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Fixed Point Theorems in Controlled Rectangular Modular Metric Spaces with Solution of Fractional Differential Equations

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Abstract. In this paper, we establish the notion of controlled rectangular modular metric space as a generalization of modular $b-$metric space and rectangular $b-$metric space. We used contraction mappings to find the existence and uniqueness of a fixed point in the framework of controlled rectangular modular metric space. We give several non-trivial examples and show the validity of contraction mappings via graphs. At the end, we utilize our main result to solve a non-linear fractional differential equation.

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Key Words and Phrases: Controlled metric space, modular metric space, fixed point; existence and uniqueness, non-linear fractional differential equations

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1. Introduction and Preliminaries

The Banach fixed-point theorem [10] ensures the existence and uniqueness of fixed points of particular self-maps of metric spaces (MSs) and provides a constructive approach to identify those fixed points. Picard's method [20] of consecutive approximations might be viewed as an abstract formulation of this method. In 1922 Banach established the following famous result.

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Definition 1. *[10] Suppose* $(\aleph, \Delta)$ *be a MS. Then a mapping* $T : \aleph \to \aleph$ *is known as contraction mapping on* $\aleph$ *if there exists* $q \in [0, 1)$ *such that*

$$\Delta(T\vartheta, Ty) \leq q\Delta(\vartheta, y)$$

for all $\vartheta, y \in \aleph$.

Theorem 1. *[10] Let* $(\aleph, \Delta)$ *be a complete MS and* $T : \aleph \to \aleph$ *be a contraction mapping. Then* T *has a unique fixed point* ϑ^* *in* $\aleph$.

Many authors established various kinds of contraction inequalities in an attempt to generalize the famous Banach contraction principle by using different generalizations of Chistyakov [13] established the notion of modular MS and proved some new results.

Definition 2. *Let* $\aleph$ *be a non-empty set and the function* $\Delta_\xi : (0, +\infty) \times \aleph \times \aleph \to [0, +\infty]$, *which satisfies the following axioms for all* $\mu, \kappa, \vartheta \in \aleph$:

(M1) $\Delta_\xi(\mu, \kappa) = 0$ *for all* $\xi > 0$ *if and only if* $\mu = \kappa$;

(M2) $\Delta_\xi(\mu, \kappa) = \Delta_\xi(\kappa, \mu)$ *for all* $\xi > 0$;

(M3) $\Delta_{\xi+\rho}(\mu, \kappa) \leq \Delta_\xi(\mu, \vartheta) + \Delta\rho(\vartheta, \kappa)$ *for all* $\xi, \rho > 0$.

Then Δ_ξ *be known as a modular metric on* $\aleph$.

Recently, Abdou [3, 4] proved various interesting fixed point results in the sense of modular MS. In 2018, Mlaiki et al. [21] established the notion of controlled type MS as follows:

Definition 3. *Consider a non-empty set* $\aleph$ *and the function* $\alpha : \aleph \times \aleph \to [0, +\infty)$. *Then a function* $\Delta : \aleph \times \aleph \to [0, +\infty)$ *is said to be controlled MS if the following axioms holds for all* $\mu, \kappa, \vartheta \in \aleph$:

(C1) $\Delta(\mu, \kappa) = 0$ *if and only if* $\mu = \kappa$;

(C2) $\Delta(\mu, \kappa) = \Delta(\kappa, \mu)$;

(C3) $\Delta(\mu, \kappa) \leq \alpha(\mu, \vartheta)\Delta(\mu, \vartheta) + \alpha(\vartheta, \kappa)\Delta(y, \kappa)$.

Then the pair $(\Delta, \aleph)$ *is called a controlled MS.*

In addition, Mlaiki et al. [21] generalized the Banach fixed point theorem.

In 2000, Branciari [11] coined the concept of rectangular (generalized) MSs as follows:

Definition 4. *Consider a non-empty set* $\aleph$ *and the mapping* $\Delta : \aleph \times \aleph \to [0, +\infty)$ *satisfies the following conditions:*

(R1) $\Delta(\mu, \kappa) = 0$ *if and only if* $\mu = \kappa$;

(R2) $\Delta(\mu,\kappa) = \Delta(\kappa,\mu)$, *for all* $\mu,\kappa \in \aleph$;

(R3) $\Delta(\mu,\kappa) \leq \Delta(\mu,\vartheta) + \Delta(\vartheta,y) + \Delta(y,\kappa)$;

for all $\mu,\kappa \in \aleph$ *and all distinct points* $\vartheta, y \in \aleph$. *Then the pair* $(\Delta,\aleph)$ *is called a rectangular MS.*

In 2021, Alamgir et al. [5] coined the concept of controlled rectangular MS and proved some fixed point results for contraction mappings.

Definition 5. *Consider a nonempty set* $\aleph$ *and the function* $\alpha : \aleph \times \aleph \to [0,+\infty)$. *Then a function* $\Delta : \aleph \times \aleph \to [0,+\infty)$ *is said to be a controlled rectangular MS if the following axioms hold:*

(CR1) $\Delta(\mu,\kappa) = 0$ *for all* $\xi > 0$ *if and only if* $\mu = \kappa$;

(CR2) $\Delta(\mu,\kappa) = \Delta(\kappa,\mu)$ *for all* $\xi > 0$;

(CR3) $\Delta(\mu,\kappa) \leq \alpha(\mu,\vartheta)\Delta(\mu,\vartheta) + \alpha(\vartheta,y)\Delta(\vartheta,y) + \alpha(y,\kappa)\Delta(y,\kappa)$,

for all $\mu,\kappa \in \aleph$ *and all distinct points* $\vartheta, y \in \aleph$. *Then the pair* $(\Delta,\aleph)$ *is called a controlled rectangular MS.*

We refer [1, 2, 9, 15, 24–27, 29] for more detail. Aydi et. al. [6] proved a fixed point theorem for set-valued quasi-contractions in $b-$metric spaces. Karapinar et. al. [16–18] proved several interesting fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces. Souayah and Mrad [28] proved some fixed point results for contraction mappings in the context of controlled partial metric type spaces. Debnath and Sen [14] proved various fixed point results of interpolative Ćirić-Reich–Rus-Type contractions in $b-$metric spaces. Roy et. al. [23] provided an extended -metric-type space and related fixed point theorems with an application to nonlinear integral equations. Rossafi and Kari [22] gave some fixed point results in the sense of controlled rectangular metric spaces. Budhia et. al. [12] provided some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations. Aydi et. al. [7] presented a common Jungck type fixed point result in extended rectangular $b-$metric spaces. Aydi et. al. [8] gave fixed-discs in rectangular metric spaces. Kari et. al. [19] established contraction mapping on complete rectangular metric spaces.

In this manuscript, we introduce the notions of rectangular modular metric space (RMMS) and controlled rectangular modular metric space (CRMMS). We generalize and prove the well-known Banach fixed point theorem in the sense of CRMMS. We also give some non-trivial examples to ensure the validity of provided fixed point results. At the end, we use a fixed point technique to ensure the existence and uniqueness of non-linear fractional differential equations. From now to onward, we use $\Delta_\xi(\mu,\kappa) = \Delta(\mu,\kappa,\xi)$, for all $\mu,\kappa \in \aleph$ which denotes the map $\Delta_\xi : \aleph \times \aleph \times (0,+\infty) \to [0,+\infty)$, where $\xi \in (0,+\infty)$.

