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# The Double ARA-Sawi Transform

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**Abstract.** This study introduces a novel integral transform derived by integrating the ARA and Sawi transforms. The paper explores the foundational properties and establishes the existence of this new transform. It presents advanced results for partial differential equations in higher dimensions and extends the double convolution theorem to two dimensions. These developments are applied to solve specific types of differential equations, demonstrating practical applications in physics and related scientific fields.

**2020 Mathematics Subject Classifications**: 44A05

**Key Words and Phrases**: The ARA transform, The Sawi transform, The double integral transform, The ARA-Sawi transform.

## 1. Introduction

Implementing Integral transform which is one of the powerful mathematical techniques, which transforms a function to another domain.

By applying the inverse of the integral transform, we return to the original space after transforming the function.

Such transforms play a crucial role in engineering (dealing with signals), economics (input-output relationships), physics (quantum mechanics), and chemistry (reaction kinetics), where they are valuable in elucidating complex real systems. As a result, mathematicians are constantly coming up with new techniques to solve an ever-growing class of differential equations.

Among the innovative integral transforms emerging in recent years are the ARA and Sawi transforms. The ARA transform, introduced in 2020 by [9], and the Sawi transform,

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introduced in 2021 by [7], have gained attention for their unique properties and applications in various fields.

Additionally, there exist several double transforms designed to handle multi-variable differential equations. In the broad spectrum of double transforms, we encounter new methods to help solve differential equations in higher dimensions. Examples of such double transforms include the Double Laplace transform [2], the Double Laplace ARA Transform [10], Double Laplace-Sawi Transform [3], the Double Laplace-Shehu transform [5], the Double Sawi transform [6], and the Double Mellin-ARA Transform [1].

In the present work, we propose a new double transform called the Double ARA-Sawi Transform (DA-SWT) aimed at globalizing differential equation analysis. We explore its fundamental properties, characterize the necessary conditions for its existence, and demonstrate its power in convolution theory and derivative operations. By applying this novel transform method, we present new strategies for dealing with partial differential equations and integral equations. The innovation in this work lies in the combination of the ARA and Sawi transforms, creating a new approach that combines the strengths of both. This combination enhances the simplicity and applicability of addressing complex mathematical problems.

## 2. The ARA and Sawi transforms

In this section, we provide an overview and highlight key properties of the single transforms, namely the ARA and Sawi transforms.

### 2.1. The ARA transform

**Definition 1.** *The ARA transform of order $k$ of a continuous function $s(\nu)$ on the interval $(0,\infty)$ is expressed as follows:*

$$A_k(s(\nu))(\rho) = S(k,\rho) = \rho\int_0^\infty \nu^{k-1}e^{-\rho\nu}s(\nu)d\nu,\ \ \rho>0,\ \ for\ k=1,2,3,....$$

*In particular, if $k=1$, the ARA transform of order 1 is expressed as*

$$A_1(s(\nu))(\rho) = S(\rho) = \rho\int_0^\infty e^{-\rho\nu}s(\nu)\ d\nu,\ \ \rho>0.$$

In the rest of the study, we denote $A_1(s(\nu))(\rho)$ by $A(s(\nu))(\rho)$.

Some basic properties of the ARA transform are now given.

Let $S(\rho) = A(s(\nu))$, then for nonzero constants $\gamma$ and $\delta$, we have

$$A(\gamma s_1(\nu) + \delta s_2(\nu)) = \gamma A(s_1(\nu)) + \delta A(s_2(\nu)), \tag{1}$$

where $s_1(\nu)$ and $s_2(\nu)$ are continuous functions on $(0,\infty)$.

$$A(\nu^{\gamma}) = \frac{\Gamma(\gamma+1)}{\rho^{\gamma}}, \tag{2}$$

$$A(e^{\gamma\nu}) = \frac{\rho}{\rho-\gamma}, \ \gamma \in \mathbb{R}, \tag{3}$$

$$A(s'(\nu)) = \rho S(\rho) - \rho s(0), \tag{4}$$

$$A(s''(\nu)) = \rho^2 S(\rho) - \rho^2 s(0) - \rho s'(0). \tag{5}$$

## 2.2. The Sawi transform

**Definition 2.** *The Sawi transform of a continuous function* $r(\sigma)$ *on* $(0,\infty)$ *expressed as follows*

$$R(\varpi) = W(r(\sigma)) = \frac{1}{\varpi^2}\int_0^{\infty} e^{-\frac{\sigma}{\varpi}} r(\sigma) d\sigma, \ \varpi > 0.$$

Let us now explore the core properties that define the Sawi transform.

