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Effective Analytical-Approximate Technique for Caputo Nonlinear Time-Fractional Systems Emerging in Shallow Water Waves and Hydrodynamic Turbulence

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Abstract. This work aims to study and analyze non-linear time-fractional systems that describe non-linear surface waves propagating and evolution equations utilizing a novel analytic-approximate technique based upon integrating two schemes with adequacy, high accuracy, straightforward of implementation, computations, and elasticity in handling with more sophisticated differential equations, which is named the Laplace transform fractional residual power series method within the Caputo-fractional derivative framework. The proposed technique had been implemented on Drinfeld-Sokolov-Wilson equation (DS-WE) and coupled viscous Burgers' equation (CVBE).The approximation solutions obtained by the LT-RFPS technique are expressed in an infinite convergent fractional series form toward the exact solution for the integer fractional order. To show the accuracy and efficiency of the proposed method, tabular simulations of the produced approximations and their absolute errors are performed, along with $2D$- and $3D$-representative graphs. The physical interpretation of solution behaviors is also discussed for various ρ values over an adequate duration. Additionally, a numerical comparison is performed with other existing techniques to show the superiority of the LT-RFPS technique. Consequently, the findings of the present work emphasize that the integration between LT and RFPS schemes has led us to a straightforward, effective, and accurate iterative analytical technique for investigating a wide variety of non-linear mathematical fractional models.

2020 Mathematics Subject Classifications: 34A08, 34A25, 34A34, 35A20

Key Words and Phrases: Drinfeld-Sokolov-Wilson equation, Coupled viscous Burgers' equation, Laplace residual fractional power series, Caputo fractional derivative, Laplace transform

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1. Introduction

Fractional calculus (FC) theory is not a new mathematical theory, that dates to the 1600s. It is a branch of mathematical analysis that generalizes the concepts of ordinary

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derivatives and integrals to fractional orders. In the last two decades, FC has attained much concentration in both theoretical and applied sciences and has been thoroughly studied as a valuable instrument for portraying genetic characteristics, memory effects, and material transfer processes in various interconnected in physics, engineering, and finance, including wave propagation, rheology, shallow water flow, bioengineering, acoustic transmission, entropy, thermodynamics,and so on for more details see [14, 29, 34, 35, 38, 40]. The key characteristic of FC is their non-local property, which emphasizes that the future state relies on both the present state and all previous states, and hence it provides a more flexible approach to modeling complicated fractional order systems. In contrast to ordinary calculus theory, which has a single definition and intelligible geometric and physical explanations. As FC theory has developed, various operators of derivatives and integrals of fractional orders have been identified in the literature to characterize the behaviors of numerous models structures. Consequently, many researchers have contributed to this field, proposing various operators, including those of Caputo and Fabrizio, Atangana and Baleanu, Grünwald, Letnikov, Riemann, Liouville, conformable, Riesz, and Caputo [12, 16, 17, 26, 39, 43].

Recently, numerous scientific studies have focused on addressing emerging nonlinear fractional models in physical systems using computer simulations and symbolic programming that allows researchers to model the inherent memory and hereditary properties better often found in physical systems governed by fractional dynamics which include a set of differential equations of fractional orders. By these techniques, researchers aim to understand and predict nonlinear behaviors with greater accuracy,and simplify the derivation of analytical solutions of the posed systems. In this context, there is no standard, accurate method that provides a closed-form analytical solution for handling nonlinear fractional systems. Consequently, there is a pressing need for advanced and effective methods to investigate analytical solutions for these models, which motivates us to pursue numerical solutions.

For the past century, finding exact solutions for both linear and nonlinear partial differential equations of fractional orders and analyzing them has been a critical challenge. Scientists have worked to develop and refine new analytical and numerical methods, to tackle these situations and to handle numerous complex categories of differential equations of fractional orders appearing in physics and applied mathematics by deriving analytical solutions that show a high level of accuracy when compared to the exact solutions [4, 6, 23, 27]. Among these notable methods, homotopy analysis method, homotopy perturbation method, Adomian decomposition method, reproducing kernel method, Shehu transform method, reduced differential transforms method, q-homotopy analysis method, spectral and collocation methods, sine-Gordon expansion method, tanh-function method, and fractional power series method, consult [2, 8, 15, 24, 32, 37, 42, 48] for a detailed discussion.

Solving ordinary, partial, integral, and integro-differential equations frequently involves the use of integral transformations. Utilizing these transformations enables an efficient strategyfor resolving initial and boundary value problems. A wide range of scientists have investigated the effects of various integral transforms on different categories of differential equations [11, 33, 47].The literature documents a wide range of integral transformations. One of the most common is the Laplace transformation (LT) operator which is a crucial mathematical tool used to tackle problems involving differential equations by altering them from one shape to another. Overall, it is influential in resolving ordinary or partial DEs via reducing a target DE into a system of algebraic equations.

The residual fractional power series (RFPS) technique is a highly effective iterative computational algorithm specifically layouts to generate analytical-approximate solutions of sophisticated nonlinear fractional orders partial DEs occurring in physics, engineering, and technology, which is based on combining fractional-residual function with generalizing of Taylor expansion for an arbitrary order. This approach was applied to view the approximation solution in convergent series expansion for both non-linear and linear ordinary DEs and fractional-order DEs. Further improvements were made to the RFPS approach and has been applied in a variety of physical applications where many works have been published [3, 7, 28, 30, 31].In [3], the RFPS technique is introduced to predict multivariable power series solutions for seventh-order, nonlinear fractional-order partial DEs in the meaning of conformable FD. Using a fuzzy RFPS method [7], the exact and approximation solutions are derived for a certain class of fuzzy initial value problems of fractional order. Khalouta and Kadem [28] proposed the RFPS technique for obtaining the approximate analytical solutions of nonlinear time-fractional wave-like equations with variable coefficients in the sense of Caputo-FD. Kumar et al. [31] implemented a new technique based on the generalized Taylor's series formula to find the approximate analytical solution of time-fractional diffusion equations.The fractional diffusion model is investigated using estimated time and spatial dependency of the concentration of tumor cells in the framework of the RFPS principle. [30]

Real-world problems can be thoroughly depicted theoretically using ordinary or fractional order partial DEs, which are inherently influenced by numerous external forces that make their behavior more complex and unpredictable. Therefore, the most effective way to handle such circumstances is to examine these problems by numerical approximations to attain the model that provides an acceptable solution. Mostly, there is no conventional technique that generates an analytical solution or closed-form traveling wave solutions for nonlinear partial DEs of fractional order. Therefore, it is imperative to investigate analytical and numerical solutions to these equations using the newest, dependable, and advanced methods. Instigated by the arguments expressed before, the inspiration for this analysis is to design and apply an advanced analytical method based on two distinguished methodologies RFPS algorithm and LT instrument, so-called LT-RFPS method. It was proposed by El-Ajou [19]and proven as a superb attractive mathematical tool for enhancing the performance of the RFPS algorithm for constructing exact solitary solutions to the nonlinear time-fractional dispersive PDEs straightforwardly and effectively. It has applied to a wide range of problems, including DEs, PDEs of fractional order [5, 20, 21]. In [21] both linear and nonlinear neutral Caputo-fractional pantograph differential equations have been considered to create exact and approximate series solutions using the LT-RFPS method. Utilizing the LT-RFPS [5] different systems of linear and non-linear fractional initial value problems are solved analytically. El-Ajou and Al-Zhou [20] have applied the

LT-RFPS method to create series solutions of a hyperbolic system of time-PDEs with variable coefficients in the light of Caputo FD sense.

The advantages of this method are that it effectively combines the strengths of the Laplace transform, residual error, and fractional power series expansion, making it highly accurate for solving linear and nonlinear differential equations. It avoids linearization and small-parameter assumptions, and it does not require Adomian polynomials or he's polynomials. Also, the LT-RFPS method depends basically on the concept of the limit in discovering the series components, not by using FDs approaches, as the other existing analytical method. As a result, a recursive formula can be found for the components of the series, which in turn contributes to reducing the computational iterations and effort spent in discovering the approximate solution pattern in the form of a fast convergent series. On the other hand, the proposed method may face convergence issues for fractional equations with singularities or highly nonlinear terms. Additionally, its accuracy heavily depends on the proper specification of initial or boundary conditions.

