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Sheffer Stroke BN-algebras and Connected Topics

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Abstract. This article introduces the concept of a Sheffer stroke BN-algebra by applying the Sheffer stroke operator | to the BN-algebra axioms and aligning it with the axioms of the Sheffer stroke groupoid. From this definition, properties of Sheffer stroke BN-algebras are derived, focusing on the relationship between the axioms and the properties of the special element 0. Furthermore, the notions of Sheffer stroke BN-subalgebras, BN-ideals, and BN-homomorphisms are defined, along with normal subsets of Sheffer stroke BN-algebras, and the relationships between these concepts are explored. It is shown that every normal subset in a Sheffer stroke BN-algebra is a Sheffer stroke BN-subalgebra, but the converse is not necessarily true. This implies that every normal BN-ideal in a Sheffer stroke BN-algebra is also a Sheffer stroke BN-subalgebra. Finally, the properties of the kernel of the Sheffer stroke BN-homomorphism are investigated.

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Key Words and Phrases: BN-algebra, Sheffer stroke, Sheffer stroke BN-algebra, BN-ideal, BN-subalgebra

1. Introduction

The Sheffer stroke operator is equivalent to the NAND logic gate, a fundamental component in digital electronics. A NAND gate is a type of integrated logic gate with two inputs and one output, essentially functioning as a combination of a NOT gate and an AND gate. NAND gates work closely with other logic gates to regulate the flow of information and instructions in computers, the behavior of devices such as electric motors and water pumps in industrial control systems, and access to buildings or spaces in security systems. In addition to having a wide range of practical applications, the Sheffer stroke operator has inspired many theoretical developments in mathematics, including in the context of algebra.

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An operator $|$ is called a Sheffer stroke operator in a groupoid T if it satisfies the following four conditions: ($\mathcal{S}1$) $a|b = b|a$, ($\mathcal{S}2$) $(a|a)|(a|b) = a$, ($\mathcal{S}3$) $a|[(b|c)|(b|c)] = [(a|b)|(a|b)]|c$, ($\mathcal{S}4$) $[a|((a|a)|(b|b))]|[a|((a|a)|(b|b))] = a$, for all $a, b, c \in T$ [2].

The functions and axioms of Boolean algebra can be expressed using the Sheffer stroke operator, as can those of Hilbert algebras [15]. The Sheffer stroke operator has also been defined for R_0-algebras [10]. An R_0-algebra contains a unary operation and three binary operations. However, after defining the Sheffer stroke operator in the R_0-algebra, the resulting set contains only two binary operations with the special properties of the Sheffer stroke operator. Further research has explored the Sheffer stroke and its properties in additional algebraic systems [1, 12–14, 16, 17].

The extensive study of BCI-algebra and BCK-algebra influenced the development of BN-algebras. A non-empty set A equipped with a constant 0 and a binary operation $*$ is called a BN-algebra if it meets the following axioms: (BN1) $a * a = 0$, (BN2) $a * 0 = a$, (BN3) $(a * b) * c = (0 * c) * (b * a)$ for all $a, b, c \in A$ [11]. Building on a work that researched ideals in BN-algebra [3], various further studies have developed the ring-theoretic concepts of ideals and normality in BN-algebra and related algebras, as seen in [4, 5, 9].

Motivated by the close relationship between ideals and normal subsets in BN-algebra, as discussed in previous research, and the author's experience in abstract algebra, especially work concerning the concept of algebraic structures, such as in [6–8], in this study, the concept of a Sheffer stroke BN-algebra is defined and its properties are determined. Then, the notions of Sheffer stroke BN-subalgebras, BN-ideals, and BN-homomorphisms and their kernels, as well as normal subsets of Sheffer stroke BN-algebras, are introduced, and the relationships between these concepts are investigated.

2. Preliminaries

This section presents several definitions and properties necessary for the construction of Sheffer stroke BN-algebras and the other concepts explored in this study.

Definition 1. *[11] A BN-algebra is a non-empty set A equipped with a constant 0 and a binary operation $*$ that meets the following axioms:*

(BN1) $a * a = 0$ *for all* $a \in A$,

(BN2) $a * 0 = 0$ *for all* $a \in A$,

(BN3) $(a * b) * c = (0 * c) * (b * a)$ *for all* $a, b, c \in A$.

