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A New Product On Category Of Welded Tangle-oids

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Abstract. In this paper, we address a new product in the category of tangle-oids that we call *additive product*. We prove that this additive product with formal addition on tangle-oids gives us associative algebra. Last we give the relationship between this new product, and tensor and compositions on tangle-oids.

2020 Mathematics Subject Classifications: 18A05, 57M25, 18D10, 16T10, 57M27

Key Words and Phrases: Welded Tangle-oid, Knotoids, additive product, Tangle.

1. Introduction

Welded tangle-oids category is defined in [2] as a free category over monoidal graphs (generators) with some relations defined by tensor product and composition. Also we can see welded tangle-oids category as generators of the category of tangle [5] and knotoids see [3, 6, 9]) with some more relations $[WT_{13}], [WT_{14}]$ that are in the definition of welded tangle-oid category in section 2. So in other words, welded tangle-oids category is a tangle category that allows the strands to break, that is, in the definition of tangle-oids the generators ! and ¡ from 0 to 1 and from 1 to 0 respectively with some more relations.

In [7], we have shown an application of welded tangle-oids to effective quantum field theories. Effective quantum field theories (EQFTs) are field theories for composite particles derived from more fundamental quantum field theories (QFTs). For example, one can derive an EQFT for protons or neutrons that are composed on three quarks from Quantum chromodynamics, the fundamental QFT for quarks and gluons, the elementary particles that make up protons or neutrons.

In the category of welded tangle-oids, the generator X will come into play. It is a contraction of a quartic vertex. Cubic vertices are also in X, when two of its four open ends are identified to be equal. It is a contraction of a quartic vertex. Cubic vertices are

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also in, when two of its four open ends are identified to be equal. A contraction will be of generator type, that leading just from one field to another within a polynomial depicted by (same indices) or within different -polynomials (different indices). Moreover, they may also depict these contractions. Distinction between and will be time ordering: will occur if fields on different times will be contracted, are contraction on equal times. In case of non-contracted variables, we have generators, open ends in graphs. This happens, if elementary quantum fields still matter in effective theories like when treating atoms or molecules, but electron and/or photon (are particles arising in the more fundamental theory of Quantum Electrodynamics) dynamics still matter. Finally, we have the general case with, where interaction vertices matter. Here, the generator will come into play. It is a contraction of a quartic vertex. Cubic vertices are also in when two of its four open ends are identified to be equal. So this category will resemble all possible contractions performed when transitioning from a fundamental theory with up to quartic vertices to an effective theory. It distinguishes between equal-time contractions (pure compositions of elementary particles, e.g. atoms and molecules) and different-time contractions (quasiparticles which carry information on specific dynamic behavior, e.g. clusters linked to certain scattering processes.

In the paper [8] we apply tangle-oids to biosystems using effective quantum field theories (EQFT). The open question whether quantum field theories can be used in biological systems will be addressed in this study. Quantum effects were reported for certain biological systems like the magnetic orientation sense in migratory birds. In quantum field theories, it is possible to derive effective quantum field theories with welded tangle-oids including braid relations that describe composite particles based on the dynamics of microscopic particle where these composite particles are made of. We observe that the generators of the tangle-oid category that will depict the Feynman graphs are generators X for a scattering vertex. This arises after carrying out the integral over bosonic fields A_μ in the partition function that will lead to a four-valent interaction vertex generated by a quartic term in the fermionic fields. With X_+ and X_- we depicted order changes like that regarding time orderings. Ordinary propagators are depicted by $\cup$ and $\cap$. Finally, the generators ! and ¡ come into play if other foreign fields are picked up.

In this paper, we define a new product over these categories and prove it is associative algebra by this product and formal addition. Here our motivation is classifying all objects in the category of tangle-oids.

In this paper in Section 2 we describe welded tangle-oid category and in Section 3 we define a new product in this category giving relations and generators using the relationship between this new product and tensor and composition. Finally, we conclude by Conclusion and Future Works section.

2. Unoriented Welded Tangle-oids

In this section we review the definition of welded tangle-oids categories defined in [2]. A monoidal category (see for example [5]) of unoriented welded tangle-oids have defined by giving a presentation by using presentation of slideable $\frac{1}{2}$-monoidal categories [2].