The main objective of the current analysis is to expand the use of the Caputo LT-RFPS method, which offers an efficient approach to obtain analytical approximate series solutions for a class of coupled systems that arise in physics. To accomplish this, we consider the following coupled partial DEs of fractional orders system in the form:

The non-linear Caputo time-fractional order Drinfeld-Sokolov-Wilson equation (DSWE):

$$\mathcal{D}^{\rho}\varphi(\varsigma,\tau)+n\psi(\varsigma,\tau)\,D_{\varsigma}\psi(\varsigma,\tau)=0,$$
$$\mathcal{D}^{\rho}\psi(\varsigma,\tau)+cD_{\varsigma}^{3}\psi(\varsigma,\tau)+p\varphi(\varsigma,\tau)\,D_{\varsigma}\psi(\varsigma,\tau)+r\psi(\varsigma,\tau)\,D_{\varsigma}\varphi(\varsigma,\tau)=0,\ \ \rho\in(0,1] \tag{1}$$

where $n,\ c$, ρ and r are non-zero constants. In the 1980s, Wilson [49] provided a basic description of DSWE after Drinfeld and Sokolov [18] initially introduced it. DSWE is one of the special shapes of Lax-paired non-linear partial DEs, which is utilized to elucidate non-linear surface waves propagating on the horizontal seabed [46]. It has an endless number of conservation regulations in this system. It is interesting to note that it contains static soliton solutions, which interact with the moving solitons without undergoing deformation [36].

The non-linear Caputo time-fractional order coupled viscous burgers' equation (CVBE) system:

$$\mathcal{D}^{\rho}\varphi(\varsigma,\tau)-D_{\varsigma}^{2}\varphi(\varsigma,\tau)-\omega\varphi(\varsigma,\tau)\,D_{\varsigma}\varphi(\varsigma,\tau)+qD_{\varsigma}\left(\varphi(\varsigma,\tau)\,\psi(\varsigma,\tau)\right)=0,$$
$$\mathcal{D}^{\rho}\psi(\varsigma,\tau)-D_{\varsigma}^{2}\psi(\varsigma,\tau)-\gamma\psi(\varsigma,\tau)\,D_{\varsigma}\psi(\varsigma,\tau)+\vartheta D_{\varsigma}\left(\varphi(\varsigma,\tau)\,\psi(\varsigma,\tau)\right)=0,\quad \rho\in(0,1]\,, \tag{2}$$

where ω and γ real constants, q and ϑ are arbitrary constants depending on system parameters. Esipov initially investigated the CVBE system as a mathematics problem of poly-dispersive sedimentation [22]. It is extremely important to explain ocean waves, which is considered one of non-linear evolution equation[25]. Additionally, CVBE is employed to observe the physical issues of hydrodynamic turbulence, vorticity transport, plasma physics, shock wave theory in a viscous fluid, and wave processes in a thermoelastic medium [44].

Herein, both DS-WE and CVBE systems are considered subject to the following initial conditions:

$$\varphi(\varsigma,0)=\varphi(\varsigma) \text{ and } \psi(\varsigma,0)=\psi(\varsigma)\,. \tag{3}$$

The pattern of the present work is structured as follows: In the next Section, a quick overview of certain essential terms and ideas related to FC theory, LT instrument, and new fractional series expansion that we will use in present disquisition. In Section 2, the future recommended technique for prediction analytical-approximate solutions of governing fractional models is described. Next, applications of the scheme to DS-WE and CVBE systems in the light of Caputo differentiation with proper initial conditions are stated. Finally, summarize our essential findings.

2. Preliminaries

In this portion, the most common notion of Caputo-FD is highlighted. It also shortly states the theory and features of the LT-RFPS under the Caputo-FD instrument to accomplish the theoretical side of the present disquisition.

Definition 1. *[14] Let $\varphi(\varsigma,\tau): I\times[0,\infty)\to\mathbb{R}$. The LT of φ is defined as:*

$$\mathcal{L}_\rho\{\varphi(\varsigma,\tau)\}=\Phi(\varsigma,\xi)=\int_0^\infty \varphi(\varsigma,\tau)\ e^{-\xi\tau}d\tau,\quad \tau>\sigma, \tag{4}$$

where σ is the exponential order of φ. Also, the inverse LT of the new function $\Phi(\varsigma,\xi)$ is defined as:

$$\mathcal{L}_\rho^{-1}\{\Phi(\varsigma,\xi)\}=\varphi(\varsigma,\tau)=\int_{\delta-i\infty}^{\delta+i\infty}\Phi(\varsigma,\xi)\ e^{\xi\tau}d\xi,\quad \delta=\mathfrak{Re}(\xi)>\delta_0. \tag{5}$$

Lemma 1. *[19] Suppose that $\varphi(\varsigma,\tau)$,and $\psi(\varsigma,\tau)$, are two exponential orders, piecewise continuous functions on $I\times[0,\infty)$. Then, the following characteristics are hold:*

(i) $\lim_{\xi\to\infty}\xi\Phi(\varsigma,\xi)\ =\varphi(\varsigma,0)$.

(ii) $\mathcal{L}_\rho\{a\varphi(\varsigma,\tau)+b\psi(\varsigma,\tau)\}=a\Phi(\varsigma,\xi)+b\Psi(\varsigma,\xi)$, *for non-zero constants any a, and b.*

(iii) $\mathcal{L}_\rho^{-1}\{a\Phi(\varsigma,\xi)+b\Psi(\varsigma,\xi)\}=a\varphi(\varsigma,\tau)+b\psi(\varsigma,\tau)$.

(iv) $\mathcal{L}_\rho\{\mathcal{D}^\rho\varphi(\varsigma,\tau)\}=\xi^\rho\Phi(\varsigma,\xi)-\sum_{i=0}^{j-1}\xi^{\rho-i-1}\partial_t^{(i)}\phi(\varsigma,0)\,,\quad j-1<\rho\le j, j\in\mathbb{N}.$

(v) $\mathcal{L}_\rho\left\{\mathcal{D}^{k\rho}\varphi\left(\varsigma,\tau\right)\right\}=\xi^{k\rho}\Phi\left(\varsigma,\xi\right)-\sum_{i=0}^{k-1}\xi^{(k-i)\rho-1}\partial_\tau^{i\rho}\phi\left(\varsigma,0\right),\quad 0<\rho\leq 1, k\in\mathbb{N}.$

where $\mathcal{L}_\rho\left\{\varphi\left(\varsigma,\xi\right)\right\}=\Phi\left(\varsigma,\xi\right)$*, and* $\mathcal{L}_\rho\left\{\psi\left(\varsigma,\xi\right)\right\}=\Psi\left(\varsigma,\xi\right)$*.*

Definition 2. *[35, 38] The ρ-th time FD in the Caputo sense of* $\varphi\left(\varsigma,\tau\right):I\times[0,\infty)\to\mathbb{R}$*, is given by:*

$$\mathcal{D}^\rho\varphi\left(\varsigma,\tau\right)=\mathcal{J}^{m-\rho}\left(\partial_\varsigma^m\varphi\left(\varsigma,\tau\right)\right):\quad \rho\in\left(m-1,m\right),\ m\in\mathbb{N},\tag{6}$$

where $\mathcal{J}^{m-\rho}$*is the Riemann-Liouville integral approach [35, 38].*

Theorem 1. *[19] Let* $\varphi\left(\varsigma,\tau\right)$ *is to exponential order, and piecewise continuous function on* $I\times[0,\infty)$*, then the new transformation function* $\Phi\left(\varsigma,\xi\right)$*, may be formulated as in the following Laplace fractional power series (LFPS) expansion:*

$$\Phi\left(\varsigma,\xi\right)=\sum_{i=0}^{\infty}\frac{\varphi_i(\varsigma)}{\xi^{i\rho+1}}\ \xi>0,\ \rho\in\left(0,1\right],\tag{7}$$

where $\varphi_i(\varsigma)=\mathcal{D}^{i\rho}\varphi\left(\varsigma,0\right)$*.*

Remark 1. *[19] The inverse LT of the new LFPS expansion* $\Phi\left(\varsigma,\xi\right)$ *in Theorem 1 takes the following infinite series shape:*

$$\varphi\left(\varsigma,\tau\right)=\sum_{i=0}^{\infty}\frac{\varphi_i(\varsigma)}{\Gamma\left(i\rho+1\right)}\tau^{j\rho},\quad \tau\geq 0,\rho\in\left(0,1\right].\tag{8}$$

Hereinafter, the road map to design the LT-RFPS method for generating analytical approximate series solutions of general nonlinear systems of partial DEs of fractional order considering the Caputo-FD sense.