2. Main Results

In this section, we will introduce the concepts of RMMS and CRMMS and develop some fixed-point results.

Definition 6. *Consider a non-empty set* $\aleph$ *and the mapping* $\Delta_\xi : \aleph \times \aleph \times (0, +\infty) \to [0, +\infty)$*, which satisfies the following axioms:*

(RM1) $\Delta_\xi(\mu, \kappa) = 0$ *if and only if* $\mu = \kappa$;

(RM2) $\Delta_\xi(\mu, \kappa) = \Delta_\xi(\kappa, \mu)$;

(RM3) $\Delta_\xi(\mu, \kappa) \leq \Delta_\xi(\mu, \vartheta) + \Delta_\xi(\vartheta, y) + \Delta_\xi(y, \kappa)$;

for all $\xi > 0$, $\mu, \kappa \in \aleph$ *and all distinct points* $\vartheta, y \in \aleph$. *Then the pair* $(\Delta_\xi, \aleph, \xi)$ *is called RMMS.*

Definition 7. *Consider a non-empty set* $\aleph$ *and* $\alpha : \aleph \times \aleph \to [0, +\infty)$. *Then a function* $\Delta_\xi : \aleph \times \aleph \times (0, +\infty) \to [0, +\infty)$ *is said to be controlled rectangular modular metric if for all* $\xi > 0, \mu, \kappa \in \aleph$ *and distinct* $\vartheta, y \in \aleph$ *the following axioms are satisfied:*

(CM1) $\Delta_\xi(\mu, \kappa) = 0$ *if and only if* $\mu = \kappa$;

(CM2) $\Delta_\xi(\mu, \kappa) = \Delta_\xi(\kappa, \mu)$ *for all* $\mu, \kappa \in \aleph$;

(CM3) $\Delta_\xi(\mu, \kappa) \leq \alpha(\mu, \vartheta)\Delta_\xi(\mu, \vartheta) + \alpha(\vartheta, y)\Delta_\xi(\vartheta, y) + \alpha(y, \kappa)\Delta_\xi(y, \kappa)$.

Then the pair $(\Delta_\xi, \aleph)$ *is called CRMMS.*

Example 1. *Let* $\aleph = N$ *define* $\Delta_\xi : \aleph \times \aleph \times (0, +\infty) \to [0, +\infty)$, *by*

$$\Delta(\mu, \kappa, \xi) = \begin{cases} 0, & \text{if} \quad \mu = \kappa; \\ \frac{4\eta}{\xi}, & \text{if } \mu, \kappa \in \{1, 2 \text{ and } \mu \neq \kappa; \\ \frac{\eta}{\xi}, & \text{if } \mu \text{ or } \kappa \notin \{1, 2, \cdots, 10\} \text{ and } \mu \neq \kappa; \end{cases}$$

where $\eta > 0$ *is a constant. Then* $(\Delta_\xi, \aleph)$ *is a CRMMS with controlled function*

$$\alpha = (\mu, \kappa) = \begin{cases} 1 & \text{if } \mu \neq \kappa; \\ 1 + \mu + \kappa & \text{if } \mu = \kappa; \end{cases}$$

but not RMMS. Let $\mu = 1$, $\kappa = 2, \vartheta = 3$ *and* $y = 4$, *then from triangular inequality (RM3), we have*

$$\Delta(\mu, \kappa, \xi) \leq \Delta(\mu, \vartheta, \xi) + \Delta(\vartheta, y, \xi) + \Delta(y, \kappa, \xi),$$

$$\Delta(1,2,\xi) \leq \Delta(1,3,\xi) + \Delta(3,4,\xi) + \Delta(4,2,\xi).$$

That is,

$$\begin{aligned} \frac{4\eta}{\xi} &\leq \frac{\eta}{\xi} + \frac{\eta}{\xi} + \frac{\eta}{\xi} \\ &\leq \frac{3\eta}{\xi} \end{aligned}$$

After simplification, we have
$4 \leq 3$,
which is contradiction. Hence, CRMMS need not to be RMMS. Also observe that it is not modular $b-$MS and rectangular $b-$MS.

Example 2. *Let $\aleph = N$ define $\Delta_\xi : \aleph \times \aleph \times (0,+\infty) \to [0,+\infty)$ by*

$$\Delta(\mu,\kappa,\xi) = \begin{cases} 0, & \text{if} \quad \mu = \kappa; \\ 10\eta\xi, & \text{if } \mu,\kappa \in \{1,2,\cdots,10\} \text{ and } \mu \neq \kappa; \\ \frac{2\eta\xi}{3}, & \text{if } \mu \text{ or } \kappa \notin \{1,2,\cdots,10\} \text{ and } \mu \neq \kappa; \end{cases}$$

where $\eta > 0$ is a constant. Then $(\Delta_\xi, \aleph)$ is a CRMMS with control function

$$\alpha = (\mu,\kappa) = \begin{cases} 3 & \text{if } \mu \neq \kappa; \\ 2(\mu+\kappa) & \text{if } \mu = \kappa; \end{cases}$$

but not RMMS.

Let $\mu = 5, \kappa = 8, \vartheta = 11$ and $y = 12$, then from triangular inequality (RM3), we have

$$\Delta(\mu,\kappa,\xi) \leq \Delta(\mu,\vartheta,\xi) + \Delta(\vartheta,y,\xi) + \Delta(y,\kappa,\xi),$$

$$\Delta(5,8,\xi) \leq \Delta(5,11,\xi) + \Delta(11,12,\xi) + \Delta(12,8,\xi).$$

That is,

$$(10\eta)\xi \leq (2\eta)\frac{\xi}{3} + (2\eta)\frac{\xi}{3} + (2\eta)\frac{\xi}{3}.$$

That is,

$$10 \leq 6,$$

which is a contradiction. Hence, CRMMS does not need to be RMMS. Also, observe that it is not modular $b-$MS and rectangular $b-$MS.

Remark 1.

a) Every rectangular MS and control MS is a CRMMS.

b) Every CRMMS need not be RMMS, but the converse is true, as shown in the above examples.

In terms of CRMMS, the concepts of convergence, Cauchy, and completeness can be easily generalized.

Definition 8. *Assume* $(\Delta_\xi, \aleph)$ *be a CRMMS. Then*

a) a sequence (ϑ_n) *in* $\aleph$ *is said to be convergent to* $\vartheta \in \aleph$ *if*

$$\lim_{n \longrightarrow +\infty} \Delta_\xi(\vartheta_n, \vartheta) = 0$$

b) a sequence (ϑ_n) *in* $\aleph$ *is called Cauchy sequence if*

$$\lim_{n,m \longrightarrow +\infty} \Delta_\xi(\vartheta_n, \vartheta_m) = 0$$

c) the pair $(\Delta_\xi, \aleph)$ *is said to be a complete CRMMS if every Cauchy sequence in* $\aleph$ *converges in* $\aleph$.