Suppose that $R_1(\varpi) = W(r_1(\sigma))$ and $R_2(\varpi) = W(r_2(\sigma))$,with $\gamma$ and $\delta$ as nonzero real numbers, the following properties hold

$$W(\gamma r_1(\sigma) + \delta r_2(\sigma)) = \gamma W(r_1(\sigma)) + \delta W(r_2(\sigma)), \tag{6}$$

$$W(\sigma^{\gamma}) = \Gamma(\gamma+1)\varpi^{\gamma-1}, \tag{7}$$

$$W(e^{\delta\sigma}) = \frac{1}{\varpi(1-\delta\varpi)}, \tag{8}$$

$$W(r'(\sigma)) = \frac{1}{\varpi}R(\varpi) - \frac{1}{\varpi^2}r(0), \tag{9}$$

$$W(r''(\sigma)) = \frac{1}{\varpi^2}R(\varpi) - \frac{1}{\varpi^3}r(0) - \frac{1}{\varpi^2}r'(0). \tag{10}$$

# 3. Double ARA-Sawi transform

This section announces the Double ARA-Sawi Transformation (DA-SWT). We start by stating the basic properties of the DA-SWT, such as linearity. Then we state a new result regarding the partial derivatives and another new result regarding the convolution theorem. We also state how we use these results to compute the DA-SWT of some basic functions. The definition of the DA-SWT is:

$$G(\lambda,\varpi) = A_\rho W_\sigma(g(\rho,\sigma)) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}} g(\rho,\sigma)\ d\rho d\sigma, \tag{11}$$

where $g(\rho,\sigma)$ is a continuous function on $(0,\infty)\times(0,\infty)$.

Clearly, $A_\rho W_\sigma(g(\rho,\sigma))$ is linear transformation. In fact, for nonzero constants $\gamma$ and $\delta$, we have

$$\begin{aligned}
&A_\rho W_\sigma(\gamma g_1(\rho,\sigma)+\delta g_2(\rho,\sigma))\\
&= \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}(\gamma g_1(\rho,\sigma)+\delta g_2(\rho,\sigma))\ d\rho d\sigma\\
&= \gamma\frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}} g_1(\rho,\sigma)\ d\rho d\sigma + \delta\frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}} g_2(\rho,\sigma)\ d\rho d\sigma\\
&= \gamma A_\rho W_\sigma(g_1(\rho,\sigma)) + \delta A_\rho W_\sigma(g_2(\rho,\sigma)).
\end{aligned}$$

If $g(\rho,\sigma)$ can be written as $g(\rho,\sigma) = s(\rho)r(\sigma)$ for some continuous functions $s$ and $r$, then $A_\rho W_\sigma(g(\rho,\sigma)) = A(s(\rho))W(r(\sigma))$. In fact

$$\begin{aligned}
A_\rho W_\sigma(g(\rho,\sigma)) &= A_\rho W_\sigma(s(\rho)r(\sigma))\\
&= \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}} s(\rho)r(\sigma)d\rho d\sigma\\
&= \left(\lambda\int_0^\infty e^{-\lambda\rho}s(\rho)d\rho\right)\left(\frac{1}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}r(\sigma)d\sigma\right)\\
&= A(s(\rho))W(r(\sigma)).
\end{aligned}$$

### 3.1. Existence condition for Double ARA-Sawi transform

**Definition 3.** *A function $g(\rho,\sigma)$ is said to be of exponential orders $\gamma$ and $\delta$ on $0 \le \rho < \infty$ and $0 \le \sigma < \infty$ if there exist $K, X, Y > 0$ such that $|g(\rho,\sigma)| \le Ke^{\gamma\rho+\delta\sigma}$, for all $\rho > X$, $\sigma > Y$.*

**Theorem 1.** *Let $g(\rho,\sigma)$ be a continuous function on the region $(0,\infty)\times(0,\infty)$ of exponential orders $\gamma$ and $\delta$. Then $G(\lambda,\varpi)$ exists for $\lambda,\varpi$ and $\gamma$ whenever $\mathrm{Re}(\lambda) > \gamma$ and $\mathrm{Re}\left(\frac{1}{\varpi}\right) > \delta$.*

*Proof.* We have

$$\begin{aligned}
|G(\lambda,\varpi)| &= \left|\frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho,\sigma)\ d\rho d\sigma\right| \leq \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\left|g(\rho,\sigma)\right| d\rho d\sigma \\
&\leq K\frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}e^{\gamma\rho+\delta\sigma}d\rho d\sigma = K\int_0^\infty\int_0^\infty \left(\lambda e^{-(\lambda-\gamma)\rho}\right)\left(\frac{1}{\varpi^2}e^{-(\frac{1}{\varpi}-\delta)\sigma}\right)d\rho d\sigma \\
&= K\left(\lambda\int_0^\infty e^{-(\lambda-\gamma)\rho}d\rho\right)\left(\frac{1}{\varpi^2}\int_0^\infty e^{-(\frac{1}{\varpi}-\delta)\sigma}d\sigma\right) \\
&= \frac{K\lambda}{\varpi\left(\lambda-\gamma\right)\left(1-\delta\varpi\right)},
\end{aligned}$$

where $\operatorname{Re}(\lambda) > \gamma$ and $\operatorname{Re}\left(\frac{1}{\varpi}\right) > \delta$.