3. Configuration for the LT-RFPS Technique

A powerful analytical approximation mathematical tool was suggested and proved by [19] especially to produce convergence series approximate solutions for a set of complex non-linear partial DEs that occur in physics. The primary idea of the present technique is based on increasing the effectiveness of FRPS algorithm via blending it with the LT tool. Numerous physical problems have been studied using the suggested methodology, including fractional generalized long wave equations [50], fractional Caudrey-Dodd Gibbon equation [1], fractional Kuramoto-Sivashinsky equation [10], the Buckmaster and Korteweg-de Vries (KdV) models [41]. The suggested method's set of guidelines is based on transforming the governing equation into the LT space, generating an approximation series solution to the new Laplace equation, and then using the inverse LT to produce an approximation series solution to the governing problem. The detecting of the expansion parameters can be carried out through a small number of calculations, with no requiring many iterations of

FD calculations during the solution stages as in the methodology of the RFPS algorithm. Our purpose in this portion of the research is to demonstrate the fundamental idea of the solution approach. In this context, let's look at the following general nonlinear systems of partial DEs of fractional order.

$$\mathcal{D}^{\rho}\varphi + \ell_1(\varphi,\psi) + \mathcal{N}_1(\varphi,\psi) = 0,$$
$$\mathcal{D}^{\rho}\psi + \ell_2(\varphi,\psi) + \mathcal{N}_2(\varphi,\psi) = 0, \rho \in (0,1] \tag{9}$$

with the initial conditions

$$\varphi(\varsigma,0) = \varphi(\varsigma), \text{ and } \psi(\varsigma,0) = \psi(\varsigma), \tag{10}$$

where $\varphi = \varphi(\varsigma,\tau)$ and, $\psi = \psi(\varsigma,\tau)$ for $\tau \geq 0, \varsigma \in I$ are two unknown analytical functions to be explored, ℓ_1, ℓ_2 are two linear differential operators and $\mathcal{N}_1, \mathcal{N}_2$ are two non-linear differential operators, $\mathcal{D}^{\rho}$ shows the Caputo-FD of order ρ. It is presumed that the solution exists and is unique.

The following is a sequential explanation of the main stages of the LT-RFPS technique: **Step I**: Running the LT tool $\mathcal{L}_{\rho}$, on the coupled equations in (9) with initial conditions 10 and using Lemma 1, part (iv), we get

$$\Phi(\varsigma,\xi) = \frac{\varphi(\varsigma)}{\xi} - \frac{1}{\xi^{\rho}}\mathcal{L}_{\rho}\left\{\ell_1\mathcal{L}_{\rho}^{-1}(\Phi, \Psi)\right\} - \frac{1}{\xi^{\rho}}\mathcal{L}_{\rho}\left\{\mathcal{N}_1\mathcal{L}_{\rho}^{-1}(\Phi, \Psi)\right\} = 0,$$
$$\Psi(\varsigma,\xi) = \frac{\psi(\varsigma)}{\xi} - \frac{1}{\xi^{\rho}}\mathcal{L}_{\rho}\left\{\ell_2\mathcal{L}_{\rho}^{-1}(\Phi, \Psi)\right\} - \frac{1}{\xi^{\rho}}\mathcal{L}_{\rho}\left\{\mathcal{N}_2\mathcal{L}_{\rho}^{-1}(\Phi, \Psi)\right\} = 0. \tag{11}$$

Step II: According to the methodology of the LT-RFPS technique, the new transformation functions have the following LFPS expansions:

$$\Phi(\varsigma,\xi) = \sum_{n=0}^{\infty} \frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\ \xi > 0,$$
$$\Psi(\varsigma,\xi) = \sum_{n=0}^{\infty} \frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\ \xi > 0. \tag{12}$$

Hither, utilizing the facts $\lim_{\xi\to\infty} \xi\Phi(\varsigma,\xi) = \varphi(\varsigma)$, and $\lim_{\xi\to\infty} \xi\Psi(\varsigma,\xi) = \psi(\varsigma)$, the j-th truncation LFPS expansions can be expressed as:

$$\Phi_j(\varsigma,\xi) = \frac{\varphi(\varsigma)}{\xi} + \sum_{n=1}^{j} \frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\ \xi > 0,$$
$$\Psi_j(\varsigma,\xi) = \frac{\psi(\varsigma)}{\xi} + \sum_{n=1}^{j} \frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\ \xi > 0, \tag{13}$$

Step III: The unknown functions $\varphi_n(\varsigma)$, and $\psi_n(\varsigma)$ for $n = 1, 2, \ldots, j$, can be located throughout resolving of $\lim_{\xi\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Phi_j\left(\varsigma, \xi\right)\right)\right) = 0$, and $\lim_{\xi\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right) = 0$, where $\mathcal{L}_\rho\left(res_{\Phi_j}\right)$, and $\mathcal{L}_\rho\left(res_{\Psi_j}\right)$ are recognizing as the jth-Laplace fractional residual error (L-FRE) functions of (11) and given by:

$$\begin{aligned}\mathcal{L}_\rho\left(\mathrm{res}(\Phi_j(\varsigma,\xi))\right) &= \Phi_j(\varsigma,\xi) - \frac{\varphi(\varsigma)}{\xi} + \frac{1}{\xi^\rho}\mathcal{L}_\rho\left\{\ell_1\mathcal{L}_\rho^{-1}(\Phi_j, \Psi_j)\right\} + \frac{1}{\xi^\rho}\mathcal{L}_\rho\left\{\mathcal{N}_1\mathcal{L}_\rho^{-1}(\Phi_j, \Psi_j)\right\}, \\ \mathcal{L}_\rho\left(\mathrm{res}(\Psi_j(\varsigma,\xi))\right) &= \Psi_j(\varsigma,\xi) - \frac{\psi(\varsigma)}{\xi} + \frac{1}{\xi^\rho}\mathcal{L}_\rho\left\{\ell_2\mathcal{L}_\rho^{-1}(\Phi_j, \Psi_j)\right\} + \frac{1}{\xi^\rho}\mathcal{L}_\rho\left\{\mathcal{N}_2\mathcal{L}_\rho^{-1}(\Phi_j, \Psi_j)\right\}.\end{aligned} \tag{14}$$

Indeed, the following is a list of important helpful facts about LFRE functions that are fundamental to determining the approximated solutions: as stated in [19]:

(i) $\lim_{j\to\infty} \mathcal{L}_\rho\left(res\left(\Phi_j\left(\varsigma, \xi\right)\right)\right) = \mathcal{L}_\rho\left(res\left(\Phi\left(\varsigma, \xi\right)\right)\right)$, and $\lim_{j\to\infty} \mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right) = \mathcal{L}_\rho\left(res\left(\Psi\left(\varsigma, \xi\right)\right)\right)$, $\varsigma \in \mathrm{I}$, $\xi > 0$.

(ii) $\mathcal{L}_\rho\left(res\left(\Phi\left(\varsigma, \xi\right)\right)\right) = 0$, and $\mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right) = 0$, $\varsigma \in \mathrm{I}$, $\xi > 0$.

(iii) $\lim_{\varsigma\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Phi_j\left(\varsigma, \xi\right)\right)\right) = 0$, and $\lim_{\xi\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right) = 0$, for $j = 1, 2 \ldots$. and $\varsigma \in \mathrm{I}$, $\xi > 0$.

Step IV: Substitute the j-th truncation LFPS expansions in Step II into the jth-L-FRE functions of in Step III such that $\mathcal{L}_\rho\left(res\left(\Phi_j\left(\varsigma, \xi\right)\right)\right)$, and $\mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right)$ could be expressd in terms of LFPS exapnsions.

Step V: Multiply the resultant algebraic equations in Step IV by the factor $\xi^{1+j\rho}$, and then looking the solutions of $\lim_{\varsigma\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Phi_j\left(\varsigma, \xi\right)\right)\right) = 0$, and $\lim_{\xi\to\infty} \xi^{1+j\rho}\mathcal{L}_\rho\left(res\left(\Psi_j\left(\varsigma, \xi\right)\right)\right) = 0$, for $j = 1, 2 \ldots$, for the requiered unkown functions $\varphi_j(\varsigma)$, and $\psi_j(\varsigma)$.