Theorem 1. *[11] Suppose $(A; *, 0)$ be a BN-algebra, then for all $a, b, c \in A$:*

(i) $0 * (0 * a) = a$,

(ii) $b * a = (0 * a) * (0 * b)$,

(iii) $(0 * a) * b = (0 * b) * a$,

(iv) *if* $a * b = 0$, *then* $b * a = 0$,

(v) *if* $0 * a = 0 * b$*, then* $a = b$*,*

(vi) $(a * c) * (b * c) = (c * b) * (c * a)$.

Definition 2. *[11] A BN-algebra* $(A; *, 0)$ *that satisfies* (D) $(a * b) * c = a * (c * b)$ *for all* $a, b, c \in A$ *is called a BN-algebra with condition (D).*

Theorem 2. *[11] Suppose* $(A; *, 0)$ *is a BN-algebra with condition* (D)*. Then, the following properties are satisfied for all* $a, b, c \in A$*:*

(i) $0 * a = a$,

(ii) $a * b = b * a$.

Definition 3. *[3] Suppose* $(A; *, 0)$ *is a BN-algebra. A non-empty subset* I *of* A *is defined as an ideal in* A *if it meets the following conditions:*

(i) $0 \in I$*, and*

(ii) for all $a, b \in A$*, if* $b \in I$ *and* $a * b \in I$*, implies* $a \in I$.

Suppose $(A; *, 0)$ is a BN-algebra. A non-empty subset S is considered a subalgebra of A if it satisfies $a * b \in S$ for all $a, b \in S$. A non-empty subset N of A is termed a normal if it satisfies $(x * a) * (y * b) \in N$ for all $x * y, a * b \in N$, where $a, b, x, y \in A$. In a BN-algebra $(A; *, 0)$, the operation $\wedge$ is defined as $a \wedge b = b * (b * a)$ for all $a, b \in A$.

Definition 4. *[2] The operator denoted by* $|$ *is called a Sheffer stroke on a groupoid* T *if the following four conditions are met: for all* $a, b, c \in T$

$(\mathcal{S}1)$ $a|b = b|a$,

$(\mathcal{S}2)$ $(a|a)|(a|b) = a$,

$(\mathcal{S}3)$ $a|((b|c)|(b|c)) = ((a|b)|(a|b))|c$,

$(\mathcal{S}4)$ $(a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a$.

3. Sheffer stroke BN-algebras

In this section, the Sheffer stroke BN-algebra is defined as an algebra of type (2,0) with the Sheffer stroke operation | and the constant element 0. In this algebra, we integrate the Sheffer stroke operation with the BN-algebra axioms by examining the interrelationships among these axioms. Two main axioms are defined for Sheffer stroke BN-algebra, (SBN1) and (SBN2). Then, its properties are determined.

Definition 5. *A Sheffer stroke BN-algebra is an algebra (*$\mathcal{P}_{SBN}; |, 0$*) of type* $(2, 0)$ *containing the constant 0, where* $|$ *is the Sheffer stroke operation in* $\mathcal{P}_{SBN}$ *and the following axioms are satisfied for all* $a, b, c \in \mathcal{P}_{SBN}$*:*

(SBN1) $(a|(0|0))|(a|(0|0)) = a$,

(SBN2) $(a|(b|b))|((0|c)|(0|c)) = [(0|(0|(c|c))|(0|(0|(c|c))]|(b|(a|a))$.

Lemma 1. *If* ($\mathcal{P}_{SBN}; |, 0$) *is a Sheffer stroke BN-algebra, then the axioms (SBN1) and (SBN2) are independent.*

Proof. Let $(\mathcal{P}_{SBN}; |, 0)$ be a Sheffer stroke BN-algebra. We give examples of groupoids $(P; |, 0)$ and $(Q; |, 0)$ to show that each axiom can be satisfied when the other is not.

(i) Independence of (SBN1):
We provide an example in which (SBN1) does not hold but (SBN2) does. Consider the groupoid $(P; |, 0)$, where $P = \{0, 1\}$, defined as in Table 1.