Double ARA-Sawi transform for some basic functions

(i)

$$A_\rho W_\sigma(1) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}d\rho d\sigma$$

$$= \left(\lambda\int_0^\infty e^{-\lambda\rho}d\rho\right)\left(\frac{1}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}d\sigma\right) = 1\times\frac{1}{\varpi} = \frac{1}{\varpi},\ \operatorname{Re}(\lambda) > 0.$$

(ii)

$$A_\rho W_\sigma(e^{\gamma\rho+\delta\sigma}) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}e^{\gamma\rho+\delta\sigma}d\rho d\sigma$$

$$= \left(\lambda\int_0^\infty e^{\gamma\rho-\lambda\rho}d\rho\right)\left(\frac{1}{\varpi^2}\int_0^\infty e^{\delta\sigma-\frac{\sigma}{\varpi}}d\sigma\right) = \frac{\lambda}{\lambda-\gamma}\times\frac{1}{\varpi\left(1-\delta\varpi\right)}$$

$$= \frac{\lambda}{\varpi\left(\lambda-\gamma\right)\left(1-\delta\varpi\right)},\ \operatorname{Re}(\lambda) > \operatorname{Re}(\gamma).$$

(iii)

$$A_\rho W_\sigma(\rho^\gamma\sigma^\delta) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\rho^\gamma\sigma^\delta d\rho d\sigma$$

$$= \left(\lambda\int_0^\infty \rho^\gamma e^{-\lambda\rho}d\rho\right)\left(\frac{1}{\varpi^2}\int_0^\infty \sigma^\delta e^{-\frac{\sigma}{\varpi}}d\sigma\right)$$

$$= \frac{\Gamma(\gamma+1)}{\lambda^{\gamma}} \times \Gamma(\delta+1)\varpi^{\delta-1}$$

$$= \frac{\varpi^{\delta-1}}{\lambda^{\gamma}}\Gamma(\gamma+1)\Gamma(\delta+1),\ \mathrm{Re}(\lambda) > 0 \text{ and } \mathrm{Re}(\gamma) > -1.$$

## 3.2. Derivatives properties

Now, we present some basic properties of the DA-SWT

Let $G(\lambda, \varpi) = A_\rho W_\sigma(g(\rho, \sigma))$ where $g(\rho, \sigma)$ is a continuous function on $(0, \infty) \times (0, \infty)$. Then

(i)

$$A_\rho W_\sigma \left( \frac{\partial g(\rho,\sigma)}{\partial \rho} \right) = \lambda G(\lambda, \varpi) - \lambda W(g(0, \sigma)), \tag{12}$$

(ii)

$$A_\rho W_\sigma \left( \frac{\partial^2 g(\rho,\sigma)}{\partial \rho^2} \right) = \lambda^2 G(\lambda, \varpi) - \lambda^2 W(g(0, \sigma)) - \lambda W(g_\rho(0, \sigma)),$$

(iii)

$$A_\rho W_\sigma \left( \frac{\partial g(\rho,\sigma)}{\partial \sigma} \right) = \frac{1}{\varpi} G(\lambda, \varpi) - \frac{1}{\varpi^2} A(g(\rho, 0)), \tag{13}$$

(iv)

$$A_\rho W_\sigma \left( \frac{\partial^2 g(\rho,\sigma)}{\partial \sigma^2} \right) = \frac{1}{\varpi^2} G(\lambda, \varpi) - \frac{1}{\varpi^3} A(g(\rho, 0)) - \frac{1}{\varpi^2} A(g_\sigma(\rho, 0)), \tag{14}$$

(v)

$$A_\rho W_\sigma \left( \frac{\partial^2 g(\rho,\sigma)}{\partial \rho \partial \sigma} \right) = \frac{\lambda}{\varpi} G(\lambda, \varpi) - \frac{\lambda}{\varpi^2} A(g(\rho, 0)) - \frac{\lambda}{\varpi} W(g(0, \sigma)) + \frac{\lambda}{\varpi^2} g(0, 0). \tag{15}$$

*Proof.* (1) $A_\rho W_\sigma \left( \frac{\partial g(\rho,\sigma)}{\partial \rho} \right) = \frac{\lambda}{\varpi^2} \int\limits_0^\infty \int\limits_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} \frac{\partial g(\rho,\sigma)}{\partial \rho}\ d\rho d\sigma = \frac{\lambda}{\varpi^2} \int\limits_0^\infty e^{-\frac{\sigma}{\varpi}} \int\limits_0^\infty e^{-\lambda\rho} \frac{\partial g(\rho,\sigma)}{\partial \rho} d\rho d\sigma.$ By integrating by parts, we get

$$A_\rho W_\sigma \left( \frac{\partial g(\rho,\sigma)}{\partial \rho} \right) = \frac{\lambda}{\varpi^2} \int\limits_0^\infty e^{-\frac{\sigma}{\varpi}} \left( -g(0, \sigma) + \lambda \int\limits_0^\infty e^{-\lambda\rho} g(\rho, \sigma)\ d\rho \right) d\sigma$$

$$= -\frac{\lambda}{\varpi^2} \int\limits_0^\infty e^{-\frac{\sigma}{\varpi}} g(0, \sigma) d\sigma + \frac{\lambda^2}{\varpi^2} \int\limits_0^\infty \int\limits_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} g(\rho, \sigma)\ d\rho d\sigma$$

$= \lambda G(\lambda, \varpi) - \lambda W(g(0,\sigma))$.