Definition 9. *Assume* $(\Delta_\xi, \aleph)$ *be CRMMS, let* $\vartheta \in \aleph$ *and* $r > 0$. *Then,*

i) the open ball denoted and defined by

$$B(\vartheta, r) = \{\vartheta_0 \in \aleph, \Delta_\xi(\vartheta, \vartheta_0) < r\},$$

ii) the mapping $g : \aleph \to \aleph$ *is said to be continuous at* $\vartheta \in \aleph$ *if for every* $\gamma > 0$ *and* $\delta > 0$ *such that* $g(B(\vartheta, \delta)) \subseteq B(g(\vartheta, \gamma))$. *If* g *is continuous at* ϑ, *then for any sequence* (ϑ_n) *converges to* ϑ, *we have*

$$\lim_{n \longrightarrow +\infty} g(\vartheta_n) = g(\vartheta).$$

Lemma 1. *Assume*$(\Delta_\xi, \aleph)$ *be a CRMMS and* (ϑ_n) *be a Cauchy sequence in* $\aleph$ *and* $(\vartheta_n) \neq (\vartheta_m)$ *whenever* $n \neq m$. *If*

$$\lim_{n,m \longrightarrow +\infty} \Delta_\xi(\vartheta_n, \vartheta_m) < +\infty$$

for all $(\vartheta_n), (\vartheta_m) \in \aleph$, *then* (ϑ_n) *has a unique fixed point.*

Proof. Assume s, t are two fixed point of the sequence (ϑ_n) in $\aleph$. Then

$$\lim_{n \longrightarrow +\infty} g(\vartheta_n) = s.$$

and

$$\lim_{n \longrightarrow +\infty} g(\vartheta_n) = t.$$

Here (ϑ_n) is a Cauchy sequence. Then from the triangular inequality (CM3) of definition (7), we have

$$\Delta_\xi(\mu,\kappa) \leq \alpha(\mu,\vartheta_n)\Delta_\xi(\mu,\vartheta_n)+\alpha(\vartheta_n,\vartheta_m)\Delta_\xi(\vartheta_n,\vartheta_m)+\alpha(\vartheta_m,\kappa)\Delta_\xi(\vartheta_m,\kappa) \to 0 \ as \ n, r \to +\infty. \tag{2.1}$$

This implies that

$$\Delta_\xi(\mu,\kappa) = 0.$$

Hence (ϑ_n) has a unique fixed point in $\aleph$.

Definition 10. *Suppose $(\Delta_\xi, \aleph)$ be a CRMMS. Then the mapping*

i) $g : \aleph \to \aleph$ defined by

$$\begin{aligned} \Omega(\vartheta, n) &= \{\vartheta, g\vartheta, g^{2\vartheta}, \cdots g^{n\vartheta}\}, \\ \Omega(\vartheta, +\infty) &= \{\vartheta, g\vartheta, g^{2\vartheta}, \cdots g^{n\vartheta} \cdots\}, \end{aligned}$$

where $\vartheta \in \aleph$ and $n \in N$. Here, $\Omega(\vartheta, +\infty)$ is known as orbit of g.

ii) $g : \aleph \to \aleph$ is known as $g-$orbitally continuous, if

$$\lim_{k \longrightarrow +\infty} g_k^n \vartheta = \vartheta$$

implies

$$\lim_{k \longrightarrow +\infty} g\left(g_k^n \vartheta\right) = g^\vartheta$$

for $\vartheta \in \aleph$.

Theorem 2. *Suppose $g : \aleph \to \aleph$ is a mapping in a CRMMS $(\Delta_\xi, \aleph)$. Assume that the following conditions hold:*

a) for all $\mu, \kappa \in \aleph$,

$$\Delta_\xi(g\mu, g\kappa) \leq \lambda \Delta_\xi(\mu, \kappa),$$

b) $\sup_{q \geq 1} \lim_{i \longrightarrow +\infty} \alpha(\mu_i, \mu_q) \left(\frac{\alpha(\mu_i, \mu_{(i+1)})}{\alpha(\mu_{(i-1)}, \mu_i)}\right) \lambda < 1$ *for any $\mu_i \in \aleph$,*

c) $(\Delta_\xi, \aleph)$ is g is orbitally complete,

d) g is orbitally continuous,

e) For each $\mu \in \aleph$ $\lim_{n\to+\infty} \alpha(\mu_n, \mu)$ *and* $\lim_{n\to+\infty} \alpha(\mu, \mu_n)$ *exist and finite.*

Then g *has a unique fixed point.*

Proof. Suppose μ_0 be any point in $\aleph$. We describe the iterative sequence (μ_n) over μ_0 as follows

$$g(\mu_0) = \mu_1, g(\mu_1) = \mu_2, g(\mu_2) = \mu_3 \cdots g^n(\mu_0) = \mu_n,$$

then, we obtain

$$\Delta_\xi(\mu_1, \mu_2) = \Delta_\xi(g(\mu_0), g^2(\mu_0)) \leq \lambda \Delta_\xi(\mu_0, g(\mu_0)) = \lambda \Delta_\xi(\mu_0, \mu_1)$$

Recursively, we get

$$\begin{aligned}
\Delta_\xi(\mu_n, \mu_{(n+1)}) &\leq \Delta_\xi(g^n(\mu_0), g^{(n+1)}(\mu_0)) \leq \lambda \Delta_\xi(g^{(n-1)}(\mu_0), g^n(\mu_0)) \\
&\vdots \\
&\leq \lambda^n \Delta_\xi(\mu_0, \mu_1)
\end{aligned}$$

Taking $\lim_{n\to+\infty} \Delta_\xi(\mu_n, \mu_{(n+1)}) = 0$. Similarly, $\lim_{n\to+\infty} \Delta_\xi(\mu_{(n+1)}, \mu_{(n+2)}) = 0$. Now, we examine that (μ_n) is a Cauchy sequence. Here, we make the following cases:

Case 1:

Suppose p be an odd number, then $p = 2n + 1$ and $r \geq 1$, we have

$$\begin{aligned}
\Delta_\xi(\mu_n, \mu_{(n+2r+1)}) &\leq \frac{\xi}{3}\alpha(\mu_n, \mu_{(n+1)})\Delta_\xi(\mu_n, \mu_{(n+1)}) + \frac{\xi}{3}\alpha(\mu_{(n+1)}, \mu_{(n+2)})\Delta_\xi(\mu_{(n+1)}, \mu_{(n+2)}) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})\Delta_\xi(\mu_{(n+2)}, \mu_{(n+2r+1)}), \\
&= \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_\xi(\mu_0, \mu_1) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})\Delta_\xi(\mu_{(n+2)}, \mu_{(n+2r+1)}), \\
&\leq \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_\xi(\mu_0, \mu_1) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})[\frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+3)})\Delta_\xi(\mu_{(n+2)}, \mu_{(n+3)}) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+3)}, \mu_{(n+4)})\Delta_\xi(\mu_{(n+3)}, \mu_{(n+4)}) + \frac{\xi}{3}\alpha(\mu_{(n+4)}, \mu_{(n+2r+1)})\Delta_\xi(\mu_{(n+4)}, \mu_{(n+2r+1)})], \\
&= \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_\xi(\mu_0, \mu_1) \\
&\quad + \frac{\xi^2}{3^2}[\alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})\alpha(\mu_{(n+2)}, \mu_{(n+3)})\lambda^{(n+2)} \\
&\quad + \alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})\alpha(\mu_{(n+3)}, \mu_{(n+4)})\lambda^{(n+3)}]\Delta_\xi(\mu_0, \mu_1) \\
&\quad + \frac{\xi^2}{3^2}\alpha(\mu_{(n+2)}, \mu_{(n+2r+1)})\alpha(\mu_{(n+4)}, \mu_{(n+2r+1)})\Delta_\xi(\mu_{(n+4)}, \mu_{(n+2r+1)}),
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\xi}{3}[\lambda^{n}\alpha(\mu_{n},\mu_{(n+1)})+\lambda^{(n+1)}\alpha(\mu_{(n+1)},\mu_{(n+2)})]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{2}}{3^{2}}[\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+2)},\mu_{(n+3)})\lambda^{(n+2)}\\
&+\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+3)},\mu_{(n+4)})\lambda^{(n+3)}]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{2}}{3^{2}}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+2r+1)})\\
&[\frac{\xi}{3}\alpha(\mu_{(n+4)},\mu_{(n+5)})\Delta_{\xi}(\mu_{(n+4)},\mu_{(n+5)})+\frac{\xi}{3}\alpha(\mu_{(n+5)},\mu_{(n+6)})\Delta_{\xi}(\mu_{(n+5)},\mu_{(n+6)})\\
&+\frac{\xi}{3}\alpha(\mu_{(n+6)},\mu_{(n+2r+1)})\Delta_{\xi}(\mu_{(n+6)},\mu_{(n+2r+1)})],\\
\leq\ & \frac{\xi}{3}[\lambda^{n}\alpha(\mu_{n},\mu_{(n+1)})+\lambda^{(n+1)}\alpha(\mu_{(n+1)},\mu_{(n+2)})]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{2}}{3^{2}}[\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+2)},\mu_{(n+3)})\lambda^{(n+2)}\\
&+\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+3)},\mu_{(n+4)})\lambda^{(n+3)}]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{2}}{3^{2}}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+2r+1)})\\
&[\frac{\xi}{3}\lambda^{(n+4)}\alpha(\mu_{(n+4)},\mu_{(n+5)})+\frac{\xi}{3}\lambda^{^{(n+5)}}\alpha(\mu_{(n+5)},\mu_{(n+6)})]\Delta_{\xi}(\mu\_o,\mu_{1})\\
&+\frac{\xi}{3}\alpha(\mu_{(n+6)},\mu_{(n+2r+1)})\Delta_{\xi}(\mu_{(n+6)},\mu_{(n+2r+1)}),\\
\leq\ & \frac{\xi}{3}[\lambda^{n}\alpha(\mu_{n},\mu_{(n+1)})+\lambda^{(n+1)}\alpha(\mu_{(n+1)},\mu_{(n+2)})]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{2}}{3^{2}}[\lambda^{(n+2)}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+2)},\mu_{(n+3)})\\
&+\lambda^{(n+3)}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+3)},\mu_{(n+4)})]\Delta_{\xi}(\mu_{0},\mu_{1})\\
&+\frac{\xi^{3}}{3^{3}}[\lambda^{(n+4)}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+5)})\\
&+\lambda^{(n+5)}\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+2r+1)})\alpha(\mu_{(n+5)},\mu_{(n+6)})]\\
&\Delta_{\xi}(\mu_{0},\mu_{1})\\
&\vdots\\
&+\frac{\xi^{m}}{3^{m}}[\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\alpha(\mu_{(n+4)},\mu_{(n+2r+1)})\dots\\
&\alpha(\mu_{(n+2r-2)},\mu_{(n+2r-1)})\lambda^{(n+2r-2)}\\
&+\alpha(\mu_{(n+2)},\mu_{(n+2r-1)})\cdots\alpha(\mu_{(n+2r-1)},\mu_{(n+2r)})\lambda^{(n+2r-1)}\\
&+\alpha(\mu_{(n+2)},\mu_{(n+2r+1)})\dots\alpha(\mu_{(n+2r)},\mu_{(n+2r+1)})\lambda^{(n+2r)}]\Delta_{\xi}(\mu_{0},\mu_{1}).
\end{aligned}
$$

From the above inequality, we get

$$\lim_{n\to+\infty}\Delta_{\xi}(\mu_{n},\mu_{(n+2r+1)})=0.$$

Case: 2

Let p be an even number then $p = 2n$ and $r \geq 1$, then

$$\begin{aligned}
\Delta_{\xi}(\mu_n, \mu_{(n+2r)}) &\leq \frac{\xi}{3}[\alpha(\mu_n, \mu_{(n+1)})\Delta_{\xi}(\mu_n, \mu_{(n+1)}) + \frac{\xi}{3}\alpha(\mu_{(n+1)}, \mu_{(n+2)})\Delta_{\xi}(\mu_{(n+1)}, \mu_{(n+2)}) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\Delta_{\xi}(\mu_{(n+2)}, \mu_{(n+2r)})] \\
&= \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\Delta_{\xi}(\mu_{(n+2)}, \mu_{(n+2r)}), \\
&\leq \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})[\frac{\xi}{3}\alpha(\mu_{(n+2)}, \mu_{(n+3)})\Delta_{\xi}(\mu_{(n+2)}, \mu_{(n+3)}) \\
&\quad + \frac{\xi}{3}\alpha(\mu_{(n+3)}, \mu_{(n+4)})\Delta_{\xi}(\mu_{(n+3)}, \mu_{(n+4)}) + \frac{\xi}{3}\alpha(\mu_{(n+4)}, \mu_{(n+2r)})\Delta_{\xi}(\mu_{(n+4)}, \mu_{(n+2r)})], \\
&\leq \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad + \frac{\xi^2}{3^2}[\lambda^{(n+2)}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\alpha(\mu_{(n+2)}, \mu_{(n+3)}) \\
&\quad + \lambda^{(n+3)}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\alpha(\mu_{(n+3)}, \mu_{(n+4)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad + \frac{\xi^2}{3^2}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\alpha(\mu_{(n+4)}, \mu_{(n+2r)})\Delta_{\xi}(\mu_{(n+4)}, \mu_{(n+2r)}), \\
&\leq \frac{\xi}{3}[\lambda^n \alpha(\mu_n, \mu_{(n+1)}) + \lambda^{(n+1)}\alpha(\mu_{(n+1)}, \mu_{(n+2)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad + \frac{\xi^2}{3^2}[\lambda^{(n+2)}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\alpha(\mu_{(n+2)}, \mu_{(n+3)}) \\
&\quad + \lambda^{(n+3)}\alpha(\mu_{(n+2)}, \mu_{(n+2r)})\alpha(\mu_{(n+3)}, \mu_{(n+4)})]\Delta_{\xi}(\mu_0, \mu_1) \\
&\quad \vdots \\
&\quad + \frac{\xi^{(m-1)}}{3^{(m-1)}}[\alpha(\mu_{(n+2)}, \mu_{(n+2r)}) \cdots \alpha(\mu_{(n+2-4)}, \mu_{(n+2r-3)})\lambda^{(n+2r-4)} \\
&\quad + \alpha(\mu_{(n+2)}, \mu_{(n+2r)}) \cdots \alpha(\mu_{(n+2-3)}, \mu_{(n+2r-2)})\lambda^{(n+2r-3)} \\
&\quad + \alpha(\mu_{(n+2)}, \mu_{(n+2r)}) \cdots \alpha(\mu_{(n+2r-2)}, \mu_{(n+2r)})\lambda^{(n+2r-2)}]\Delta_{\xi}(\mu_0, \mu_1).
\end{aligned}$$

From the above inequality, we get $\lim_{n\to+\infty} \Delta_{\xi}(\mu_n, \mu_{(n+2r+1)}) = 0$.