Definition 1. *[2, definition 7.2.1] Consider the monoidal graph*

$$\beta = (\mathbb{N}, E(\beta), \otimes_0, 0, \delta_1, \delta_2),$$

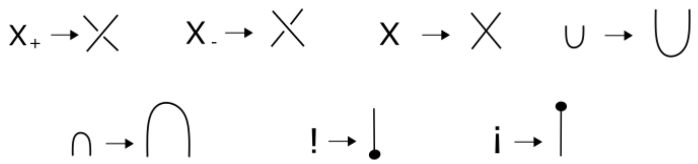
where for all $m, n \in \mathbb{N}$, $m \otimes_0 n = m + n$, *and*

$$E(\beta) = \{X_+, X_-, X, \cup, \cap, ¡, !\},$$

the incidence maps

$$\begin{array}{cccc} \delta_1 X_+ = 2, & \delta_2 X_+ = 2, & \delta_1 X_- = 2, & \delta_2 X_- = 2, \\ \delta_1 X = 2, & \delta_2 X = 2, & \delta_1 \cup = 0, & \delta_2 \cup = 2, \\ \delta_1 \cap = 2, & \delta_2 \cap = 0, & \delta_1 ¡ = 1, & \delta_2 ¡ = 0, \\ \delta_1 ! = 0, & \delta_2 ! = 1. & & \end{array}$$

These generators can be presented geometrically as



Consider the path category, see for example ([4], over β^*, the extent of the monoidal graph β.

$$P(\beta^*) = (\mathbb{N}, \hom_{P(\beta^*)}(n, m), \bullet, \phi_{\_}).$$

Therefore

$$\Omega(\beta) = (P(\beta^*), \otimes_0, 0, {}_n\#, \#_m)$$

is a $\frac{1}{2}$-monoidal category, whose set of objects is the set of natural numbers, where for all $n, m, k \in \mathbb{N}$;

$${}_n\#_m(k) = n \otimes_0 k \otimes_0 m = n + k + m,$$

and for all generating morphism $(f\colon k \to k') \in E(\beta)$, we have

$${}_n\#_m(f) = n + k + m \xrightarrow{n \Theta f \Theta m} n + k' + m.$$

Then we have the free-$\frac{1}{2}$-monoidal category-triple

$$(\beta, \Omega(\beta), \delta).$$

Definition 2 (Unoriented welded tangle-oids category)**.** *The* unoriented welded tangle-oids category *UWTC is the strict monoidal category formally presented by*

$$\mathfrak{F}\Big(\Omega(\beta) \Big/ \underline{\overline{W}}\Big),$$

where $\Omega(\beta)$ *defined in [2, Section 7.2] and* $\underline{\overline{W}}$ *is the* $\frac{1}{2}$*-monoidal closure of the congruence template* W *that is defined as follows.*

Given $m, n \in \mathbb{N}$*, then* $W_{m,n}$ *is the relation in* $\hom_{P(\beta^*)}(m,n)$*, defined as (the picture will follow)*

In $\hom_{P(\beta^*)}(1,1)$*, we have the only relations*

- $[WT_1]$: $(\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1)$.
- $[WT_2] : (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_3] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1)$.
- $[WT_4] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(\cup \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(2,2)$*, we have the only relation*

- $[WT_5] : X_- X_+ \sim_{W_{2,2}} \mathrm{id}_2 \sim_{W_{2,2}} X_+ X_-$.

In $\hom_{P(\beta^*)}(3,3)$*, we have the only relations*

- $[WT_6] : (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_7] : (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_8] : (X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)$.

In $\hom_{P(\beta^*)}(3,1)$*, we have the only relations*

- $[WT_9] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)$.
- $[WT_9]' : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)$.
- $[WT_9]'' : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(1,3)$*, we have the only relations*

- $[WT_{10}] : (\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]' : (\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]'' : (\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.