(2) $A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial \rho^2}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\frac{\partial^2 g(\rho,\sigma)}{\partial\rho^2}\, d\rho d\sigma = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\int_0^\infty e^{-\lambda\rho}\frac{\partial^2 g(\rho,\sigma)}{\partial\rho^2}d\rho d\sigma.$

By integrating by parts, we get

$A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial \rho^2}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\left(-g_\rho(0,\sigma) - \lambda g(0,\sigma) + \lambda^2\int_0^\infty e^{-\lambda\rho}g(\rho,\sigma)d\rho\right)d\sigma$

$= -\frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}g_\rho(0,\sigma)d\sigma - \frac{\lambda^2}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}g(0,\sigma)d\sigma + \frac{\lambda^3}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho,\sigma)d\rho d\sigma$

$= \lambda^2 G(\lambda,\varpi) - \lambda^2 W(g(0,\sigma)) - \lambda W(g_\rho(0,\sigma)).$

(3) $A_\rho W_\sigma\left(\frac{\partial g(\rho,\sigma)}{\partial \sigma}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\frac{\partial g(\rho,\sigma)}{\partial\sigma}\, d\rho d\sigma = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\frac{\partial g(\rho,\sigma)}{\partial\sigma}d\sigma d\rho.$

By integrating by parts, we get

$A_\rho W_\sigma\left(\frac{\partial g(\rho,\sigma)}{\partial \sigma}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}\left(-g(\rho,0) + \frac{1}{\varpi}\int_0^\infty e^{-\frac{\sigma}{\varpi}}g(\rho,\sigma)d\sigma\right)d\rho$

$= -\frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}g(\rho,0)d\rho + \frac{\lambda}{\varpi^3}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho,\sigma)\, d\sigma d\rho$

$= \frac{1}{\varpi}G(\lambda,\varpi) - \frac{1}{\varpi^2}A(g(\rho,0)).$

(4) $A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial \sigma^2}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\frac{\partial^2 g(\rho,\sigma)}{\partial\sigma^2}\, d\rho d\sigma = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\frac{\partial^2 g(\rho,\sigma)}{\partial\sigma^2}d\sigma d\rho.$

By integrating by parts, we get

$A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial \sigma^2}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}\left(-g_\sigma(\rho,0) - \frac{1}{\varpi}g(\rho,0) + \frac{1}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}g(\rho,\sigma)d\sigma\right)d\rho$

$= -\frac{\lambda}{\varpi^2}\int_0^\infty e^{-\lambda\rho}g_\sigma(\rho,0)d\rho - \frac{\lambda}{\varpi^3}\int_0^\infty e^{-\lambda\rho}g(\rho,0)d\rho + \frac{\lambda}{\varpi^4}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho,\sigma)d\sigma d\rho$

$= \frac{1}{\varpi^2}G(\lambda,\varpi) - \frac{1}{\varpi^3}A(g(\rho,0)) - \frac{1}{\varpi^2}A(g_\sigma(\rho,0)).$

(5) $A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial\rho\partial \sigma}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}\frac{\partial^2 g(\rho,\sigma)}{\partial\rho\partial\sigma}d\rho d\sigma = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\int_0^\infty e^{-\lambda\rho}\frac{\partial^2 g(\rho,\sigma)}{\partial\rho\partial\sigma}d\rho d\sigma$

By integrating by parts, we get

$A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial\rho\partial \sigma}\right) = \frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}\left(-g_\sigma(0,\sigma) + \lambda\int_0^\infty e^{-\lambda\rho}g_\sigma(\rho,\sigma)\, d\rho\right)d\sigma$

$= -\frac{\lambda}{\varpi^2}\int_0^\infty e^{-\frac{\sigma}{\varpi}}g_\sigma(0,\sigma)d\sigma + \frac{\lambda^2}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g_\sigma(\rho,\sigma)d\rho d\sigma$

$= -\lambda W(g_\sigma(0,\sigma)) + \lambda A_\rho W_\sigma\left(g_\sigma(\rho,\sigma)\right)$

Using Equations 9 and 13 we get

$A_\rho W_\sigma\left(\frac{\partial^2 g(\rho,\sigma)}{\partial\rho\partial\sigma}\right) = \frac{\lambda}{\varpi}G(\lambda,\varpi) - \frac{\lambda}{\varpi^2}A(g(\rho,0)) - \frac{\lambda}{\varpi}W(g(0,\sigma)) + \frac{\lambda}{\varpi^2}g(0,0).$

### 3.3. Convolution Theorem of Double ARA-Sawi transform

Let $H(\rho,\sigma)$ represent the Heaviside unit step function, which is defined as follows:

$$H(\rho-\gamma,\sigma-\delta) = \begin{cases} 1, & \rho>\gamma \text{ and } \sigma>\delta \\ 0, & \text{otherwise} \end{cases}$$

Then we have the following lemma

**Lemma 1.** *Let $g(\rho,\sigma)$ be a continuous function on $(0,\infty)\times(0,\infty)$ and $H(\rho,\sigma)$ be the Heaviside unit step function. Then $A_\rho W_\sigma(g(\rho-\gamma,\sigma-\delta)H(\rho-\gamma,\sigma-\delta)) = e^{-\lambda\gamma-\frac{\delta}{\varpi}}A_\rho W_\sigma(g(\rho,\sigma)$.*