Table 1: Composition table for $(P; |, 0)$

$\vert$	0	1
0	0	1
1	0	0

It can be demonstrated that $(P; |, 0)$ fulfills (SBN2), but it fails to satisfy (SBN1) when $a = 1$, since $(1|(0|0))|(1|(0|0)) = (1|0)|(1|0) = 0|0 = 0 \neq 1$.

(ii) Independence of (SBN2):
We present an example in which (SBN2) does not apply while (SBN1) remains valid. Take the groupoid $(Q; |, 0)$, where $Q = \{0, 1\}$, defined as in Table 2.

Table 2: Composition table for $(Q; |, 0)$

$\vert$	0	1
0	0	0
1	1	1

It can be shown that $(P; |, 0)$ satisfies (SBN2), but it does not fulfill (SBN1) when $a = 1$, $b = 1$, and $c = 0$ since the left-hand side $(1|(1|1))|((0|0)|(0|0)) = (1|1)|(0|0) = 1|0 = 1$, whereas the right-hand side
$[(0|(0|(0|0))|(0|(0|(0|0))]|(1|(1|1)) = (0|(0|0))|(0|(0|0))])|(1|1) = 0|1 = 0$.

Next, an example of an algebra that satisfies the Sheffer stroke BN-algebra property is presented, showing the structure of the algebra with elements $\{0, p, q, 1\}$ that satisfy the specified axioms.

Example 1. *Consider* ($\mathcal{P}_{SBN}; |, 0$)*, an algebra with the Sheffer stroke operation and the constant 0, defined as in Table 3.*

Table 3: Composition table for $(\mathcal{P}_{SBN};|,0)$

$\vert$	0	p	q	1
0	1	1	1	1
p	1	q	q	q
q	1	q	p	p
1	1	q	p	0

It can be shown that $(\mathcal{P}_{SBN};|,0)$ is a Sheffer stroke BN-algebra.

By applying the Sheffer stroke operation to the elements in the algebra $(\mathcal{P}_{SBN};|,0)$, we can discover relationships that apply to all elements in $\mathcal{P}_{SBN}$. These relationships show how the elements interact through the operation $|$ and constant 0, for example, how this operation relates to the identity element or changes when applied to the same or different elements. In the following theorem, some basic properties are established to help understand the structure of the Sheffer stroke BN-algebra and provide a solid basis for further analysis of the relationships between elements in this algebra.

Theorem 3. *Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra. Then, the following properties are satisfied for all $a,b\in\mathcal{P}_{SBN}$:*

(i) $a|a = a|(0|0)$,

(ii) $[(a|b)|(a|b)]|b = a|b$,

(iii) $(a|(a|a))|(a|a) = a$,

(iv) $0|(0|(a|a)) = 0|(a|a)$,

(v) $0|(a|(b|b)) = 0|(b|(a|a))$,

(vi) $0|(a|a) = 0|a$.

Proof. Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra.

(i) By substituting $b := a$ and $c := 0|0$ into $(\mathcal{S}3)$, for all $a\in\mathcal{P}_{SBN}$, we have

$$a|[(a|(0|0))|(a|(0|0))] = ((a|a)|(a|a))|(0|0).$$

Then, by applying axioms (SBN1) on the left-hand side and $(\mathcal{S}2)$ on the right-hand side, we obtain $a|a = a|(0|0)$.

(ii) By substituting $c := b$ into $(\mathcal{S}3)$, we have $[(a|b)|(a|b)]|b = a|[(b|b)|(b|b)]$. Then, by using $(\mathcal{S}2)$ we obtain $[(a|b)|(a|b)]|b = a|b$ for all $a,b\in\mathcal{P}_{SBN}$.

(iii) By using axioms $(\mathcal{S}1)$ and $(\mathcal{S}2)$, we find that

$$(a|(a|a))|(a|a) = (a|a)|(a|(a|a)) = a,$$

for all $a\in\mathcal{P}_{SBN}$.