Step VI: Collect the obtained results $\varphi_j(\varsigma)$, and $\psi_j(\varsigma)$ from the previous step and replacing them with the j-th truncation LFPS expansions of (13) to attain the j-th approximated solutions of (11). At j tends to infinity, one can get the approximated closed-form solutions $\Phi\left(\varsigma, \xi\right)$, and $\Psi\left(\varsigma, \xi\right)$ of (9).

Step VII: The analytical-approximated solutions $\varphi\left(\varsigma, \tau\right)$ and, $\psi\left(\varsigma, \tau\right)$ of the studied problem (9) can be predicted by performing the invers LT tool of the attained results in Step VI.

To test our proposed method. The next section shows the applicability and performance of the LT-RFPS method of two coupled non-linear fractional evolution systems.

4. Solutions for Governing Models.

In the study of non-linear physical problems occurring in nature, solutions investigation of non-linear evolution problems of fractional order plays a significant role. In this portion, the Caputo LT-RFPS technique is applied to solve common coupled non-linear fractional evolution systems in physical sciences, including time-fractional DS-WE and time-fractional CVBE systems. Besides, the simulation of these problems is discussed. The Mathematica computing system is used to do the calculations and generate a visual representation of solution behavior.

4.1. Solution of non-linear Caputo time-fractional DS-WE [13] is considered in the present piece can be investigated along with following initial conditions:

$$\varphi(\varsigma,0)=3sech^{2}(\varsigma)\ ,\ \text{and}\ \ \psi(\varsigma,0)=2sech(\varsigma) \tag{15}$$

As stated in the last discussion. Considering the posed model (1) along with initial conditions (3). Then, the Laplace equations of (1) will be in the form:

$$\begin{aligned}\Phi(\varsigma,\xi)-\frac{\varphi(\varsigma)}{\xi}+\frac{c}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\{\Psi\}D_{\varsigma}\mathcal{L}_{\rho}^{-1}\{\Psi\}\right)=0,\\ \Psi(\varsigma,\xi)-\frac{\psi(\varsigma)}{\xi}+\frac{n}{\xi^{\rho}}\mathcal{L}_{\rho}\left(D_{\varsigma}^{3}\mathcal{L}_{\rho}^{-1}\{\Psi\}\right)+\frac{p}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\{\Phi\}D_{\varsigma}\mathcal{L}_{\rho}^{-1}\{\Psi\}\right)\\ +\frac{r}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\{\Psi\}D_{\varsigma}\mathcal{L}_{\rho}^{-1}\{\Phi\}\right)=0.\end{aligned} \tag{16}$$

To obtain the j-th truncation LFPS expansions series solution of (16), we expand the functions $\Phi_j(\varsigma,\xi)$, and $\Psi_j(\varsigma,\xi)$ as follows:

$$\begin{aligned}\Phi_j(\varsigma,\xi)=\frac{\varphi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\ ,\\ \Psi_j(\varsigma,\xi)=\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}.\end{aligned} \tag{17}$$

Subsequently, the jth-L-FRE functions of (16) will be identified as follows:

$$\begin{aligned}\mathcal{L}_{\rho}\left(\mathrm{res}_{\Phi_j}(\varsigma,\xi)\right)=\sum_{n=1}^{j}\frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\\ +\frac{c}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\left\{\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\right\}\mathcal{L}_{\rho}^{-1}\left\{D_{\varsigma}\left(\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\right)\right\}\right),\end{aligned} \tag{18}$$

$$\mathcal{L}_{\rho}\left(\mathrm{res}_{\Psi_j}(\varsigma,\xi)\right)=\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}+\frac{n}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\left\{D_{\varsigma}^{3}\left(\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\right)\right\}\right)$$

$$+\frac{b}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\left\{\frac{\varphi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\right\}\mathcal{L}_{\rho}^{-1}\left\{D_{\varsigma}\left(\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\right)\right\}\right)$$
$$+\frac{r}{\xi^{\rho}}\mathcal{L}_{\rho}\left(\mathcal{L}_{\rho}^{-1}\left\{\frac{\psi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\psi_n(\varsigma)}{\xi^{n\rho+1}}\right\}\mathcal{L}_{\rho}^{-1}\left\{D_{\varsigma}\left(\frac{\varphi(\varsigma)}{\xi}+\sum_{n=1}^{j}\frac{\varphi_n(\varsigma)}{\xi^{n\rho+1}}\right)\right\}\right).$$

By multiplying both sides of (18) by $\xi^{j\rho+1}$, and by performing a few algebraic simplifications, yields

$$\xi^{j\rho+1}\mathcal{L}_{\rho}\left(res_{\Phi j}\left(\varsigma,\xi\right)\right)=c\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma){\psi'}_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}+\varphi_j(\varsigma), \tag{19}$$

$$\xi^{j\rho+1}\mathcal{L}_{\rho}\left(res_{\Psi_j}\left(\varsigma,\xi\right)\right)=n\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma){\psi'}_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}+$$
$$p\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}+r\psi_{j-1}^{(3)}(\varsigma)+\psi_j(\varsigma).$$

To this end, we can resolve (19) for $\varphi_j(\varsigma)$, and $\psi_j(\varsigma)$, as the following reccurrence relations:

$$\varphi_j\left(\varsigma\right)=-c\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}, \tag{20}$$

$$\begin{aligned}\psi_j\left(\varsigma\right)=-&\left(n\Gamma\left((j-1)\,\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i\left(\varsigma\right)\psi'_{-i+j-1}\left(\varsigma\right)}{\Gamma\left(i\rho+1\right)\Gamma\left(\left(j-1-i\right)\rho+1\right)}\right)\\&+p\Gamma\left((j-1)\,\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i\left(\varsigma\right)\varphi'_{-i+j-1}\left(\varsigma\right)}{\Gamma\left(i\rho+1\right)\Gamma\left(\left(j-1-i\right)\rho+1\right)}\\&+r\psi_{j-1}^{(3)}\left(\varsigma\right).\end{aligned}$$

Corollary 1. *For $\rho\in(0,1]$, the analytical-approximate series solutions of non-linear Caputo time-fractional DS-WE* (1) *and* (3) *can be expressed as follows:*

$$\varphi\left(\varsigma,\tau\right)=\varphi\left(\varsigma\right)+\sum_{j=1}^{\infty}\left(-c\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}\right)\frac{\tau^{j\rho}}{\Gamma\left(j\rho+1\right)},$$

$$\psi(\varsigma,\tau)=\psi(\varsigma)-\sum_{j=1}^{\infty}\left(n\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\,\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((j-1-i)\rho+1\right)}\right.$$
$$+p\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\,\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((j-1-i)\rho+1\right)}$$
$$\left.+r\psi_{j-1}^{(3)}(\varsigma)\right)\frac{\tau^{j\rho}}{\Gamma(j\rho+1)}. \tag{21}$$

Proof. According to principle of the LT-RFPS scheme for predicting the analytical-approximate series solution of the governing model, firstly, the LFPS solutions have been obtained in Laplace space of (20) as follows:

$$\Phi(\varsigma,\xi)=\frac{\varphi(\varsigma)}{\xi}+\sum_{j=1}^{\infty}\left(-c\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}\right)\frac{1}{\xi^{j\rho+1}}\ ,$$

$$\Psi(\varsigma,\xi)=\frac{\psi(\varsigma)}{\xi}$$
$$-\sum_{j=1}^{\infty}\left(n\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\,\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((j-1-i)\rho+1\right)}\right.$$
$$+p\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\,\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((j-1-i)\rho+1\right)}$$
$$\left.+r\psi_{j-1}^{(3)}(\varsigma)\right)\frac{1}{\xi^{j\rho+1}}. \tag{22}$$

The analytically approximated series solution $\varphi(\varsigma,\tau)$, and $\psi(\varsigma,\tau)$ of (1) and (3) may be attained in terms of Taylor's infinite series expansions by running the inverse LT instrument into (22) as the following shapes:

$$\varphi(\varsigma,\tau)=\varphi(\varsigma)+\sum_{j=1}^{\infty}\left(-c\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma(((j-1)-i)\rho+1)}\right)\frac{\tau^{j\rho}}{\Gamma(j\rho+1)},$$

$$\psi(\varsigma,\tau)=\psi(\varsigma)$$
$$-\sum_{j=1}^{\infty}\left(n\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\,\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\,\Gamma\left((j-1-i)\rho+1\right)}\right.$$
$$+p\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\,\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\,\Gamma\left((j-1-i)\rho+1\right)}$$

$$+ r\psi_{j-1}^{(3)}(\varsigma)\Big) \frac{\tau^{j\rho}}{\Gamma(j\rho+1)}. \tag{23}$$