*Proof.* We have

$$A_\rho W_\sigma(g(\rho-\gamma,\sigma-\delta)H(\rho-\gamma,\sigma-\delta)) \tag{16}$$

$$= \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho-\gamma,\sigma-\delta)H(\rho-\gamma,\sigma-\delta)d\rho d\sigma$$

$$= \frac{\lambda}{\varpi^2}\int_\gamma^\infty\int_\delta^\infty e^{-\lambda\rho-\frac{\sigma}{\varpi}}g(\rho-\gamma,\sigma-\delta)d\rho d\sigma.$$

Now, by making the substitution $z=\rho-\gamma$ and $w=\sigma-\delta$, equation 3.3 becomes:

$$\begin{aligned} A_\rho W_\sigma(g(\rho-\gamma,\sigma-\delta)H(\rho-\gamma,\sigma-\delta)) &= \frac{\lambda}{\varpi^2}\int_0^\infty\int_0^\infty e^{-\lambda(z+\gamma)-\frac{(w+\delta)}{\varpi}}g(z,w)dzdw \\ &= e^{-\lambda\gamma-\frac{\delta}{\varpi}}A_\rho W_\sigma(g(\rho,\sigma)). \end{aligned}$$

**Definition 4.** *Let $g(\rho,\sigma)$ and $k(\rho,\sigma)$ be continuous functions. We define the convolution in the DA-SWT as*

$$(g**k)(\rho,\sigma) = \int_0^\rho\int_0^\sigma g(\rho-\gamma,\sigma-\delta)k(\gamma,\delta)d\gamma d\delta.$$

In the following theorem, we compute DA-SWT of the convolution of two functions

**Theorem 2.** *Let* $G(\lambda, \varpi) = A_\rho W_\sigma(g(\rho, \sigma))$ *and* $K(\lambda, \varpi) = A_\rho W_\sigma(k(\rho, \sigma))$*. Then*

$$A_\rho W_\sigma((g * *k)(\rho, \sigma)) = \frac{\varpi^2}{\lambda} G(\lambda, \varpi) K(\lambda, \varpi).$$

*Proof.*

$A_\rho W_\sigma((g{*}{*}k)(\rho, \sigma))$

$$\begin{aligned} &= \frac{\lambda}{\varpi^2} \int_0^\infty \int_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} (g * *k)(\rho, \sigma) d\rho d\sigma \\ &= \frac{\lambda}{\varpi^2} \int_0^\infty \int_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} \left( \int_0^\rho \int_0^\sigma g(\rho - \gamma, \sigma - \delta) k(\gamma, \delta) d\gamma d\delta \right) d\rho d\sigma. \end{aligned} \tag{17}$$

Using the Heaviside unit step function, We can write equation 17 as

$A_\rho W_\sigma((g{*}{*}g)(\rho, \sigma))$

$$\begin{aligned} &= \frac{\lambda}{\varpi^2} \int_0^\infty \int_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} \left( \int_0^\infty \int_0^\infty g(\rho - \gamma, \sigma - \delta) H(\rho - \gamma, \sigma - \delta) k(\gamma, \delta)) d\gamma d\delta \right) d\rho d\sigma \\ &= \int_0^\infty \int_0^\infty k(\gamma, \delta) \left( \frac{\lambda}{\varpi^2} \int_0^\infty \int_0^\infty e^{-\lambda\rho - \frac{\sigma}{\varpi}} g(\rho - \gamma, \sigma - \delta) H(\rho - \gamma, \sigma - \delta) d\rho d\sigma \right) d\gamma d\delta. \end{aligned}$$

So by Lemma 1, We have

$$\begin{aligned} A_\rho W_\sigma((g * *k)(\rho, \sigma)) &= G(\lambda, \varpi) \int_0^\infty \int_0^\infty k(\gamma, \delta) e^{-\lambda\gamma - \frac{\delta}{\varpi}} d\gamma d\delta \\ &= \frac{\varpi^2}{\lambda} G(\lambda, \varpi) K(\lambda, \varpi). \end{aligned}$$

In Table 1, we have the DAHT of some basic functions.

