negative. If $(N, P^{'})$ is not parahyperbolic, then by Theorem 4 there exists a $P^{'}$-average function h on N, $0 < h < 1$. Since $-h$ is lower bounded, $-h \geq 0$ a contradiction.

Corollary 2. *Suppose h is a $P^{'}$-average function bounded on one side in N. If h takes both positive and negative values in N, then there exists a bounded $P^{'}$-average function H, $0 < H < 1$, on N, hence N is bounded hyperbolic.*

5. Relation between bounded $P^{'}$ and P-average functions

In a random walk the constant function 1 is P-average on N. The question is: what can we say about the existence of bounded or just positive P-average functions on N that are not constants? We have examples of $\{N, P\}$ on which there are no non-constant bounded or just positive P-average functions. In this section we try to assert the existence of such functions on $\{X, P\}$ if similar functions exist on $\{X, P^{'}\}$ where $P^{'}$ is subordinate to P.
If there are non-zero bounded $P^{'}$-average functions on N, then the constant 1 is not a $P^{'}$-potential, hence there are bounded positive $P^{'}$-average functions on N. In this section we investigate the relation between bounded $P^{'}$-average functions and bounded P-average functions on N.

Theorem 6. *Let (N, P) be hyperbolic with its Green's potential $G_y(x)$ satisfying the condition $sup_{z\in N} G_z(z) \leq M$. If $\sum\limits_x [1 - p^{'}(x)] < \infty$, then N has bounded positive $P^{'}$-average functions on N.*

Proof. If 0 is the only bounded positive $P^{'}$-average function on N, then constant 1 is a $P^{'}$-potential in N and

$$\begin{aligned}
1 =& \sum_y [1 - p^{'}(y)] G^{'}_y(x) \; for \; x \in N \\
\leq& \sum_y [1 - p^{'}(y)] G_y(x) \\
\leq& \sum_y [1 - p^{'}(y)] G_y(y) \\
\leq& M \sum_y [1 - p^{'}(y)] \\
<& \infty.
\end{aligned}$$

Hence $u(x) = \sum_y [1 - p^{'}(y)] G_y(x)$ should be a P-potential. But this is not possible since $u(x)$ maximizes the P-average function 1.

Theorem 7. *Let B (respectively $B^{'}$) be the set of all bounded non-negative P-(respectively $P^{'}$-) average functions in N. Then there is an injective map $S : B^{'} \rightarrow B$ such that $S(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 S(h_1) + \alpha_2 S(h_2)$ where α_1, α_2 are non-negative constants and h_1, h_2 are in $B^{'}$.*

Proof. Let $h \in B^{'}$. Then h is a bounded P-subaverage function. Let $S(h)$ be the least P-average majorant of h. Then $S(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 S h_1 + \alpha_2 S h_2$. Suppose $S(h_1) = S(h_2)$. Note that for $h \in B^{'}$, $S(h) - h$ is a P-potential and hence a $P^{'}$-potential. Consequently, if $S(h_1) = S(h_2)$, then $|h_1 - h_2| = |[S(h_1) - h_1] - [S(h_2) - h_2]| \leq p_1 + p_2$ where p_1 and p_2 are $P^{'}$-potential on N. since $|h_1 - h_2|$ is $P^{'}$-subaverage function on N, $h_1 = h_2$.

Corollary 3. *If there are non-proportional bounded non-negative $P^{'}$-average functions in N, then there is atleast one non-constant bounded P-average function in N.*

Proof. If h_1 and h_2 are non-proportional in $B^{'}$, then $S(h_1)$ and $S(h_2)$ are non-proportional bounded P-average functions in N. Hence atleast one of them is non-constant.

Lemma 3. *Let h be a $P^{'}$-average function in N, such that $|h| \leq s$ where s is $P^{'}$-superaverage on N. Then $h = h_1 - h_2$ where h_1 and h_2 are non-negative $P^{'}$-average functions such that $h_1 - h^+$ and $h_2 - h^-$ are $P^{'}$-potentials. This decomposition is unique.*

Proof. Let h_1 be the least $P^{'}$-average majorant of h^+ and h_2 be the least $P^{'}$-average majorant of h^-. Then $p_1 = h_1 - h^+$ and $p_2 = h_2 - h^-$ are $P^{'}$-potentials on N. Hence $h = h^+ - h^- = (h_1 - h_2) - (p_1 - p_2)$. Then by the uniqueness of Riesz decomposition, $h = h_1 - h_2$ on N.
Suppose $h = u_1 - u_2$ is another such decomposition. Since $u_1 - h^+$ and $h_1 - h^+$ are potentials, so $u_1 = h_1$ and then $u_2 = h_2$.

Theorem 8. *If there exists a bounded $P^{'}$-average function on N that takes both positive and negative values, then there is atleast one bounded non-constant P-average function on N.*

Proof. Let h be a bounded $P^{'}$-average function, write $h = h_1 - h_2$ as in Lemma 3, since h takes both positive and negative values by the assumption, h_1 and h_2 are positive. Suppose $h_1 = \lambda h_2$. Then $h = (\lambda - 1)h_2$, contradicting the assumption that h takes both positive and negative values on N. Since h_1 and h_2 are non-proportional, by Corollary 3, there is atleast one non-constant bounded P-average function on N.

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