Now, considering the initial conditions (15) into the obtained recurrence relations (20), set $c = 3, n = p = 2$, and $r = 1$, the 3^{rd}- approximated solutions of the posed model (15) will be formulated as:

$$\begin{aligned}\varphi_3(\varsigma,\tau) &= 3\,\mathrm{sech}^2(\varsigma)\\ &+ \left(12\,\mathrm{sech}^2(\varsigma)\tanh(\varsigma)\right)\frac{\tau^{\rho}}{\Gamma(\rho+1)}\\ &+ \left(24\,\mathrm{sech}^4(\varsigma)\left(\cosh(2\varsigma)-2\right)\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}\\ &+ \mathrm{sech}^4(\varsigma)\tanh(\varsigma)\Bigg(\frac{72\,\Gamma(2\rho+1)}{\Gamma^2(\rho+1)}\\ &\quad + \frac{24\,\Gamma(2\rho+1)\cosh(2\varsigma)}{\Gamma^2(\rho+1)} + 48\cosh(2\varsigma) - 336\\ &\quad \Bigg)\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\end{aligned}$$

$$\begin{aligned}\psi_3(\varsigma,\tau) &= 2\,\mathrm{sech}(\varsigma)\\ &+ \left(4\,\mathrm{sech}(\varsigma)\tanh(\varsigma)\right)\frac{\tau^{\rho}}{\Gamma(\rho+1)}\\ &+ \left(4\,\mathrm{sech}^3(\varsigma)\,\left(\cosh(2\varsigma)-3\right)\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}\\ &+ \tanh(\varsigma)\,\mathrm{sech}^5(\varsigma)\,(438 - 232\cosh(2\varsigma) + 2\cosh(4\varsigma)\\ &\quad -\frac{240\,\Gamma(2\rho+1)}{\Gamma^2(\rho+1)} + \frac{96\,\Gamma(2\rho+1)\cosh(2\varsigma)}{\Gamma^2(\rho+1)}\Bigg)\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\end{aligned} \tag{24}$$

Furthermore, the values of j-th truncation approximate solutions for each $j \geq 4$ may be constructed in a similar manner. In what follows, the achieved j-terms in the shape of an infinite series guides us to the analytical-approximate solutions $\varphi(\varsigma,\tau)$, and $\psi(\varsigma,\tau)$ for the studied model. Particularly, the solutions of fractional DW-SE (1) at $\rho = 1$ a can be expressed in the forms:

$$\begin{aligned}\varphi(\varsigma,\tau) &= 3\,\mathrm{sech}^2(\varsigma)\\ &+ \left(12\,\mathrm{sech}^2(\varsigma)\tanh(\varsigma)\right)\tau\\ &+ \left(24\,\mathrm{sech}^4(\varsigma)\left(\cosh(2\varsigma)-2\right)\right)\frac{\tau^2}{2}\end{aligned}$$

$$+ \operatorname{sech}^4(\varsigma)\,\tanh(\varsigma)\,(96\cosh(2\varsigma) - 292)\,\frac{\tau^3}{6}$$
$$+ \ldots$$

$$\begin{aligned}\psi(\varsigma,\tau) = {} & 2\operatorname{sech}(\varsigma) \\ & + (4\operatorname{sech}(\varsigma)\tanh(\varsigma))\,\tau \\ & + \left(4\operatorname{sech}^3(\varsigma)\,(\cosh(2\varsigma) - 3)\right)\frac{\tau^2}{2} \\ & + \tanh(\varsigma)\,\operatorname{sech}^5(\varsigma)\,(2\cosh(4\varsigma) + 40\cosh(2\varsigma) - 48)\,\frac{\tau^3}{6} \\ & + \ldots \end{aligned} \tag{25}$$

which agree with the first three terms of the Maclaurin series of the exact solutions $\varphi(\varsigma,\tau) = 3\text{sech}^2(\varsigma - 2\tau)$, and $\psi(\varsigma,\tau) = 2\text{sech}(\varsigma - 2\tau)$ [9].

In what follows, some comparisons of numerical simulations of LT-ERPS outcomes are provided in Tables 1, and 2. The exact and 4-th approximate solution $\varphi_4(\varsigma,\tau)$, for the fractional DW-SE (1) as well the absolute errors $|\varphi - \varphi_4|$ are listed in Table 1 based on the future our method and MGMLFM [9]. In Table 2, absolute errors $|\varphi - \varphi_4|$ are given for different for diverse values of ρ. Here, high precision up to eleven digits in the outcomes is observed even in four-term approximation, which indicates the validity of the present procedure. Moreover, some graphic representations achieved by the proposed approach for the governing DW-SE (1) are displayed in Figures 1, 2 and 3. The acquired 4-th approximate solutions are portrayed in Figure 1 for diverse values of ρ that are given as: 1, 0.88, and 0.77, respectively with fix $\varsigma = 3$ over a temporal domain $\tau \in [0,1]$, where the moving of fractional curves of the solutions for the target model (1) are provided in 2D-diagram. From this diagram, one may note the immense impact of the fractional-order parameter ρ on the solutions' coincide and homogeneity concerning the time variable. In Figure 2, 3D-Surface plots of the exact versus the obtained solutions $\varphi_4(\varsigma,\tau)$, and $\psi_4(\varsigma,\tau)$ behavior' via our used method over a large enough space-time domain $(\varsigma,\tau) = [-5,5] \times [0,0.1]$ for $\rho \in \{0.85,\ 1\}$, are presented.Further, the coupled 3D-Surface plots of exact versus obtained solutions $\varphi_4(\varsigma,\tau)$, and $\psi_4(\varsigma,\tau)$ over a large enough space-time domain $(\varsigma,\tau) = [-3,3] \times [0,0.1]$ for diverse values of ρ are presented in Figure 3. Clearly, the diagrams highlight that the outcomes of the LT-RFPS approach are almost identical to one another and align with the exact solutions stated in reference [46] for the studied fractional DS-WE (1). In summary, the outcomes indicate that even with fewer iterations, LT-RFPS solutions demonstrate high accuracy and efficiency compared to other well-known numerical solvers. These are the most crucial features of the presented technique, which provides more precise and reliable approximations over the considered domain.

Table 1: **Comparisons of numerical results of the fractional DS-WE (1.1) at** $\rho = 1$**.**

τ	$\varphi(\varsigma,\tau)$	**LT-FRPS**		**MGMLFM [46]**	
		$\varphi_4(\varsigma,\tau)$	$\|\varphi-\varphi_4\|$	$\varphi_4(\varsigma,\tau)$	$\|\varphi-\varphi_4\|$
0.02	0.0005901158348	0.0005901158198	1.50329×10^{-11}	0.000590116	1.40822×10^{-10}
0.04	0.0006392596155	0.0006392591279	4.87584×10^{-10}	0.000639261	1.26010×10^{-9}
0.06	0.0006929455290	0.0006929417537	3.75334×10^{-9}	0.000692949	3.70758×10^{-9}
0.08	0.0007501642395	0.0007501482041	1.60354×10^{-8}	0.000750173	8.37625×10^{-8}
0.1	0.0008126347567	0.0008125851368	4.96199×10^{-8}	0.000812651	1.62924×10^{-8}
τ	$\psi(\varsigma,\tau)$	**LT-FRPS**		**MGMLFM [46]**	
		$\psi_4(\varsigma,\tau)$	$\|\psi-\psi_4\|$	$\psi_4(\varsigma,\tau)$	$\|\psi-\psi_4\|$
0.02	0.0280503317828	0.0280503317597	2.28943×10^{-11}	0.0280504	5.44042×10^{-8}
0.04	0.0291496797578	0.0291496724634	7.37439×10^{-10}	0.0291952	2.32744×10^{-7}
0.06	0.0303863023564	0.0303862947963	7.56003×10^{-9}	0.0303869	5.60288×10^{-7}
0.08	0.0316262388864	0.0316262149734	2.29116×10^{-8}	0.0316273	1.06608×10^{-6}
0.1	0.0329167587856	0.0329166853227	7.34584×10^{-8}	0.0329185	1.78351×10^{-6}