Table 1: Table of DAHT

| $g(\rho,\sigma)$ | $A_\rho W_\sigma(g(\rho,\sigma))$ |
|---|---|
| 1 | $\frac{1}{\varpi}$, $\mathrm{Re}(\lambda)>0$ |
| $\rho^\gamma\sigma^\delta$ | $\frac{\varpi^{\delta-1}}{\lambda^\gamma}\Gamma(\gamma+1)\Gamma(\delta+1)$, $\mathrm{Re}(\lambda)>0$ and $\mathrm{Re}(\gamma)>-1$ |
| $e^{\gamma\rho+\delta\sigma}$ | $\frac{\lambda}{\varpi(\lambda-\gamma)(1-\delta\varpi)}$, $\mathrm{Re}(\lambda)>\mathrm{Re}(\gamma)$ |
| $e^{i(\gamma\rho+\delta\sigma)}$ | $\frac{i\lambda}{\varpi(\lambda-i\gamma)(i+\delta\varpi)}$, $\mathrm{Im}(\gamma)+\mathrm{Re}(\lambda)>0$ |
| $\sin(\gamma\rho+\delta\sigma)$ | $\frac{\lambda(\gamma+\lambda\varpi\delta)}{\varpi(\lambda^2+\gamma^2)(1+\delta^2\varpi^2)}$, $\lvert\mathrm{Im}(\gamma)\rvert<\mathrm{Re}(\lambda)$ |
| $\cos(\gamma\rho+\delta\sigma)$ | $\frac{\lambda(\lambda-\varpi\gamma\delta)}{\varpi(\lambda^2+\gamma^2)(1+\delta^2\varpi^2)}$, $\lvert\mathrm{Im}(\gamma)\rvert<\mathrm{Re}(\lambda)$ |
| $\sinh(\gamma\rho+\delta\sigma)$ | $\frac{\lambda(\gamma+\lambda\varpi\delta)}{\varpi(\lambda^2-\gamma^2)(1-\delta^2\varpi^2)}$, $\mathrm{Re}(\lambda)>\mathrm{Re}(\gamma)$ and $\mathrm{Re}(\lambda)+\mathrm{Re}(\gamma)>0$ |
| $\cosh(\gamma\rho+\delta\sigma)$ | $\frac{\lambda(\lambda+\varpi\gamma\delta)}{\varpi(\lambda^2-\gamma^2)(1-\delta^2\varpi^2)}$, $\mathrm{Re}(\lambda)>\mathrm{Re}(\gamma)$ and $\mathrm{Re}(\lambda)+\mathrm{Re}(\gamma)>0$ |
| $s(\rho)r(\sigma)$ | $A(s(\rho))W(r(\sigma))$ |
| $g(\rho-\gamma,\sigma-\delta)H(\rho-\gamma,\sigma-\delta)$ | $e^{-\lambda\gamma-\frac{\delta}{\varpi}}A_\rho W_\sigma(g(\rho,\sigma)$ |
| $(g**f)(\rho,\sigma)$ | $\frac{\varpi^2}{\lambda}A_\rho W_\sigma(g(\rho,\sigma))A_\rho W_\sigma(f(\rho,\sigma))$ |
| $J_0\left(c\sqrt{\rho\sigma}\right)$ | $\frac{4\lambda}{\varpi(4\lambda+c^2\varpi)}$, $\mathrm{Re}\left(\lambda+\frac{c^2\varpi}{4}\right)>0$ |

## 4. Applications

In this section, we use the DA-SWT for solving PDEs and Integro PDEs

**Example 1.** *Consider the heat equation*

$$g_\sigma - g_{\rho\rho} = 2g + 6\sigma - 3, \text{ where } \rho,\sigma>0,$$

*With ICs*
$g(\rho,0)=\sin\rho$,
*and BCs*
$g(0,\sigma)=-3\sigma$, $g_\rho(0,\sigma)=e^\sigma$.

**Solution 1.** *By applying the single ARA transform to the ICs and the single Sawi transform to the BCs, we get*
$A(g(\rho,0))=\frac{\lambda}{1+\lambda^2}$, $W(g(0,\sigma))=-3$, $W(g_\rho(0,\sigma))=\frac{1}{\varpi(1-\varpi)}$.
*Apply the DA-SWT to Equation 1, we get*

$$\begin{aligned}&\frac{1}{\varpi}G(\lambda,\varpi)-\frac{1}{\varpi^2}A(g(\rho,0))-\lambda^2G(\lambda,\varpi)\\&+\lambda^2W(g(0,\sigma))+\lambda W(g_\rho(0,\sigma))\\=\;&2G+6-\frac{3}{\varpi}.\end{aligned}$$

*So,*

$$\frac{1-\lambda^2\varpi-2\varpi}{\varpi}\times G(\lambda,\varpi) \;=$$

$$\frac{1}{\varpi^2} \times \frac{\lambda}{1+\lambda^2} - \lambda^2 \times -2 - \lambda \times \frac{1}{\varpi\,(1-\varpi)} + 6 - \frac{3}{\varpi}.$$

*By simplifying, we get,*

$$G(\lambda, \varpi) = \frac{\lambda}{\varpi\,(1+\lambda^2)\,(1-\varpi)} - 3.$$

*Therefore,*

$$g(\rho, \sigma) = A_\rho^{-1} W_\sigma^{-1} \left( \frac{\lambda}{\varpi\,(1+\lambda^2)\,(1-\varpi)} - 3 \right) = e^\sigma \sin\rho - 3\sigma.$$