(iv) By substituting $c := a$ and $b := 0$ into (SBN2), and by using (i), (ii), ($\mathcal{S}1$), and ($\mathcal{S}2$), for all $a \in \mathcal{P}_{SBN}$, we obtain

$$\begin{aligned}(a|(0|0))|((0|a)|(0|a)) &= [(0|(0|(a|a))|(0|(0|(a|a))]|(0|(a|a)),\\ (a|(0|0))|((0|a)|(0|a)) &= 0|(0|(a|a)),\\ (a|a)|((0|a)|(0|0)) &= 0|(0|(a|a)),\\ (a|a)|((0|0)|(0|a)) &= 0|(0|(a|a)),\\ (a|a)|0 &= 0|(0|(a|a)),\\ 0|(a|a) &= 0|(0|(a|a)).\end{aligned}$$

(v) By substituting $c := 0$ into (SBN2), and by using ($\mathcal{S}2$), (iv), (SBN1), and ($\mathcal{S}1$), for all $a, b \in \mathcal{P}_{SBN}$, we have

$$\begin{aligned}(a|(b|b))|((0|0)|(0|0)) &= [(0|(0|(0|0))|(0|(0|(0|0))]|(b|(a|a)),\\ (a|(b|b))|0 &= [(0|(0|0))|(0|(0|0))]|(b|(a|a)),\\ (a|(b|b))|0 &= 0|(b|(a|a)),\\ 0|(a|(b|b)) &= 0|(b|(a|a)).\end{aligned}$$

(vi) By substituting $b := a|a$ into (v), we have

$$0|(a|((a|a)|(a|a))) = 0|((a|a)|(a|a)),$$

for all $a \in \mathcal{P}_{SBN}$. Then, by using ($\mathcal{S}2$) we obtain $0|(a|a) = 0|a$ for all $a \in \mathcal{P}_{SBN}$.

In a Sheffer stroke BN-algebra, we will observe how certain conditions involving the Sheffer stroke operation and constant 0 can yield important conclusions about the relationships between elements. Theorem 4 describes a special case in which, if element b has a certain association with element a through this operation, then the result of the Sheffer stroke operation between 0 and b will be the same as the result of the operation between 0 and a.

Theorem 4. *Let* ($\mathcal{P}_{SBN}; |, 0$) *be a Sheffer stroke BN-algebra. If* $a = b|(a|a)$ *for all* $a, b \in \mathcal{P}_{SBN}$*, then* $0|b = 0|a$.

Proof. Let ($\mathcal{P}_{SBN}; |, 0$) be a Sheffer stroke BN-algebra. Since $a := b|(a|a)$ for all $a, b \in \mathcal{P}_{SBN}$, by using (v), ($\mathcal{S}1$), and ($\mathcal{S}2$), we have

$$\begin{aligned}0|(b|(a|a)) &= 0|(a|(b|b)),\\ 0|a &= 0|((b|(a|a))|(b|b)),\\ 0|a &= 0|((b|b)|(b|(a|a))),\\ 0|a &= 0|b.\end{aligned}$$

We next define some additional concepts, but first let us look at the role of subsets in the study of Sheffer stroke BN-algebra, namely, Sheffer stroke BN-subalgebras, normal subsets, and Sheffer stroke BN-ideals, have an important function in maintaining the regularity of this algebra. A Sheffer stroke BN-subalgebra is a part of the algebra that is closed under the Sheffer stroke operation, meaning that the result of each such operation will be in the subset itself. A normal subset is a special subset that makes an algebraic structure more symmetrical, preserving order among certain elements. A Sheffer stroke BN-ideal is a subset that contains the constant 0 and meets certain rules so that it remains consistent under Sheffer stroke operations. These three types of subsets help understand patterns and stability in Sheffer stroke BN-algebra.

Definition 6. *Let* $(\mathcal{P}_{SBN}; |, 0)$ *be a Sheffer stroke BN-algebra and* $\mathcal{S}$ *a non-empty subset of* $\mathcal{P}_{SBN}$*. The set* $\mathcal{S}$ *is called a Sheffer stroke BN-subalgebra of* $\mathcal{P}_{SBN}$ *if it satisfies* $(a|(b|b))|(a|(b|b)) \in \mathcal{S}$ *for all* $a, b \in \mathcal{S}$*.*