Table 2: **Comparisons of** $|\varphi-\varphi_4|$**, for fractional DS-WE (1.1) at different values of** $\rho = 1$ **with fix** $\varsigma = 5$**.**

τ	$\rho = 1$	$\rho = 0.95$	$\rho = 0.85$	$\rho = 0.75$
0.02	1.50329×10^{-11}	$1.1633362824\times10^{-5}$	$4.5246511821\times10^{-5}$	$1.0123340205\times10^{-4}$
0.04	4.87584×10^{-10}	$2.1232437835\times10^{-5}$	$8.1287772719\times10^{-5}$	$1.7985154209\times10^{-4}$
0.06	3.75334×10^{-9}	$3.0975754334\times10^{-5}$	$1.1821394808\times10^{-4}$	$2.6161210872\times10^{-4}$
0.08	1.60354×10^{-8}	$4.1300429669\times10^{-5}$	$1.5776568399\times10^{-4}$	$3.5023773725\times10^{-4}$
0.1	4.96199×10^{-8}	$5.2433675062\times10^{-5}$	$2.0083445301\times10^{-4}$	$4.4753470392\times10^{-4}$

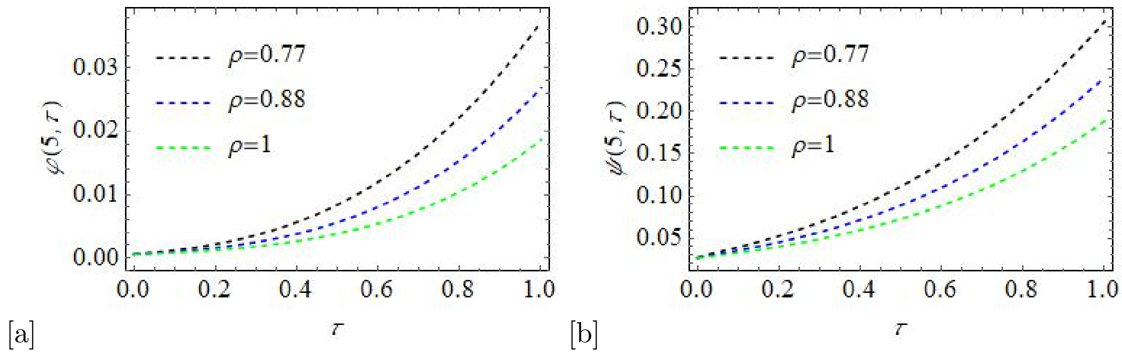
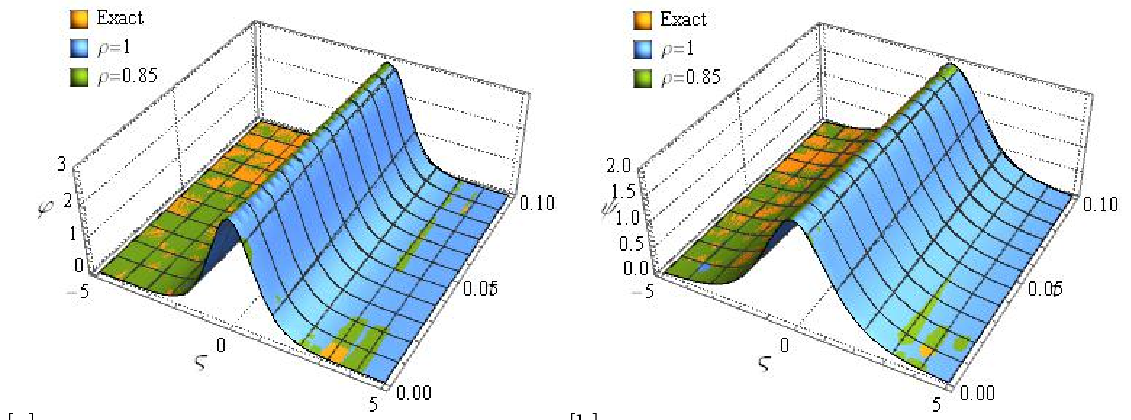


Figure 1: Fractional-order curves of $\varphi_4(\varsigma,\tau)$ and $\psi_4(\varsigma,\tau)$ for the time-fractional DS-WE (1) at various values of ρ, where $\varsigma = 5$ and $\tau \in [0,1]$.

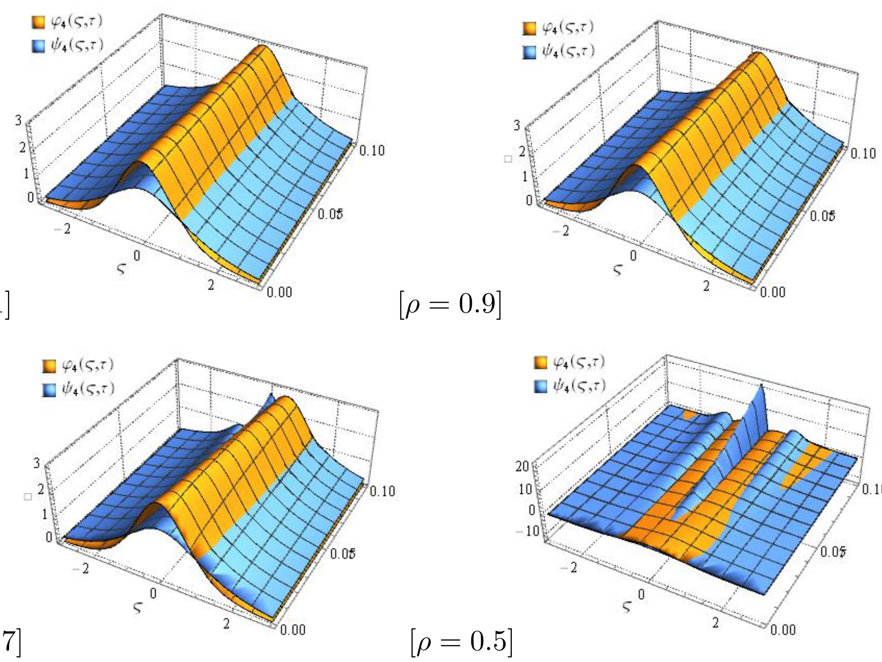
4.2. Solution of non-linear Caputo time-fractional order CVBE system [13] is considered in the present piece can be investigated along with following initial conditions:

$$\varphi(\varsigma,0) = \psi(\varsigma,0) = sin(\varsigma)\ . \tag{26}$$



[a] [b]

Figure 2: Surface plots of exact versus obtained solutions $\varphi_4(\varsigma, \tau)$ and $\psi_4(\varsigma, \tau)$ for the time-fractional DS-WE (1) at various values of ρ, where $\varsigma \in [-5, 5]$ and $\tau \in [0, 0.1]$.



[$\rho = 1$] [$\rho = 0.9$] [$\rho = 0.7$] [$\rho = 0.5$]

Figure 3: Surface plots of the obtained solutions $\varphi_4(\varsigma, \tau)$ and $\psi_4(\varsigma, \tau)$ for the time-fractional DS-WE (1) at various values of ρ.