In $\hom_{P(\beta^*)}(1,0)$*, we have the only relation*

- $[WT_{11}] : \cap(\mathrm{id}_1 \otimes !) \sim_{W_{1,0}} ¡ \sim_{W_{1,0}} \cap(! \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(0,1)$*, we have the only relation:*

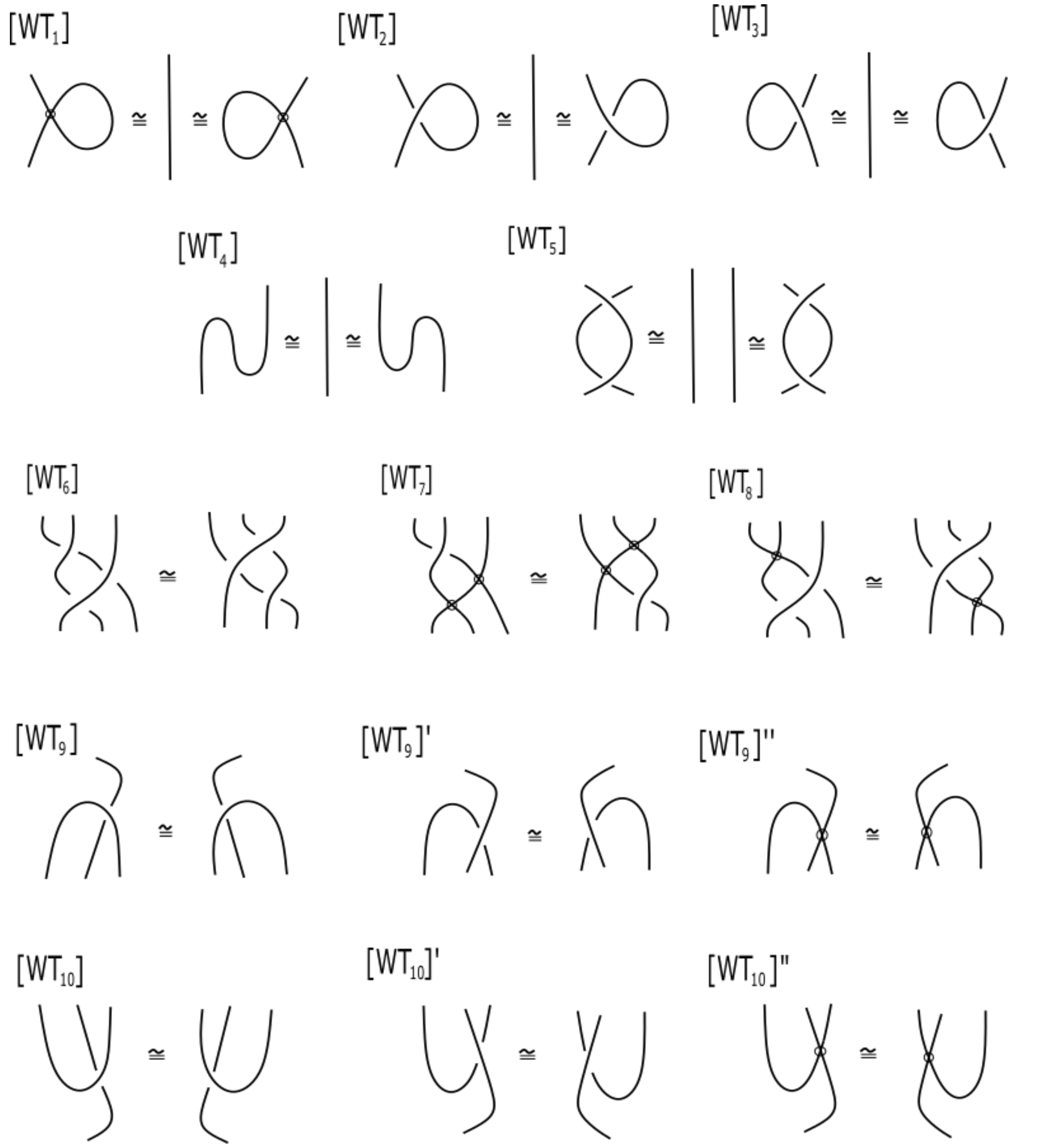
- $[WT_{12}] : (\mathrm{id}_1 \otimes ¡)\cup \sim_{W_{0,1}} ! \sim_{W_{0,1}} (¡ \otimes \mathrm{id}_1)\cup$.