Hence, both cases show that $\{\mu_n\}$ is a Cauchy sequence. As $\aleph$ is $g-$orbitally complete, so there exist $\mu \in \aleph$ such that

$$\lim_{n\to+\infty} \mu_n = \mu.$$

Now we show that μ is a fixed point of g. As $\aleph$ is $g-$orbitally continuous, we get

$$\Delta_{\xi}(\mu, g\mu) \leq \frac{\xi}{3}\alpha(\mu, \mu_n)\Delta_{\xi}(\mu, \mu_n) + \frac{\xi}{3}\alpha(\mu, \mu_{(n+1)})\Delta_{\xi}(\mu, \mu_{(n+1)}) + \frac{\xi}{3}\alpha(\mu_{(n+1)}, g\mu)\Delta_{\xi}(\mu_{(n+1)}, g\mu).$$

Since for each $\mu \in \aleph$, $\lim_{n\to+\infty} \alpha(\mu_n, \mu)$ and $\lim_{n\to+\infty} \alpha(\mu, \mu_n)$ exist and finite, so by taking limit and utilizing

$$\lim_{n\to+\infty} \Delta_\xi(\mu_n, \mu_{(n+1)}) = 0.$$

We get

$$\lim_{n\longrightarrow+\infty} \Delta_\xi(\mu, g\mu) = 0.$$

That is,

$$g\mu = \mu.$$

Hence μ is a fixed point of g. In view of Lemma (2), μ is unique fixed point of g.

Corollary 1. *Suppose $g : \aleph \to \aleph$ be a mapping on a complete CRMMS $(\Delta_\xi, \aleph)$. Assume that the following conditions hold:*

a) For all $\mu, \kappa \in \aleph$ we have

$$\Delta_\xi(g\mu, g\kappa) \leq \lambda \Delta_\xi(\mu, \kappa), \qquad \lambda \in [0, 1),$$

b) $sup_{(q\geq1)} \lim_{n\longrightarrow+\infty} \alpha(\mu_i, \mu_q) \left(\frac{\alpha(\mu_{(i+1)}, \mu_{(i+2)})}{\alpha(\mu_{(i-1)}, \mu_i)}\right) \lambda < 1$, for any $\mu_i \in \aleph$,

c) g is continuous.

Then g has a unique fixed point.

Example 3. *Suppose $\aleph = R$ and a mapping $\Delta_\xi : \aleph \times \aleph \times (0, +\infty) \longrightarrow \aleph$ define by*

$$\Delta(\mu, \kappa, \xi) = \frac{|\mu - \kappa|}{\xi + |\mu - \kappa|}$$

Then $(\Delta_\xi, \aleph)$ is a complete CRMMS with controlled function

$$\alpha = (\mu, \kappa) = \begin{cases} 1 & if \ \ \mu \neq \kappa; \\ 1 + \mu + \kappa & if \ \mu = \kappa; \end{cases}$$

but not RMMS. Define a mapping $g : \aleph \longrightarrow \aleph$ by

$$g(\mu) = \frac{\mu}{5} + 7$$

Now, we examine the contraction condition. Let $\frac{3}{4} \leq \lambda < 1$, then

$$\begin{aligned} \Delta(g\mu, g\kappa, \xi) &= \frac{|g\mu - g\kappa|}{\xi + |g\mu - g\kappa|} = \frac{|\frac{\mu}{5} - \frac{\kappa}{5}|}{\xi + |\frac{\mu}{5} - \frac{\kappa}{5}|} \\ &= \frac{|\mu - \kappa|}{5\xi + |\mu - \kappa|} \leq \lambda \frac{|\mu - \kappa|}{\xi + |\mu - \kappa|} = \lambda \Delta(\mu, \kappa, \xi). \end{aligned}$$

Observe that all circumstances of Corollary 1 are fulfilled and $\frac{35}{4}$ is a unique fixed point of g. See Figures 1, 2 and 3 for more details.

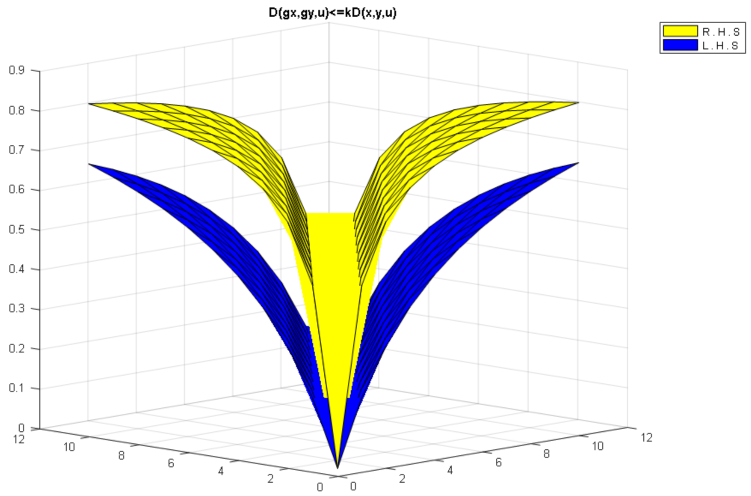

Figure 1: First view of contraction mapping $\Delta(g\mu, g\kappa, \xi) \leq \lambda\Delta(\mu, \kappa, \xi)$ when $\xi = 1$ and $\frac{3}{4} \leq \lambda < 1$.

Table 1: The matrix of values of $\lambda\Delta(\mu, \kappa, \xi)$

0	0.4500	0.6000	0.6750	0.7200	0.7500	0.7714	0.7875	0.8000	0.8100	0.8181
0.4500	0	0.4500	0.6000	0.6750	0.7200	0.7500	0.7714	0.7875	0.8000	0.8100
0.6000	0.4500	0	0.4500	0.6000	0.6750	0.7200	0.7500	0.7714	0.7875	0.8000
0.6750	0.6000	0.4500	0	0.4500	0.6000	0.6750	0.7200	0.7500	0.7714	0.7875
0.7200	0.6750	0.6000	0.4500	0	0.4500	0.6000	0.6750	0.7200	0.7500	0.7714
0.7500	0.7200	0.6750	0.6000	0.4500	0	0.4500	0.6000	0.6750	0.7200	0.7500
0.7714	0.7500	0.7200	0.6750	0.6000	0.4500	0	0.4500	0.6000	0.6750	0.7200
0.7875	0.7714	0.7500	0.7200	0.6750	0.6000	0.4500	0	0.4500	0.6000	0.6750
0.8000	0.7875	0.7714	0.7500	0.7200	0.6750	0.6000	0.4500	0	0.4500	0.6000
0.8100	0.8000	0.7875	0.7714	0.7500	0.7200	0.6750	0.6000	0.4500	0	0.4500
0.8181	0.8100	0.8000	0.7875	0.7714	0.7500	0.7200	0.6750	0.6000	0.4500	0