For integer case $\rho = 1$, the exact solutions of (2) and (26) are $\varphi(\varsigma, \tau) = \sin(\varsigma)\, e^{-\tau}$. Considering the LT-RFPS layout that was discussed in the earlier application. The governing model can be solved directly without the requirement for any worthy restrictions on the model's structure. Consequently, the jth-L-FRE functions of the converted coupled CVBE system (2) into Laplace space will be identified as follows:

$$\begin{aligned}
\mathcal{L}_\rho\left(\mathrm{res}_{\Phi_j}(\varsigma,\xi)\right) = {} & \Phi_j(\varsigma,\xi) - \frac{\varphi(\varsigma)}{\xi} - \frac{1}{\xi^\rho}\partial_{\varsigma\varsigma}\Phi_j(\varsigma,\xi) \\
& - \frac{\omega}{\xi^\rho}\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Phi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Phi_j\}\right) \\
& + \frac{q}{\xi^\rho}\left(\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Phi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Psi_j\}\right)\right. \\
& \left.+\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Psi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Phi_j\}\right)\right), \\
\mathcal{L}_\rho\left(\mathrm{res}_{\Psi_j}(\varsigma,\xi)\right) = {} & \Psi_j(\varsigma,\xi) - \frac{\psi(\varsigma)}{\xi} - \frac{1}{\xi^\rho}\partial_{\varsigma\varsigma}\Psi_j(\varsigma,\xi) \\
& - \frac{\gamma}{\xi^\rho}\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Psi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Psi_j\}\right) \\
& + \frac{\vartheta}{\xi^\rho}\left(\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Phi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Psi_j\}\right)\right. \\
& \left.+\mathcal{L}_\rho\left(\mathcal{L}_\rho^{-1}\{\Psi_j\}\mathcal{L}_\rho^{-1}\{\partial_\varsigma\Phi_j\}\right)\right). \qquad (27)
\end{aligned}$$

where the the j-th truncation $\Phi_j(\varsigma,\xi)$, and $\Psi_j(\varsigma,\xi)$ are provided in (13). After recall these j-th LFPS expansions into the jth-truncation L-FRE functions in (27), and then multiply the resultant equations by the factor $\xi^{j\rho+1}$, and solving them for the unknown functions $\varphi_j(\varsigma)$, and $\psi_j(\varsigma)$ one can reach to the following reccurence relations:

$$\begin{aligned}
\varphi_j(\varsigma) = {} & \varphi''_{j-1}(\varsigma) - q\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)} \\
& - q\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)} \\
& + \omega\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)}, \\
\psi_j(\varsigma) = {} & \psi''_{j-1}(\varsigma) - \vartheta\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)} \\
& - \vartheta\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)} \\
& + \gamma\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma((-i+j-1)\rho+1)}. \qquad (28)
\end{aligned}$$

Corollary 2. *For $\rho \in (0,1]$, the analytical-approximated series solutions of the non-linear Caputo time-fractional order coupled CVBE system* (2) *and* (3) *can be expressed as follows:*

$$\varphi(\varsigma,\tau) = \varphi(\varsigma) + \sum_{j=1}^{\infty}\left[\varphi''_{j-1}(\varsigma) - q\,\Gamma((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\,\Gamma((-i+j-1)\rho+1)}\right.$$

$$- q\,\mathit{\Gamma}((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\mathit{\Gamma}(i\rho+1)\,\mathit{\Gamma}((-i+j-1)\rho+1)}$$
$$+ \omega\,\mathit{\Gamma}((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\mathit{\Gamma}(i\rho+1)\,\mathit{\Gamma}((-i+j-1)\rho+1)}\Bigg]\frac{\tau^{j\rho}}{\mathit{\Gamma}(j\rho+1)}.$$

$$\psi(\varsigma,\tau) = \psi(\varsigma) + \sum_{j=1}^{\infty}\Bigg[\psi''_{j-1}(\varsigma) - \vartheta\mathit{\Gamma}((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\mathit{\Gamma}(i\rho+1)\,\mathit{\Gamma}((-i+j-1)\rho+1)}$$
$$- \vartheta\mathit{\Gamma}((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\mathit{\Gamma}(i\rho+1)\,\mathit{\Gamma}((-i+j-1)\rho+1)}$$
$$+ \gamma\mathit{\Gamma}((j-1)\rho+1)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\mathit{\Gamma}(i\rho+1)\,\mathit{\Gamma}((-i+j-1)\rho+1)}\Bigg]\frac{\tau^{j\rho}}{\mathit{\Gamma}(j\rho+1)}. \quad (29)$$

Proof. Based upon the methodology of the LT-RFPS approach in creating the analytical-approximated series solution of the target problem(2), we find out the LFPS approximate solutions of the new converted Laplace equation of the posed model in the Laplace space as follows:

$$\Phi(\varsigma,\xi) = \frac{\varphi(\varsigma)}{\xi} + \sum_{j=1}^{\infty}\Bigg[\varphi''_{j-1}(\varsigma) - q\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$- q\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$+ \omega\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}\Bigg]\frac{1}{\xi^{j\rho+1}}.$$

$$\mathit{\Psi}(\varsigma,\xi) = \frac{\psi(\varsigma)}{\xi} - \sum_{j=1}^{\infty}\Bigg[\sum_{j=1}^{\infty}\Bigg(\psi''_{j-1}(\varsigma) - \vartheta\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$- \vartheta\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$+ \gamma\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$]\frac{1}{\xi^{j\rho+1}}. \quad (30)$$

The analytical-approximate series solutions $\varphi(\varsigma,\tau)$, and $\psi(\varsigma,\tau)$ of (2) along with (3) may be attained in terms of Taylor's infinite series expansions via running the inverse LT instrument into (30) as the following shapes:

$$\varphi(\varsigma,\tau)=\varphi(\varsigma)+\sum_{j=1}^{\infty}\Bigg[\varphi''_{j-1}(\varsigma)-q\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$-q\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$+\omega\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$\Bigg]\frac{\tau^{j\rho}}{\Gamma(j\rho+1)}.$$

$$\psi(\varsigma,\tau)=\psi(\varsigma)+\sum_{j=1}^{\infty}\Bigg(\psi''_{j-1}(\varsigma)-\vartheta\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\varphi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$-\vartheta\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\varphi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}$$
$$+\gamma\Gamma\left((j-1)\rho+1\right)\sum_{i=0}^{j-1}\frac{\psi_i(\varsigma)\psi'_{-i+j-1}(\varsigma)}{\Gamma(i\rho+1)\Gamma\left((-i+j-1)\rho+1\right)}\Bigg)\frac{\tau^{j\rho}}{\Gamma(j\rho+1)}. \quad (31)$$

Now, considering the initial conditions (26) into the obtained recurrence relations (28), set $\omega=\gamma=2$, and $q=\vartheta=1$ the analytical-approximate solutions of (2) will be formulated as:

$$\begin{aligned}\varphi(\varsigma,\tau)&=\sin(\varsigma)-\sin(\varsigma)\frac{\tau^{\rho}}{\Gamma(\rho+1)}+\sin(\varsigma)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-\sin(\varsigma)\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\cdots\\&=\sin(\varsigma)\sum_{j=0}^{\infty}\frac{(-1)^j\tau^{j\rho}}{\Gamma(j\rho+1)}\\&=\sin(\varsigma)E_\rho(-\tau^\rho),\end{aligned}$$

$$\begin{aligned}\psi(\varsigma,\tau)&=\sin(\varsigma)-\sin(\varsigma)\frac{\tau^{\rho}}{\Gamma(\rho+1)}+\sin(\varsigma)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-\sin(\varsigma)\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\cdots\\&=\sin(\varsigma)\sum_{j=0}^{\infty}\frac{(-1)^j\tau^{j\rho}}{\Gamma(j\rho+1)}\\&=\sin(\varsigma)E_\rho(-\tau^\rho).\end{aligned} \quad (32)$$

In case $\rho = 1$, the obtained analytical-approximate solutions of the non-linear Caputo time-fractional order coupled CVBE system (2) along with (26) reduces to the following infinite Maclaurin series expansions:

$$\varphi(\varsigma,\tau) = sin(\varsigma)\left(1 - \tau + \frac{\tau^2}{\Gamma(3)} - \frac{\tau^3}{\Gamma(4)} + \frac{\tau^4}{\Gamma(5)} + \dots\right) = sin(\varsigma)\sum_{j=0}^{\infty}\frac{(-)^j\tau^j}{\Gamma(j+1)},$$

$$\psi(\varsigma,\tau) = sin(\varsigma)\left(1 - \tau + \frac{\tau^2}{\Gamma(3)} - \frac{\tau^3}{\Gamma(4)} + \frac{\tau^4}{\Gamma(5)} + \dots\right) = sin(\varsigma)\sum_{j=0}^{\infty}\frac{(-)^j\tau^j}{\Gamma(j+1)}, \tag{33}$$

which concur with the analytical solution gained by Laplace decomposition method (CLDM) [13], and Laplace Homotopy perturbation method (LPM) [45], so that $\varphi(\varsigma,\tau) = \psi(\varsigma,\tau) = \sin(\varsigma)\ e^{-\tau}$.