In $\hom_{P(\beta^*)}(2,1)$*, we have the only relations*

- $[WT_{13}] : (¡ \otimes \mathrm{id}_1)X_+ \sim_{W_{2,1}} \mathrm{id}_1 \otimes ¡$.
- $[WT_{13}]' : (\mathrm{id}_1 \otimes ¡)X_- \sim_{W_{2,1}} ¡ \otimes \mathrm{id}_1$.
- $[WT_{14}] : (¡ \otimes \mathrm{id}_1)X \sim_{W_{2,1}} \mathrm{id}_1 \otimes ¡$.
- $[WT_{14}]' : (\mathrm{id}_1 \otimes ¡)X \sim_{W_{2,1}} ¡ \otimes \mathrm{id}_1$.

Note that we do not impose that in $\hom_{P(\beta^*)}(2,1)$***:***
$(¡ \otimes \mathrm{id}_1)X_- \nsim_{W_{2,1}} \mathrm{id}_1 \otimes ¡$.

These relations can be present geometrically as (note we read the diagram from bottom to top)


$[WT_1]$
$[WT_2]$
$[WT_3]$
$[WT_4]$
$[WT_5]$
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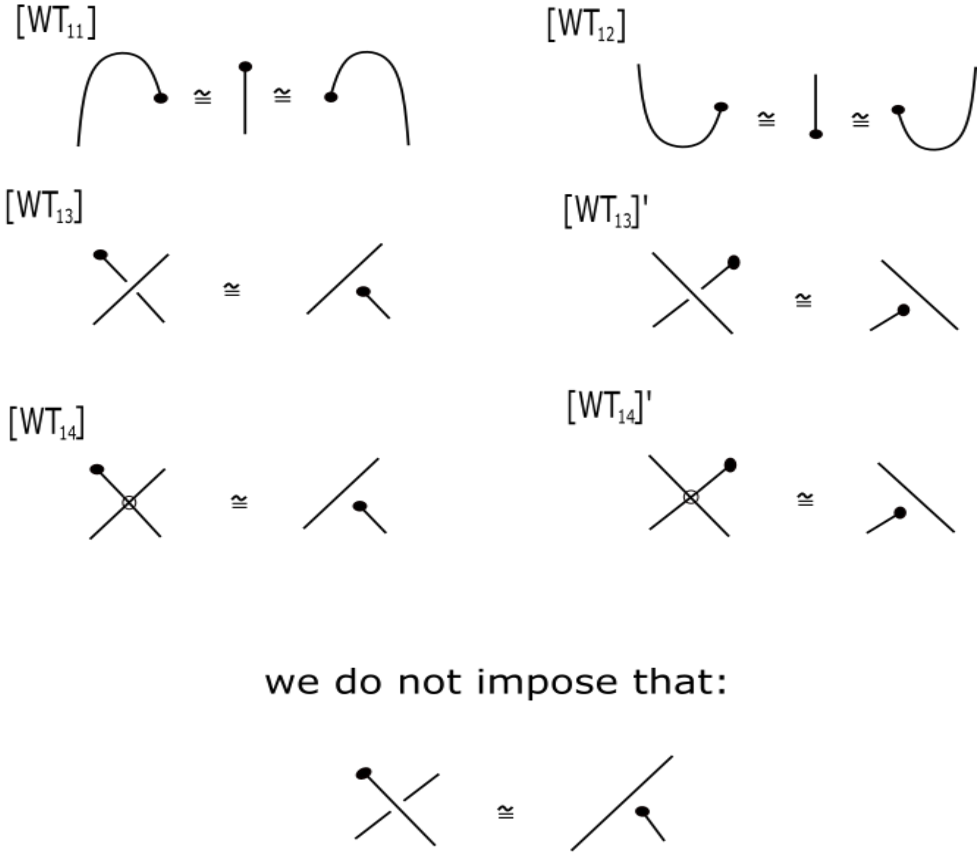
$[WT_7]$
$[WT_8]$
$[WT_9]$
$[WT_9]'$
$[WT_9]''$
$[WT_{10}]$
$[WT_{10}]'$
$[WT_{10}]''$



3. Additive Product

In this section, we will define the additive product in welded tangle-oids categories and explain the algebraic structure of it, for more about algebraic structure (see for example [1]).