Theorem 3. *Suppose $g : \aleph \to \aleph$ be a mapping on a CRMMS $(\Delta_\xi, \aleph)$. Assume that the following conditions hold:*

a) For all $\mu, \kappa \in \aleph$,

$$\Delta_\xi(g\mu, g\kappa) \leq \lambda[\Delta_\xi(\mu, g\mu) + \Delta_\xi(\kappa, g\kappa)], \lambda \in \left[0, \frac{1}{2}\right) \tag{A}$$

b) $sup_{(q\geq1)} \lim\limits_{i\longrightarrow+\infty} \alpha(\mu_i, \mu_q)\left(\frac{\alpha(\mu_{(i)}, \mu_{(i+1)})}{\alpha(\mu_{(i-1)}, \mu_i)}\right)\lambda < 1$, for any $\mu_i \in \aleph$, where $\lambda \neq \frac{1}{\alpha(\mu_1, \mu_2)}$ for each $\mu_1, \mu_2 \in \aleph$,

c) For each $\mu \in \aleph$ $\lim\limits_{n\longrightarrow+\infty} \alpha(\mu_n, \mu_{(n+1)}) \leq 1$, $\lim\limits_{n\longrightarrow+\infty} \alpha(\mu, \mu_n)$ and $\lim\limits_{n\longrightarrow+\infty} \alpha(\mu_n, \mu)$ exist and finite.

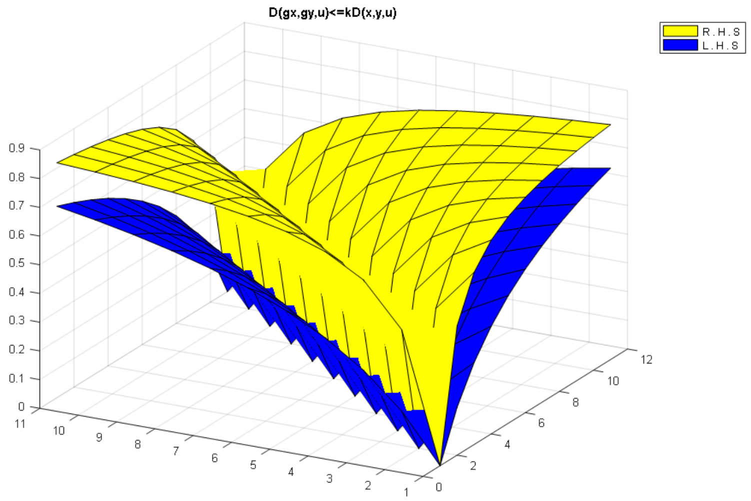

Figure 2: Second view of contraction mapping $\Delta(g\mu, g\kappa, \xi) \leq \lambda\Delta(\mu, \kappa, \xi)$ when $\xi = 1$ and $\frac{3}{4} \leq \lambda < 1$.

Table 2: The matrix of values of $\Delta(g\mu, g\kappa, \xi)$

0	0.1666	0.2857	0.3750	0.4444	0.5000	0.5454	0.5833	0.6153	0.6428	0.6666
0.1666	0	0.1666	0.2857	0.3750	0.4444	0.5000	0.5454	0.5833	0.6153	0.6428
0.2857	0.1666	0	0.1666	0.2857	0.3750	0.4444	0.5000	0.5454	0.5833	0.6153
0.3750	0.2857	0.1666	0	0.1666	0.2857	0.3750	0.4444	0.5000	0.5454	0.5833
0.4444	0.3750	0.2857	0.1666	0	0.1666	0.2857	0.3750	0.4444	0.5000	0.5454
0.5000	0.4444	0.3750	0.2857	0.1666	0	0.1666	0.2857	0.3750	0.4444	0.5000
0.5454	0.5000	0.4444	0.3750	0.2857	0.1666	0	0.1666	0.2857	0.3750	0.4444
0.5833	0.5454	0.5000	0.4444	0.3750	0.2857	0.1666	0	0.1666	0.2857	0.3750
0.6153	0.5833	0.5454	0.5000	0.4444	0.3750	0.2857	0.1666	0	0.1666	0.2857
0.6428	0.6153	0.5833	0.5454	0.5000	0.4444	0.3750	0.2857	0.1666	0	0.1666
0.6666	0.6428	0.6153	0.5833	0.5454	0.5000	0.4444	0.3750	0.2857	0.1666	0

Then g has a unique fixed point in $\aleph$.

Proof. Suppose μ_0 be any point in $\aleph$. We describe the iterative sequence (μ_n) over $\mu_0, g(\mu_0) = \mu_1, g(\mu_1) = \mu_2, g(\mu_2) = \mu_3 \cdots g^n(\mu_0) = \mu_n$, then from (A), we obtain

$$\begin{aligned}
\Delta_\xi(\mu_1, \mu_2) &= \Delta_\xi(g\mu_0, g\mu_1), \\
\Delta_\xi(\mu_1, \mu_2) &\leq \lambda[\Delta_\xi(\mu_0, g\mu_0) + \Delta_\xi(\mu_1, g\mu_1)] \\
\Delta_\xi(\mu_1, \mu_2) &= \lambda[\Delta_\xi(\mu_0, g\mu_0) + \Delta_\xi(\mu_1, \mu_2)] \\
\Delta_\xi(\mu_1, \mu_2) - \lambda\Delta_\xi(\mu_1, \mu_2) &= \lambda\Delta_\xi(\mu_0, \mu_1), \\
\Delta_\xi(\mu_1, \mu_2) &\leq \frac{\lambda}{1-\lambda}\Delta_\xi(\mu_0, \mu_1).
\end{aligned}$$

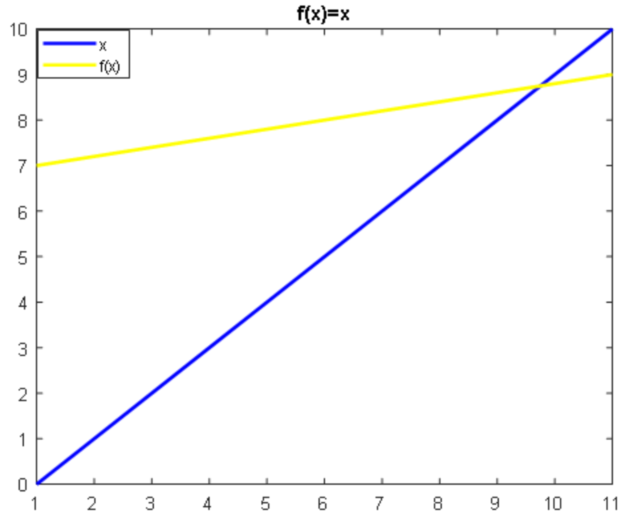

Figure 3: Graph of $g(\mu) = \mu$. It is easy to see that shows that $\frac{35}{4}$ is a unique fixed point.