Definition 7. *Let* $(\mathcal{P}_{SBN}; |, 0)$ *be a Sheffer stroke BN-algebra and* $\mathcal{N}$ *a non-empty subset of* $\mathcal{P}_{SBN}$*. The set* $\mathcal{N}$ *is called a normal subset of the Sheffer stroke BN-algebra* $\mathcal{P}_{SBN}$ *if it satisfies*

$$(((a|(p|p))|(a|(p|p)))|(b|(q|q)))|(((a|(p|p))|(a|(p|p)))|(b|(q|q))) \in \mathcal{N},$$

for any $(a|(b|b))|(a|(b|b))$*,* $(p|(q|q))|(p|(q|q)) \in \mathcal{N}$*.*

Definition 8. *Let* $(\mathcal{P}_{SBN}; |, 0)$ *be a Sheffer stroke BN-algebra and* $\mathcal{I}$ *a non-empty subset of* $\mathcal{P}_{SBN}$*. The set* $\mathcal{I}$ *is called a Sheffer stroke BN-ideal of* $\mathcal{P}_{SBN}$ *if it satisfies the following conditions:*

(i) $0 \in \mathcal{I}$,

(ii) $b \in \mathcal{I}$ *and* $(a|(b|b))|(a|(b|b)) \in \mathcal{I}$ *imply* $a \in \mathcal{I}$ *for all* $a, b \in \mathcal{P}_{SBN}$*.*

Before deriving results about Sheffer stroke BN-ideals, Sheffer stroke BN-subalgebras, and normal subsets in Sheffer stroke BN-algebras, we will look at a concrete example. This example aims to show how the elements in the algebra fit into these categories of subset categories according to the properties of the Sheffer stroke operation.

Example 2. *Consider the same Sheffer stroke BN-algebra* $(\mathcal{P}_{SBN}; |, 0)$ *defined in Example 1. The set of Sheffer stroke BN-ideals of* $\mathcal{P}_{SBN}$ *consists of* $\{0\}$*,* $\{0, p\}$*,* $\{0, q\}$*, and* $\mathcal{P}_{SBN}$*. The subsets* $\{0\}$*,* $\{0, p\}$*,* $\{0, 1\}$*,* $\{0, p, q\}$*, and* $\mathcal{P}_{SBN}$ *are Sheffer stroke BN-subalgebras of* $\mathcal{P}_{SBN}$*. The normal subsets in* $\mathcal{P}_{SBN}$ *are* $\{0\}$*,* $\{0, p\}$*, and* $\mathcal{P}_{SBN}$*.*

In the following theorem, we will see how the existence of a Sheffer stroke BN-ideal can affect the membership of other elements. The theorem shows that if an element is in the Sheffer stroke BN-ideal and fulfills certain relationships, then there are other elements that are also included in the Sheffer stroke BN-ideal.

Theorem 5. *Let* $(\mathcal{P}_{SBN}; |, 0)$ *be a Sheffer stroke BN-algebra and let* $\mathcal{I}$ *be a Sheffer stroke BN-ideal of* $\mathcal{P}_{SBN}$*. If* $b \in \mathcal{I}$ *and* $(a|(b|b))|(a|(b|b)) = 0$*, then* $a \in \mathcal{I}$ *for all* $a, b \in \mathcal{P}_{SBN}$*.*

Proof. Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra. Since $\mathcal{I}$ is a Sheffer stroke BN-ideal of $\mathcal{P}_{SBN}$, we have $0 \in \mathcal{I}$. Given that

$$b \in \mathcal{I}, (a|(b|b))|(a|(b|b)) \in \mathcal{I},$$

it follows that $a \in \mathcal{I}$. Moreover, since $(a|(b|b))|(a|(b|b)) = 0 \in \mathcal{I}$, we conclude that $a \in \mathcal{I}$ for all $a, b \in \mathcal{P}_{SBN}$.

The following theorem addresses the relationship between Sheffer stroke BN-subalgebras, normal subsets, and Sheffer stroke BN-ideals.