Definition 3. *The additive product* $\oplus$ *in welded tangle-oids categories is defined as follows, on objects, let* $n, m \in N$,

- $\forall m \in \mathbb{N}, m \oplus 1 = m = 1 \oplus m,$
- $\forall\ m, n \in \mathbb{N},\ m \oplus n = m + n - 1,$
- $0 \oplus 0$ *is not defined.*

Theorem 1. *The additive product* $\oplus$ *in welded tangle-oids categories satisfy the following properties.*

On morphisms;

- *let* $n, n', m, m' \in \mathbb{N}$, *and* $f\colon n \to n'$ *and* $g\colon m \to m'$,

$$f \oplus g\colon n \oplus m \to n' \oplus m'.$$

Note that the additive not defined in case $n = m = 0$ *or* $n' = m' = 0$.

- *the identity morphisms* $\mathrm{id}_1\colon 1 \to 1$,

- *the product is well defined, let* $f, f'\colon n \to n'$, $g, g'\colon m \to m'$ *and* $f \cong f'$, $g \cong g'$. *Then,*

$$f \oplus g\colon n \oplus m \to n' \oplus m'$$
$$f' \oplus g'\colon n \oplus m \to n' \oplus m',$$

Then

$$f \oplus g \cong f' \oplus g'.$$

- *the product is commutative, on objects,* $\forall m, n$,

$$m \oplus n = m + n - 1 = n \oplus m,$$

- *associativity, on objects let* $m, n, u \in \mathbb{N}$,

$$\begin{aligned}(m \oplus n) \oplus u &= m + (n-1) \oplus u = m + n - 1 + u - 1 = m + n + u - 2,\\ m \oplus (n \oplus u) &= m \oplus n + (u-1) = m + n + u - 1 - 1 = m + n + u - 2.\end{aligned}$$

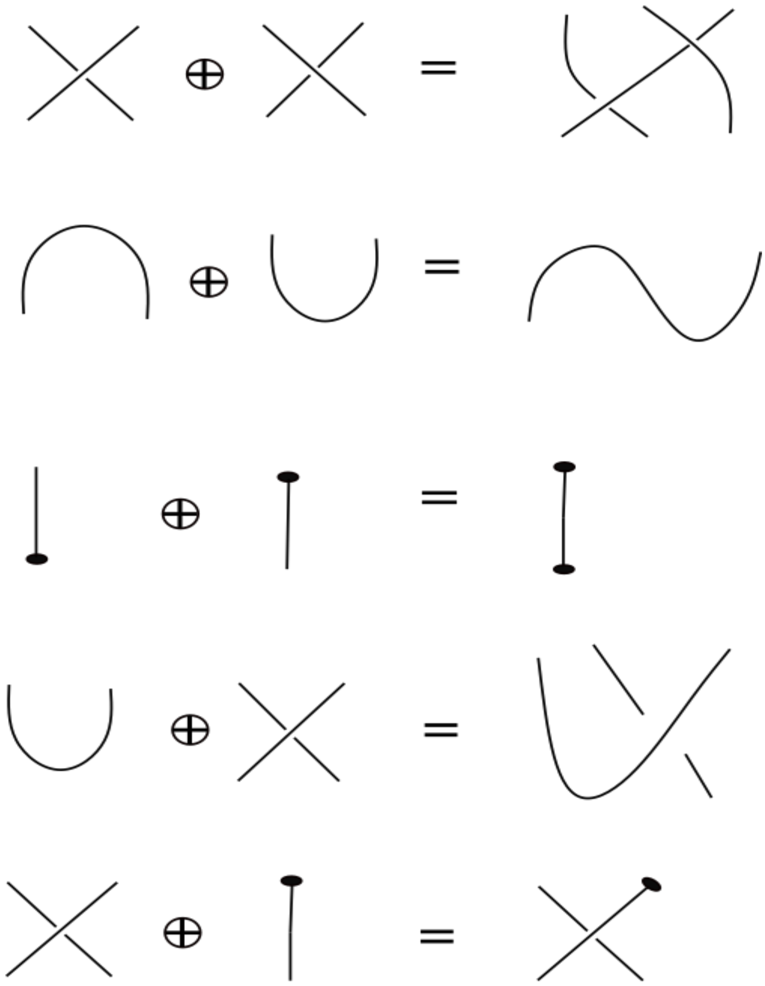
let $f\colon n \to n', g\colon m \to m'$ *and* $h\colon u \to u'$, *we have*

$$\begin{aligned}(f \oplus g) \oplus h&\colon (n \oplus m) \oplus u \to (n' \oplus m') \oplus u'\\ f \oplus (g \oplus h)&\colon n \oplus (m \oplus u) \to n' \oplus (m' \oplus u'),\end{aligned}$$

then

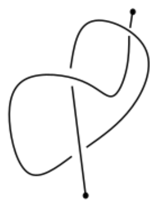
$$(f \oplus g) \oplus h = f \oplus (g \oplus h).$$

See the following example:



Theorem 2. *The category of welded tangle-oids is an associative algebra by formal addition and by additive product* $\oplus$.