Let, $\frac{\lambda}{(1-\lambda)} = \alpha < 1$, as $\lambda \leq \frac{1}{2}$. Then by continuously applying (A), we obtain

$$\Delta_\xi(\mu_n, \mu_{(n+1)}) \leq \alpha^n \Delta_\xi(\mu_0, \mu_1).$$

Taking limit on both sides, we get

$$\lim_{n \longrightarrow +\infty} \Delta_\xi(\mu_n, \mu_{(n+1)}) = 0. \tag{B}$$

Now again from (A), we have

$$\begin{aligned} \Delta_\xi(\mu_n, \mu_{(n+2)}) &= \Delta_\xi(g\mu_{(n-1)}, g\mu_{(n+1)}), \\ \Delta_\xi(\mu_1, \mu_2) &\leq \lambda[\Delta_\xi(\mu_{(n-1)}, g\mu_{(n-1)}) + \Delta_\xi(\mu_{(n+1)}, g\mu_{(n+1)})], \\ &\leq \lambda[\Delta_\xi(\mu_{(n-1)}, \mu_n) + \Delta_\xi(\mu_{(n+1)}, \mu_{(n+2)})] \end{aligned}$$

Again, applying limit on the both sides, we get

$$\lim_{n \longrightarrow +\infty} \Delta_\xi(\mu_n, \mu_{(n+1)}) = 0. \tag{C}$$

Now we prove that the sequence $\{\mu_n\}$ is a Cauchy sequence. Using equations (B) and (C), repeat the method as in Theorem (0.2.6), we examine that $\{\mu_n\}$ is a Cauchy sequence. As $\aleph$ is complete, so there exist $\mu \in \aleph$ such that

$$\lim_{n \longrightarrow +\infty} \Delta_\xi(\mu_n, \mu) = 0. \tag{D}$$

Now we examine that μ is a fixed point of g

$$\Delta_\xi(\mu, g\mu) \leq \frac{\xi}{3}[\alpha(\mu, \mu_n)\Delta_\xi(\mu, \mu_n) + \alpha(\mu_n, \mu_{(n+1)})\Delta_\xi(\mu_n, \mu_{(n+1)})$$

$$
\begin{aligned}
& +\alpha(\mu_{(n+1)}, g\mu)\Delta_\xi(\mu_{(n+1)}, g\mu)] \\
\leq\ & \frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n) + \alpha(\mu_n,\mu_{(n+1)})\Delta_\xi(\mu_n,\mu_{(n+1)}) \\
& +\alpha(\mu_{(n+1)}, g\mu)\Delta_\xi(g\mu_n, g\mu)], \\
\leq\ & \frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n) + \alpha(\mu_n,\mu_{(n+1)})\Delta_\xi(\mu_n,\mu_{(n+1)}) \\
& +\alpha(\mu_{(n+1)}, g\mu)\lambda(\Delta_\xi(\mu_n, g\mu_n) + \Delta_\xi(\mu, g\mu))] \\
\leq\ & \frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n)] + \frac{\xi}{3}[\alpha(\mu_n,\mu_{(n+1)})\Delta_\xi(\mu_n,\mu_{(n+1)})] \\
& \quad +\frac{\xi}{3}\alpha(\mu_{(n+1)}, g\mu)\lambda\Delta_\xi(\mu_n, g\mu_n) \\
& +\frac{\xi}{3}\lambda\alpha(\mu_{(n+1)}, g\mu)\Delta_\xi(\mu, g\mu),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_\xi(\mu, g\mu) - \frac{\xi}{3}\lambda\alpha(\mu_{(n+1)}, g\mu)\Delta_\xi(\mu, g\mu) \leq\ & \frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n)] \\
& +\frac{\xi}{3}[\alpha(\mu_n,\mu_{(n+1)})\Delta_\xi(\mu_n,\mu_{(n+1)})] \\
& +\frac{\xi}{3}\alpha(\mu_{(n+1)}, g\mu)\lambda\Delta_\xi(\mu_n, g\mu_n), \\
\Delta_\xi(\mu, g\mu)(1-\frac{\xi}{3}\lambda\alpha(\mu_{(n+1)}, g\mu)) \leq\ & \frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n)] + \frac{\xi}{3}[\alpha(\mu_n,\mu_{(n+1)})\Delta_\xi(\mu_n,\mu_{(n+1)})] \\
& +\frac{\xi}{3}\alpha(\mu_{(n+1)}, g\mu)\lambda\Delta_\xi(\mu_n, g\mu_n), \\
\Delta_\xi(\mu, g\mu) \leq\ & \frac{(\frac{\xi}{3}[\alpha(\mu,\mu_n)\Delta_\xi(\mu,\mu_n)])}{(1-\frac{\xi}{3}\lambda\alpha(\mu_{(n+1)}, g\mu))}E \qquad (1)\\
& +\frac{(\frac{\xi}{3}[\alpha(\mu_n,\mu_{(n+1)}) + \lambda\alpha(\mu_{(n+1)}, g\mu)])}{(1-\frac{\xi}{3}\lambda\alpha(\mu_{(n+1)}, g\mu))}
\end{aligned}
$$

for each $\mu \in \aleph$, $\lim_{n\longrightarrow+\infty}\Delta_\xi(\mu_n,\mu_{(n+1)}) \leq 1$, $\lim_{n\longrightarrow+\infty}\Delta_\xi(\mu_n,\mu)$,and $\lim_{n\longrightarrow+\infty}\Delta_\xi(\mu,\mu_n)$ exist and finite. Therefore, by taking $\lim_{n\longrightarrow+\infty}$ in (E) and using (B) and (C), we obtain

$$\Delta_\xi(\mu, g\mu) = 0. \tag{F}$$

This shows that

$$\mu = g\mu.$$

For uniqueness, consider $\mu \neq \kappa$ with κ be another fixed point of g then from (A), we obtain

$$
\begin{aligned}
\Delta_\xi(\mu,\kappa) &= \Delta_\xi(g\mu, g\kappa), \\
&\leq \lambda[\Delta_\xi(\mu, g\mu) + \Delta_\xi(\kappa, g\kappa)] \\
&\leq \lambda[\Delta_\xi(\mu,\mu) + \Delta_\xi(\kappa,\kappa)]
\end{aligned}
$$

We get, $\Delta_\xi(\mu,\kappa) = 0$, where $\Delta_\xi(\mu,\mu) = 0$ and $\Delta_\xi(\kappa,\kappa) = 0$. Hence $\Delta_\xi(\mu,\kappa) = 0$, this implies that $\mu = \kappa$. Hence μ is a unique fixed point of g.