Numerical and graphical simulations were performed to demonstrate the effectiveness and reliability of the proposed approach in resolving the coupled CVBE system (2). The results are presented in Table 3, as well as Figures 4 and 5. Specifically, Table 3 lists the jth-approximate solutions for various values of ρ inclding $\{0.7,\ 0.8,\ 0.9,\ 1\}$ with fix values of ς over the time domain $\tau \in [0,2]$. Also, the table shows the absolute errors $|\varphi - \varphi_6|$ which displays superb solutions within a small iterationsand which confirm the validity, and efficiency of the present procedure. Figure 5 illustrates how the number of iterations of approximate solutions affects on the behavior of the acquired findings at different values of the fractional order parameter with exact solutions. Also, ρth-time Caputo-FD of the 6-th truncation approximate solutions $\varphi_6\ (\varsigma,\tau)$ and $\psi_6(\varsigma,\tau)$ at various values of ρ are displayed in 2D-diagrams as shown in Figure 4 for different values of $\rho \in \{0.8, 0.9, 1\}$. It is noted from this simulation clearly shows that the behavior of the LT-RFPS solutions for various ρ-levels provide more precise and reliable approximations over the considered domain in harmony with one another and tends constantly to the classical ordered $\rho = 1$ in a consistency manner. Particularly when the approximation solutions' terms are increased.

Table 3: **Comparisons of numerical results of the fractional coupled CVBE system (1.2).**

ς_i	τ_i	$\varphi(\varsigma,\tau)$	$\varphi_6(\varsigma,\tau)$	$\lvert\varphi-\varphi_6\rvert$	$\varphi_6(\varsigma,\tau)$		
					$\rho=0.9$	$\rho=0.8$	$\rho=0.7$
3	0.5	0.0855936116	0.08559381738	2.5081×10^{-7}	0.08221970849	0.07936203908	0.07706652200
	1.0	0.0519151497	0.05194000296	2.4853×10^{-5}	0.05316525164	0.05493971096	0.05746574405
	1.5	0.0314881299	0.03188981432	4.01684×10^{-4}	0.03720797799	0.03449808776	0.05198482368
	2.0	0.0190985163	0.02195200125	2.85348×10^{-3}	0.03217494285	0.04573073471	0.06505821149
5	0.5	-0.581616973	-0.5816183713	1.39844×10^{-6}	-0.5586909710	-0.5392728274	-0.5236745461
	1.0	-0.352768526	-0.3529374066	1.68881×10^{-4}	-0.3612630913	-0.3733207162	-0.3904853814
	1.5	-0.213964927	-0.2166944112	2.72948×10^{-3}	-0.2528318543	-0.2955737661	-0.3532419768
	2.0	-0.129776288	-0.1491659987	1.93897×10^{-2}	-0.2186318875	-0.3107448207	-0.4420769182

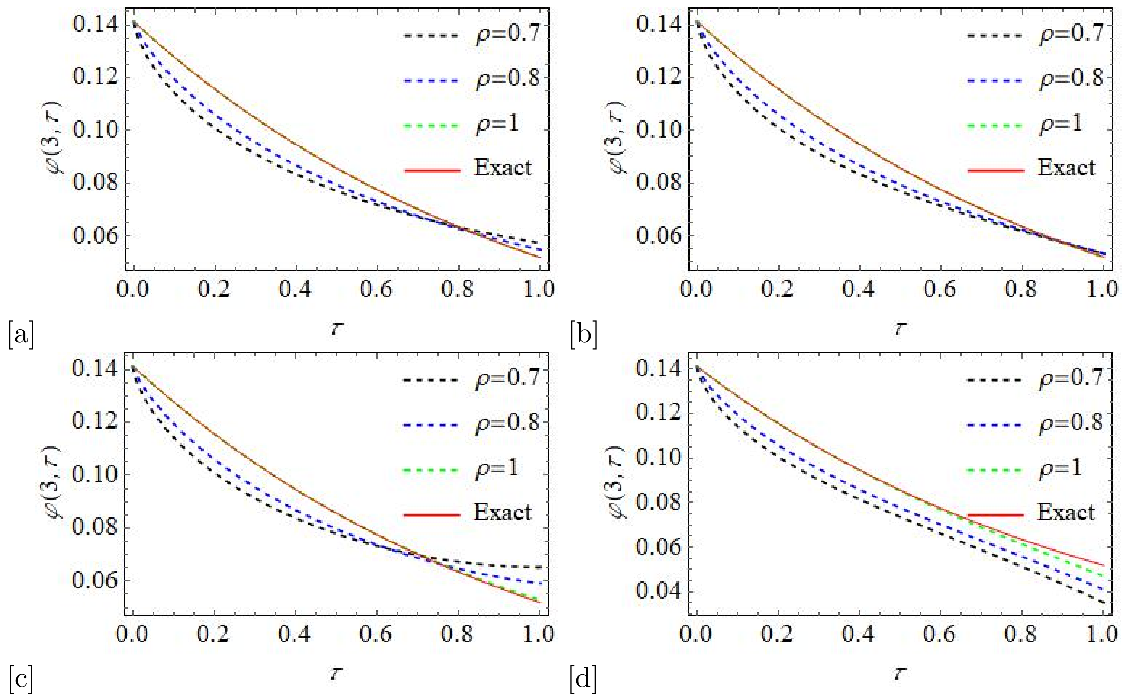


Figure 4: Profile solutions of the exact and different j-th approximate solutions of $\varphi(\varsigma,\tau)$ at various values of ρ: (a) $j=6$, (b) $j=5$, (c) $j=4$, and (d) $j=3$, where $\varsigma=3$ and $\tau\in[0,1]$.

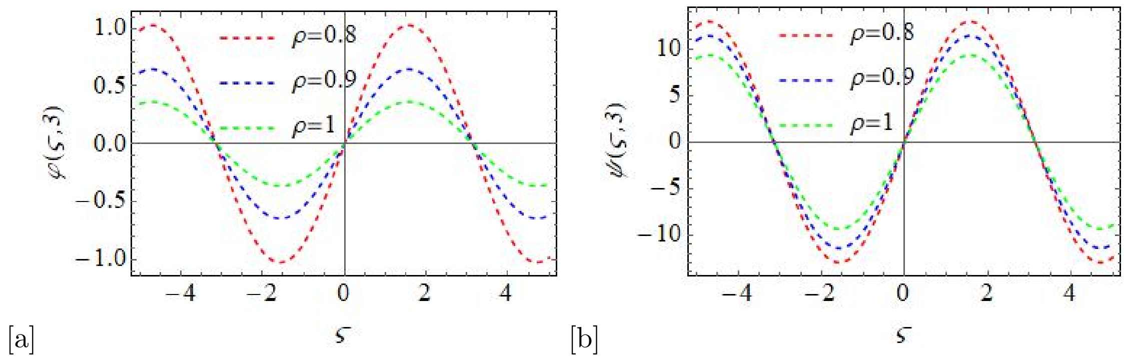


Figure 5: Profile solutions of the 6-th truncation approximate solutions $\varphi_6(\varsigma, \tau)$ and $\psi_6(\varsigma, \tau)$ at various values of ρ, with fixed $\tau = 3$ and $-5 \leq \varsigma \leq 5$

5. Conclusions

In the present disquisition, the LT-RFPS technique was proposed to investigate the non-linear time-fractional DS-WE and CVBE systems. The Caputo-FD as a time-fractional approach was used for the mathematical formalism of these models. The strategy of the projected technique is regarded as more effective than other analytical schemes because of its limited number of approximations. The employed technique involves performing the LT explicitly to the governing models and then simulating the FRPS algorithm in a new space. The inverse LT is then implemented to find out the approximate solutions of the studied models. The capabilities of the proposed approach have been demonstrated through numerical simulation. This simulation finds that the target model's solutions are astonishingly near to the provided exact solutions when the time value is decreased. Additionally, the amplitude of the model's solitary wave increases as the value of parameter ρ is lessened. As well, the results show that for small values of the time variable, the absolute error becomes fewer with increasing the space variable values. On another aspect, the impact of varied values of the parameter ρ, and changing the values of space and time considered domain on the posed models has been discussed graphically. From these representations, it is seen that the behavior of the obtained solutions is consistent with the integer value and harmonious for various fractional values of ρ. As a result, it can be claimed that the LT-RFPS technique is indeed effective and suitable to solve such models. Consequently, due to its accuracy, and straightforwardness and minimal calculation effort, we recommend employing the future technique in investigations of fractional models in mathematical physics and engineering. Ideally, this investigation will be helpful to scholars, in the future, to treat higher dimensions complex nonlinear spacetime partial DEs in the framework of the conformable RFPS technique methodology and linking it to one of the well-known integral transformations.

Conflict of interest

The authors declare no conflict of interest.

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