Theorem 6. *Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra. If $\mathcal{N}$ is a normal subset of $\mathcal{P}_{SBN}$, then $\mathcal{N}$ is a Sheffer stroke BN-subalgebra of $\mathcal{P}_{SBN}$.*

Proof. Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra and $a, b \in \mathcal{N}$. By using axiom (SBN1), we obtain $(a|(0|0))|(a|(0|0)) \in \mathcal{N}$ and $(b|(0|0))|(b|(0|0)) \in \mathcal{N}$. Then, by using axiom (SBN1), Theorem 3 (iv), and the fact that $\mathcal{N}$ is a normal subset of $\mathcal{P}_{SBN}$, we obtain

$$(a|(b|b))|(a|(b|b)) = (((a|(b|b))|(a|(b|b)))|(0|0))|(((a|(b|b))|(a|(b|b)))|(0|0)),$$

such that,

$$(((a|(b|b))|(a|(b|b)))|(0|(0|0)))|(((a|(b|b))|(a|(b|b)))|(0|(0|0))) \in \mathcal{N}.$$

Thus, it has been proven that $\mathcal{N}$ is a BN-subalgebra of $\mathcal{P}_{SBN}$.

The converse of Theorem 6 does not hold in general.

In Sheffer stroke BN-algebras, it is important to understand that normal sets always form BN-subalgebras. Theorem 6 shows that if a subset is normal, then its elements will remain connected via the Sheffer stroke operation, satisfying the condition for it to be a BN-subalgebra. However, not all BN-subalgebras are normal, as shown in Example 2. The subset $\mathcal{N} = \{0, 1\}$ is a Sheffer stroke BN-subalgebra of $\mathcal{P}_{SBN}$, but it is not a normal subset of $\mathcal{P}_{SBN}$, since

$$(1|(1|1))|1|(1|1)) = (1|0)|(1|0) = 1|1 = 0 \in \mathcal{I},$$

and

$$(0|(p|p))|(0|(p|p)) = (0|q)|(0|q) = 1|1 = 0 \in \mathcal{I}.$$

However,

$$((1|(0|0))|(1|(0|0)))|(1|(p|p)) = ((1|1)|(1|1))|(1|q) = (0|0)|p = 1|p = q,$$

and hence

$$((1|(0|0))|(1|(0|0)))|(1|(p|p))|((1|(0|0))|(1|(0|0)))|(1|(p|p)) = q|q = p \in \mathcal{N}.$$

Corollary 1. *Let* $(\mathcal{P}_{SBN};|,0)$ *be a Sheffer stroke BN-algebra. If* I *is a Sheffer stroke normal BN-ideal of* $\mathcal{P}_{SBN}$*, then* $\mathcal{I}$ *is a Sheffer stroke BN-subalgebra of* $\mathcal{P}_{SBN}$*.*

Proof. Let $(\mathcal{P}_{SBN};|,0)$ be a Sheffer stroke BN-algebra. Since $\mathcal{I}$ is both a BN-ideal and normal in $\mathcal{P}_{SBN}$, it follows from Theorem 6 that $\mathcal{I}$ is a Sheffer stroke BN-subalgebra of $\mathcal{P}_{SBN}$.

The converse of Corollary 1 does not hold in general.

Corollary 1 reinforces the idea that in a Sheffer stroke BN-algebra, sets that are both BN-ideal and normal necessarily form BN-subalgebras. This underlines that the normality and ideality properties in this algebra have a direct effect on the structure of its BN-subalgebra. However, not all BN-subalgebras are both BN-ideal and normal. Example 2 shows that although the sets $\{0,1\}$ and $\{0,p,q\}$ are BN-subalgebras, they do not qualify as BN-ideals or normal subsets in the Sheffer stroke BN-algebra. This emphasizes that normal subsets and BN-ideals have additional structure compared with ordinary BN-subalgebras.

For a Sheffer stroke BN-algebra, we will define a mapping called a Sheffer stroke BN-homomorphism. This mapping connects two Sheffer stroke BN-algebras by ensuring that the Sheffer stroke operations in one set are applied consistently in the other set. The following definition explains the conditions that must be met by this mapping.