3. Application fractional calculus

In this section, we use Corollary 1 to find the existence and uniqueness of a solution of nonlinear fractional differential equations given by

$$D_c^{\alpha}\mu(l) = f(l, \mu(l)) \qquad (l \in (0,1), \alpha \in (1,2]),$$

with boundary conditions $\mu(0) = 0, \mu'(0) = I\mu(l) l \in (0,1)$,where D_c^{α} means Caputo fractional derivative of order α, defined by

$$D_c^{\alpha} f(l) = \frac{1}{(\Gamma(n-\alpha))}\int_0^l (l-\bar{\omega})^{(n-\alpha-1)} f^n(\bar{\omega})\Delta\overline{\omega} \qquad (n-1 < \alpha < n, \quad n = [\alpha]+1)$$

and $f : [0,1] \times R \longrightarrow R^+$ is a continuous function. We assume $\aleph = C([0,1], \mathbb{R})$ from$[0,1]$ into $\mathbb{R}$ with supremum $|\mu| = \sup_{l\in[0,1]} |\mu(l)|$.The Riemann-Liouville fractional integral of order α is given by

$$I^{\alpha} f(l) = \frac{1}{\Gamma(\alpha)}\int_0^l (l-\overline{\omega})^{(\alpha-1)} f(\overline{\omega})\Delta\overline{\omega}. \qquad (\alpha > 0)$$

Initially, we give reasonable form of a nonlinear fractional differential equation and then inquest the existence of a solution. Now, we assume the fractional differential equation given by

$$D_c^{\alpha}\mu(l) = f(l, \mu(l)) \qquad (l \in (0,1), \quad \alpha \in (1,2]), \tag{3.1}$$

with the boundary conditions

$$\mu(0) = 0, \mu'(0) = I\mu(l)(l \in (0,1)),$$

where

i $f : [0,1] \times R \longrightarrow R^+$ is a continuous function,

ii $\mu(l) : [0,1] \longrightarrow R$ is continuous,

meet the below conditions

$$|f(l,\mu) - f(l,\kappa)| \leq L|\mu - \kappa|,$$

for all $l \in [0,1]$, L is a constant with $L\Pi < 1$, where

$$\Pi = \frac{1}{\Gamma(\alpha+1)} + \frac{2\kappa^{(\alpha+1)}\Gamma(\alpha)}{(2-\kappa^2)\Gamma(\alpha+1)}.$$

Then the equation (3.1) has a unique solution.

Proof. We define a CRMMS by

$$\Delta(\mu,\kappa,\xi)=\frac{|\mu(l)-\kappa(l)|}{(\xi+|\mu(l)-\kappa(l)|)}$$

for all $\mu,\kappa\in\aleph$, we consider $|\mu-\kappa|=\sup\limits_{(l\in[0,1])}|\mu(l)-\kappa(l)|$. We define a mapping $\psi:\aleph\to\aleph$ by

$$\mu(l)=\frac{1}{\Gamma(\alpha)}\int_0^l(l-\overline{\omega})^{(\alpha-1)}f(\overline{\omega},\mu(\overline{\omega}))\Delta\overline{\omega}+\frac{2l}{(2-\kappa^2)\Gamma(\alpha)}\int_0^\kappa\left(\int_0^{\overline{\omega}}(\overline{\omega}-m)^{(\alpha-1)}f(m,\mu(m))\Delta m\right)\Delta\overline{\omega} \tag{3.2}$$

for all $l\in[0,1]$. An equation (3.1) has a solution, for a function $\mu\in\aleph$ iff $\mu(l)=\psi\mu(l)$ for all $l\in[0,1]$. For all $l\in[0,1]$, we have

$$\Delta(\psi\mu,\psi\kappa,\xi)=\frac{|\psi\mu(l)-\psi\kappa(l)|}{(\xi+|\psi\mu(l)-\psi\kappa(l)|)}. \tag{3.3}$$

Now,

$$\begin{aligned}
|\psi\mu(l)-\psi\kappa(l)| &= \frac{1}{\Gamma(\alpha)}\int_0^l(l-\overline{\omega})^{(\alpha-1)}f(\overline{\omega},\mu(\overline{\omega}))\Delta\overline{\omega}\\
&\quad+\frac{2l}{(2-\kappa^2)\Gamma(\alpha)}\int_0^\kappa\left(\int_0^{\overline{\omega}}(\overline{\omega}-m)^{(\alpha-1)}f(m,\mu(m))\Delta m\right)\Delta\overline{\omega}\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^l(l-\overline{\omega})^{(\alpha-1)}f(\overline{\omega},\kappa(\overline{\omega}))\Delta\overline{\omega}\\
&\quad+\frac{2l}{(2-\kappa^2)\Gamma(\alpha)}\int_0^\kappa\left(\int_0^{\overline{\omega}}(\overline{\omega}-m)^{(\alpha-1)}f(m,\kappa(m))\Delta m\right)\Delta\overline{\omega}\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^l(l-\overline{\omega})^{(\alpha-1)}\left|f(\overline{\omega},\mu(\overline{\omega}))-f(\overline{\omega},\kappa(\overline{\omega}))\right|\Delta\overline{\omega}\\
&\quad+\frac{2l}{(2-\kappa^2)\Gamma(\alpha)}\int_0^\kappa\left(\int_0^{\overline{\omega}}(\overline{\omega}-m)^{(\alpha-1)}\left|f(\overline{\omega},\mu(m))-f(\overline{\omega},\kappa(m))\right|\Delta m\right)\\
&\le \frac{L|\mu-\kappa|}{\Gamma(\alpha)}\int_0^l(l-\overline{\omega})^{(\alpha-1)}\Delta\overline{\omega}+\frac{2L|\mu-\kappa|}{\Gamma(\alpha)}\int_0^\kappa\left(\int_0^{\overline{\omega}}(\overline{\omega}-m)^{(\alpha-1)}\Delta m\right)\Delta\overline{\omega}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{L\,|\mu-\kappa|}{\Gamma(\alpha+1)} + \frac{2\kappa^{\alpha+1}L\,|\mu-\kappa|\,\Gamma(\alpha)}{(2-\kappa^2)\,\Gamma(\alpha+2)} \\
&\le L\,|\mu-\kappa|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{2\kappa^{\alpha+1}\Gamma(\alpha)}{(2-\kappa^2)\,\Gamma(\alpha+2)}\right) \\
&= L\Pi|\mu-\kappa|.
\end{aligned}
$$

From the fact $L\Pi < 1$, $\frac{3}{4} \le \lambda < 1$, and (3.3), we get

$$
\begin{aligned}
\Delta(\psi\mu, \psi\kappa, \xi) &= \frac{|\psi\mu(l) - \psi\kappa(l)|}{(\xi + |\psi\mu(l) - \psi\kappa(l)|)} \\
&\le \frac{L\Pi|\mu-\kappa|}{(\xi + L\Pi|\mu-\kappa|)} \\
&\le \frac{\lambda|\mu-\kappa|}{(\xi + |\mu-\kappa|)} \\
&= \lambda\Delta(\mu, \kappa, \xi).
\end{aligned}
$$

Observe that all conditions of Corollary 1 are fulfilled. This implies that $\mu(l)$ is a unique fixed point of ψ.

Open Problem 3.1 Introduce a new notion to combine the structure of fuzzy sets and CRMMS and then prove Theorem (2) and Theorem (3) in the context of fuzzy CRMMS.

4. Conclusion

In this manuscript, we established the notions of rectangular modular metric space and controlled rectangular modular metric space and proved several fixed point results. Also, we provide some non-trivial examples with some graphical views and an application to fractional calculus. These results are in more generalized form in the existing literature. This work can easily be extended in the combined structure of fuzzy sets and controlled rectangular modular metric space.

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