*Its graph is*

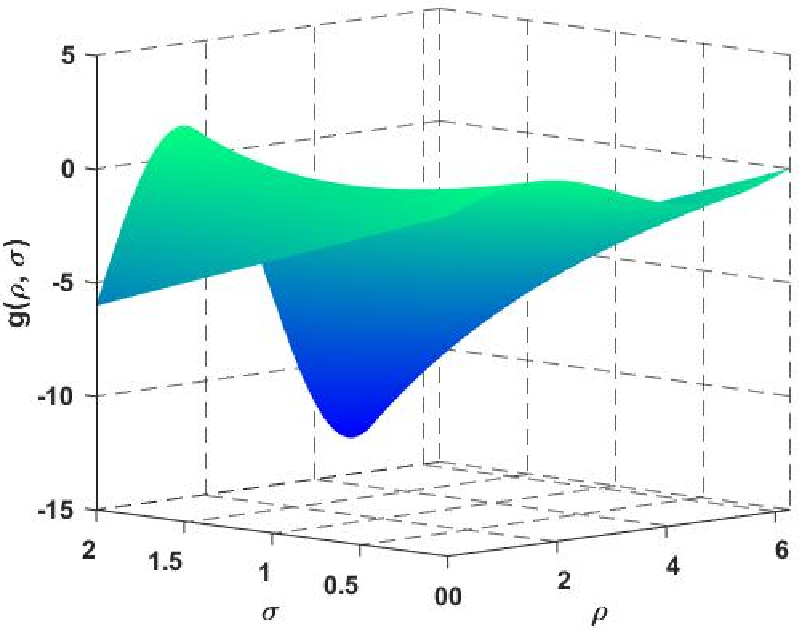


Figure 1: The solution of Example 1

**Example 2.** *Consider the telegraph equation*

$$g_{\rho\rho} + g_\rho - 2g_{\sigma\sigma} = 2g(\rho, \sigma), \text{ where } \rho, \sigma > 0,$$

*With ICs*
$g(\rho, 0) = e^\rho + 1,\ g_\sigma(\rho, 0) = 0,$
*and BCs*
$g\,(0, \sigma) = 1 + \cos\sigma,\ g_\rho\,(0, \sigma) = 1.$

**Solution 2.** *By applying the single ARA transform and the single Sawi transform to the ICs, we get*
$A\left(g(\rho,0)\right)=\frac{\lambda}{\lambda-1}+1,\ A\left(g_{\sigma}(\rho,0)\right)=0,\ W\left(g\left(0,\sigma\right)\right)=\frac{1}{\varpi}+\frac{1}{\varpi(1+\varpi^2)},\ W\left(g_{\rho}\left(0,\sigma\right)\right)=\frac{1}{\varpi}.$

*Apply the DA-SWT to Equation 2, we get*

$$\begin{aligned}\lambda^2G(\lambda,\varpi)-\lambda^2W(g(0,\sigma))-\lambda W(g_\rho(0,\sigma))+\lambda G(\lambda,\varpi)\\ -\lambda W(g(0,\sigma))-2\frac{1}{\varpi^2}G(\lambda,\varpi)+2\frac{1}{\varpi^3}A(g(\rho,0))\\ +2\frac{1}{\varpi^2}A(g_\sigma(\rho,0))=2G.\end{aligned}$$

*So,*

$$\begin{aligned}\frac{\lambda^2\varpi^2+\lambda\varpi^2-2\varpi^2-2}{\varpi^2}\times G(\lambda,\varpi) &= \\ &\lambda^2\times\left(\frac{1}{\varpi}+\frac{1}{\varpi\left(1+\varpi^2\right)}\right)+\lambda\times\frac{1}{\varpi}\\ &+\lambda\times\left(\frac{1}{\varpi}+\frac{1}{\varpi\left(1+\varpi^2\right)}\right)-\frac{2}{\varpi^3}\times\left(\frac{\lambda}{\lambda-1}+1\right).\end{aligned}$$

*By simplifying, we get,*

$$G(\lambda,\varpi)=\frac{\lambda}{\varpi\left(\lambda-1\right)}+\frac{1}{1+\varpi^2}.$$

*Therefore,*

$$g(\rho,\sigma)=A_\rho^{-1}W_\sigma^{-1}\left(\frac{\lambda}{\varpi\left(\lambda-1\right)}+\frac{1}{1+\varpi^2}\right)=e^\rho+\cos\sigma.$$

*Its graph is*

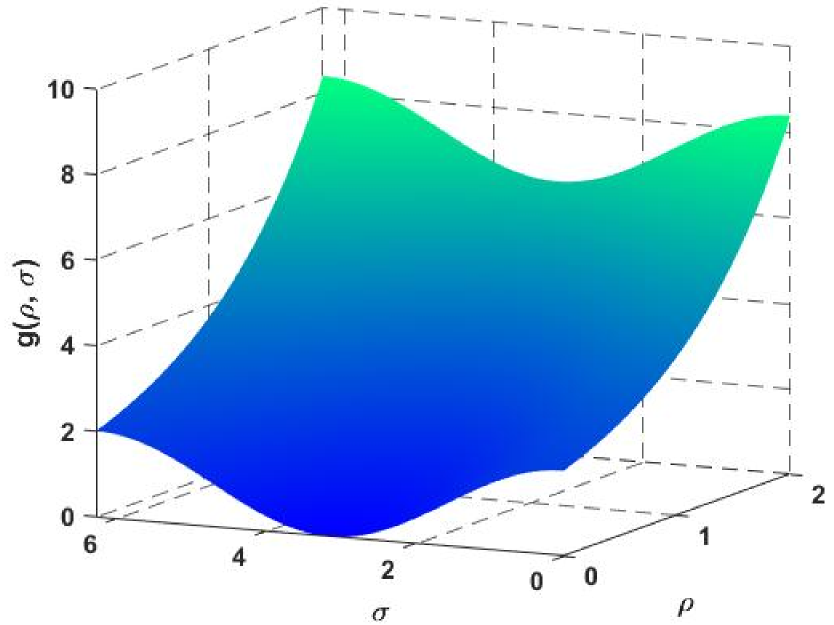


Figure 2: The solution of Example 2

**Example 3.** *Consider the equation of Volterra Integro PDE.*

$$g_\rho + g_\sigma - \cosh\sigma - \rho\sinh\sigma - \rho^2\sinh\sigma = 2\int_0^\rho\int_0^\sigma g(\gamma,\delta))d\gamma d\delta, \text{ where } \rho,\sigma > 0, \tag{18}$$