Definition 9. *Let* $(\mathcal{P}_{SBN};|_P,0)$ *and* $(\mathcal{Q}_{SBN};|_Q,0)$ *be two Sheffer stroke BN-algebras. A map* $\varphi:\mathcal{P}_{SBN}\to\mathcal{Q}_{SBN}$ *is called a Sheffer stroke BN-homomorphism if it satisfies* $\varphi(a|_Pb)=\varphi(a)|_Q\varphi(b)$ *for all* $a,b\in\mathcal{P}_{SBN}$*. We assume* $\varphi(0)=0$*.*

In Sheffer stroke BN algebras, homomorphisms are mappings that keep algebraic operations consistent. For example, if we have two Sheffer stroke BN-algebras, $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$, and a mapping $\varphi:\mathcal{P}_{SBN}\to\mathcal{Q}_{SBN}$ that satisfies the properties of a Sheffer stroke BN-homomorphism, we can examine the kernel of this mapping. The kernel is the set of elements that map to the zero elements in the goal algebra. We define $Ker\varphi=\{a\in\mathcal{P}_{SBN}:\varphi(a)=0\}$.

The following theorem will show that the kernel of a Sheffer stroke BN-homomorphism always forms a BN-subalgebra in the source algebra. In other words, the elements that map to zero form the corresponding algebraic structure in the source algebra.

Theorem 7. *Let* $(\mathcal{P}_{SBN};|_P,0)$ *and* $(\mathcal{Q}_{SBN};|_Q,0)$ *be two Sheffer stroke BN-algebras. If* $\varphi:\mathcal{P}_{SBN}\to\mathcal{Q}_{SBN}$ *is a Sheffer stroke BN-homomorphism, then* $Ker\varphi$ *is a Sheffer stroke BN-subalgebra of* $\mathcal{P}_{SBN}$*.*

Proof. Let $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$ be two Sheffer stroke BN-algebras and $\varphi:\mathcal{P}_{SBN}\to\mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. Let $a,b\in Ker\varphi$, we have $\varphi(a)=0$ and $\varphi(b)=0$. By using (SBN1), we obtain

$$\begin{aligned}\varphi((a|_P(b|_Pb))|_P(a|_P(b|_Pb))) &= \varphi(a|_P(b|_Pb))|_Q\varphi(a|_P(b|_Pb)),\\ &= (\varphi(a)|_Q(\varphi(b)|_Q(b)))|_Q(\varphi(a)|_Q(\varphi(b)|_Q\varphi(b))),\\ &= (0|_Q(0|_Q0))|_Q(0|_Q(0|_Q0)),\end{aligned}$$

$$\varphi((a|_P(b|_Pb))|_P(a|_P(b|_Pb))) = 0.$$

Hence, we have $((a|_P(b|_Pb))|_P(a|_P(b|_Pb))) \in Ker\varphi$. This shows that $Ker\varphi$ is a Sheffer stroke BN-subalgebra of $\mathcal{P}_{SBN}$.

In Theorem 8, the kernel of the Sheffer stroke BN-homomorphism φ is the set of elements in $\mathcal{P}_{SBN}$ that map to zero elements in $\mathcal{Q}_{SBN}$. The property proven by Theorem 8 is that the kernel of a Sheffer stroke BN-homomorphism not only forms a Sheffer stroke BN-subalgebra of $\mathcal{P}_{SBN}$ but also forms a BN-ideal. It provides an understanding of the algebraic structures preserved and changed by homomorphisms, as well as the importance of kernels in preserving ideal properties and algebraic structures in the context of homomorphisms.

Theorem 8. *Let ($\mathcal{P}_{SBN};|_P,0$) and ($\mathcal{Q}_{SBN};|_Q,0$) be two Sheffer stroke BN-algebras. If $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ is a Sheffer stroke BN-homomorphism, then $Ker\varphi$ is a Sheffer stroke BN-ideal of $\mathcal{P}_{SBN}$.*

Proof. Let $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$ be two Sheffer stroke BN-algebras and $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. Since $\varphi(0) = 0$, we have $0 \in Ker\varphi$. Let $b \in Ker\varphi$, we have $\varphi(b) = 0$. Let $(a|(b|b))|(a|(b|b)) \in Ker\varphi$, then by using (SBN1), we obtain

$$\begin{aligned}
\varphi((a|_P(b|_Pb))|_P(a|_P(b|_Pb))) &= 0,\\
&= (\varphi(a)|_Q(\varphi(b)|_Q\varphi(b)))|_Q(\varphi(a)|_Q(\varphi(b)|_Q\varphi(b))),\\
&= (\varphi(a)|_Q(0|_Q0))|_Q(\varphi(a)|_Q(0|_Q0)),\\
\varphi(a) &= 0.
\end{aligned}$$

We have $a \in Ker\varphi$. Hence, $Ker\varphi$ is a Sheffer stroke BN-ideal of $\mathcal{P}_{SBN}$.