Example 1. *We can write any welded tangle-oids as composition and tensor or as composition, tensor and sum. Here is an example.*

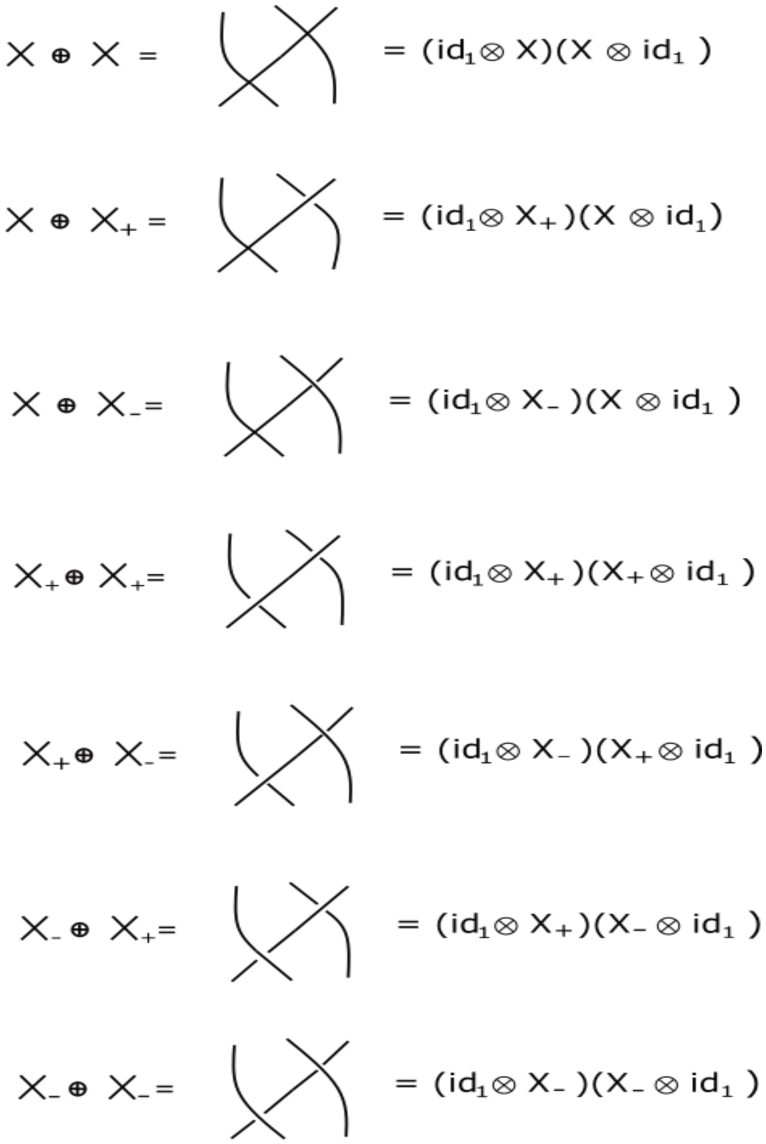


Represent the example algebraically in two different ways,

1. $(\cap \otimes \cap \otimes ¡)(\mathrm{id}_1 \otimes X_- \otimes X_-)(\mathrm{id}_2 \otimes \cup \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-)(\cup \otimes !)$,
2. $(\cap \otimes ¡) \otimes (\cap \oplus X_- \otimes X_-)(\mathrm{id}_1 \otimes \cup \otimes \mathrm{id}_1)(\cup, \oplus X_- \oplus !)$.

Then we have the following result for the tangle-oids.