*With ICs*
$g(\rho,0) = \rho,\ g(0,\sigma) = 0.$

**Solution 3.** *By applying the single ARA transform and the single Sawi transform to the ICs, we get*
$A\left(g(\rho,0)\right) = \frac{1}{\lambda},\ W\left(g\left(0,\sigma\right)\right) = 0.$
*By Definition 4 and Theorem 2, we have*

$$\int_0^\rho\int_0^\sigma g(\gamma,\delta))d\gamma d\delta = \left(1 ** g\right)\left(\rho,\sigma\right). \tag{19}$$

*Apply the DA-SWT to Equation* 18*, we get*

$$\lambda G(\lambda,\varpi) - \lambda W(g(0,\sigma)) + \frac{1}{\varpi}G(\lambda,\varpi) - \frac{1}{\varpi^2}A(g(\rho,0))$$

$$-\frac{1}{\varpi\left(1-\varpi^2\right)}-\frac{1}{\lambda\left(1-\varpi^2\right)}-\frac{2}{\lambda^2\left(1-\varpi^2\right)} \quad = \quad \frac{2\varpi}{\lambda}G(\lambda,\varpi).$$

*So,*

$$\frac{\lambda^2\varpi+\lambda-2\varpi^2}{\lambda\varpi}\times G(\lambda,\varpi) \quad = \quad \frac{1}{\varpi^2}\times\frac{1}{\lambda}+\frac{1}{\varpi\left(1-\varpi^2\right)}+\frac{1}{\lambda\left(1-\varpi^2\right)}+\frac{2}{\lambda^2\left(1-\varpi^2\right)}.$$

*By simplifying, we get,*

$$G(\lambda,\varpi)=\frac{1}{\lambda\varpi\left(1-\varpi^2\right)}.$$

*Therefore,*

$$g(\rho,\sigma)=A_\rho^{-1}W_\sigma^{-1}\left(\frac{1}{\lambda\varpi\left(1-\varpi^2\right)}\right)=\rho\cosh\sigma.$$

*Its graph is*

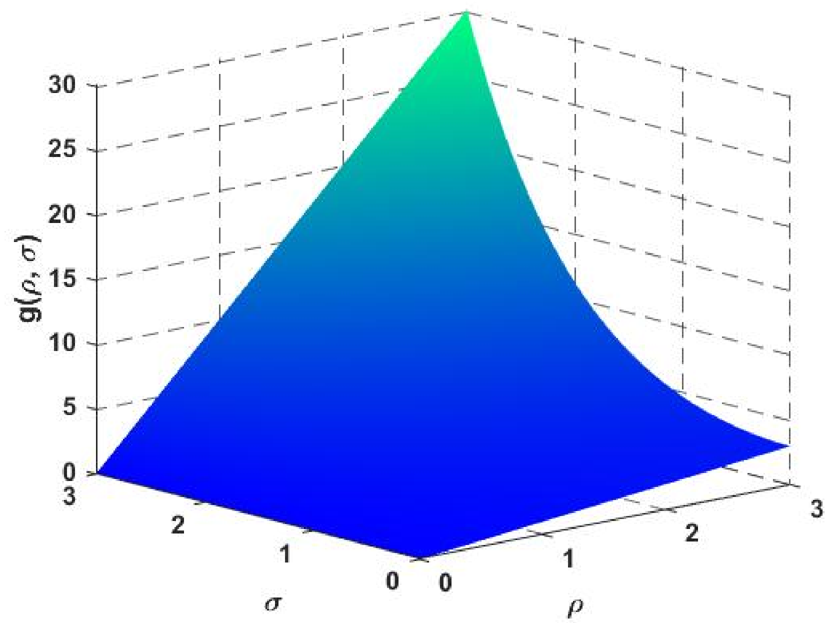


Figure 3: The solution of Example 3

## 5. Conclusion

In this paper, we introduced the Double ARA-Sawi Transform (DA-SWT) and thoroughly explored its foundational properties, rigorously characterizing the necessary conditions for its existence. Through this investigation, we demonstrated the transformative potential of these properties in convolution theory and derivative operations.

By establishing a robust theoretical framework and validating its applicability, we highlighted the practical advantages of the DA-SWT in problem-solving. Where relevant, we connected earlier numerical procedures that benefited from our previous research, showcasing how the DA-SWT builds upon and enhances existing methodologies.

We foresee significant potential for the DA-SWT in addressing fractional and conformable partial differential equations (PDEs) and integro-PDEs with variable coefficients. This innovative transform method paves the way for future advancements in solving complex mathematical and scientific problems. Additional results related to conformable PDEs and Integro PDEs are available in references [4, 8].

**Author contribution statement**

The authors listed have significantly contributed to the development and the writing of this article.

**Data availability statement**

No data was used for the research described in the article.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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