The following two theorems further characterize the properties of the kernel of a Sheffer stroke BN-homomorphism $Ker\varphi$. Theorem 9 shows that if an element a is in $Ker\varphi$, then the result of a particular operation involving a and other elements in $\mathcal{P}_{SBN}$ remains within $Ker\varphi$. The same applies to pairs of elements a and b that are both in $Ker\varphi$, as shown by Theorem 10. These two theorems provide further insight into the relationship between kernels and operations in a Sheffer stroke BN-algebra, as well as how elements in the kernel behave under certain operations in the algebraic structure.

Theorem 9. *Let ($\mathcal{P}_{SBN};|_P,0$) and ($\mathcal{Q}_{SBN};|_Q,0$) be two Sheffer stroke BN-algebras and $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. If $a \in Ker\varphi$, then $(0|_P(a|_Pa))|_P(a|_Pb) \in Ker\varphi$.*

Proof. Let $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$ be two Sheffer stroke BN-algebras and $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. Let $a \in Ker\varphi$, we have $\varphi(a) = 0$. By using Theorem 3 (vi) and $(\mathcal{S}2)$, we obtain

$$\varphi((0|_P(a|_Pa))|_P(a|_Pb)) = \varphi(0|_P(a|_Pa))|_Q(\varphi(a|_Pb),$$

$$
\begin{aligned}
&= (\varphi(0)|_Q(\varphi(a)|_Q\varphi(a)))|_Q((\varphi(a)|_Q(\varphi(b)),\\
&= (0|_Q(0|_Q0))|_Q((0|_Q(\varphi(b)),\\
&= (0|_Q0)|_Q((0|_Q(\varphi(b)),\\
\varphi((0|_P(a|_Pa))|_P(a|_Pb)) &= 0.
\end{aligned}
$$

Therefore, we have $((0|_P(a|_Pa))|_P(a|_Pb)) \in Ker\varphi$.

Theorem 10. *Let $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$ be two Sheffer stroke BN-algebras and $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. If $a, b \in Ker\varphi$, then $(b|_P(b|_Pb))|_P(a|_Pa) \in Ker\varphi$.*

Proof. Let $(\mathcal{P}_{SBN};|_P,0)$ and $(\mathcal{Q}_{SBN};|_Q,0)$ be two Sheffer stroke BN-algebras and $\varphi : \mathcal{P}_{SBN} \to \mathcal{Q}_{SBN}$ a Sheffer stroke BN-homomorphism. Let $a, b \in Ker\varphi$, we have $\varphi(a) = 0$ and $\varphi(b) = 0$. By using Theorem 3 (vi) and $(\mathcal{S}2)$, we obtain

$$
\begin{aligned}
\varphi((b|_P(b|_Pb))|_P(a|_Pa)) &= \varphi(b|_P(b|_Pb))|_Q\varphi(a|_Pa),\\
&= (\varphi(b)|_Q(\varphi(b)|_Q\varphi(b)))|_Q((\varphi(a)|_Q(\varphi(a)),\\
&= (0|_Q(0|_Q0))|_Q((0|_Q(0),\\
\varphi((b|_P(b|_Pb))|_P(a|_Pa)) &= 0.
\end{aligned}
$$

Hence, it is proven that $((b|_P(b|_Pb))|_P(a|_Pa)) \in Ker\varphi$.

4. Conclusions

In this paper, the concept of a Sheffer stroke BN-algebra is introduced, and its key properties are thoroughly examined. These include the independence of the axioms and the special properties related to the element 0. Furthermore, the paper defines the notions of Sheffer stroke BN-subalgebras, BN-ideals, and BN-homomorphisms, along with normal subsets of Sheffer stroke BN-algebras. The interrelationships between these concepts are explored, providing a clear framework for understanding their connections. Finally, the paper outlines potential avenues for future research, particularly focusing on the exploration of the filter and derivation concepts within the context of Sheffer stroke BN-algebras.

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