Theorem 3. *There is a relationship between composition, $\otimes$ in the definition of welded tangle-oids category and $\oplus$, we address this relation on the category's generators as follows.*



$X \oplus \cup =$  $= (X \otimes id_1)(id_1 \otimes \cup)$

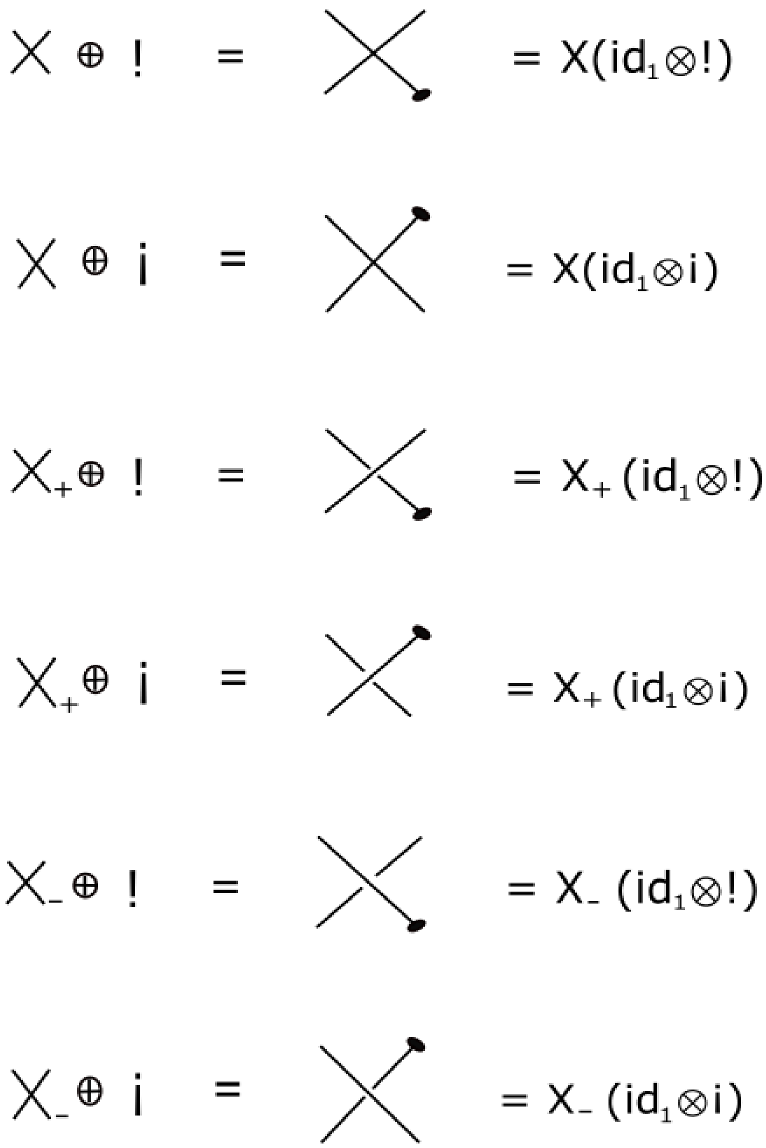
$X \oplus \cap =$  $= (id_1 \otimes \cap)(X \otimes id_1)$

$X_+ \oplus \cup =$  $= (X_+ \otimes id_1)(id_1 \otimes \cup)$

$X_+ \oplus \cap =$  $= (id_1 \otimes \cap)(X_+ \otimes id_1)$

$X_- \oplus \cup =$  $= (X_- \otimes id_1)(id_1 \otimes \cup)$

$X_- \oplus \cap =$  $= (id_1 \otimes \cap)(X_- \otimes id_1)$



4. Conclusion and Future Work

In this paper, we defined a new product in the category of tangle-oids which we call *additive product*. We proved that this additive product with formal addition on tangle-oids gives us associative algebra. Finally, we give the relationship between this new product, and tensor and compositions on tangle-oids. Here our motivation was classifying all objects in the category of tangle-oids using algebraic binary operations.

As a future work, we would like to calculate explicitly the ring structure of this algebra by generators and ideals. Also want to give this algebraic structure by matrices and graphs and we want to find a correspondence between tangle-oids, Federov polytopes and associated graphs and matrices with applications to broken DNA's. Especially the rational tangle-oids and their invariants play an important role in this idea.

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