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A Novel Problem and Algorithm for Solving Cordial Labeling of Some Fifth Powers of Graphs

Atef Abd El-hay[1,*], Khalid A. Alsatami[2], Ashraf ELrokh[1]

[1] *Mathematics and Computer Science Department, Faculty of Science Menoufia University, Menoufia, Egypt.*
[2] *Department of Mathematics, College of Science, Qassim University, Buraydah, KSA.*

Abstract. In this paper we introduce a novel application of cordial labeling using the fifth power of graphs, demonstrating its potential for understanding and studying specific graph structures. The resulting cordial labeling scheme for the fifth power of paths, cycles, fans, wheels, lemniscate and the union of fifth power of paths and cycles graphs can provide insights into the properties and structures of these graphs. It can be used to analyze its connectivity, symmetry, and other graph-theoretical characteristics.

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1. Introduction

A graph labeling is a way of assigning numbers to the vertices or edges of a graph, or both, according to some rules. Graph labeling is useful for studying various properties and applications of graphs, such as symmetry, coloring, coding, and communication. It is universally recognizable that graph theory has applications in many other academic disciplines, particularly computer science [5, 21], and also in physics, chemistry, biology, communication, psychology, sociology, and economics.

Graph labelling is one area of graph theory that has seen a great deal of ongoing research. Labeled graphs are effective models for a variety of applications, including data base administration, X-ray crystallography, radar, circuit design, and coding theory [20]. When a specific type of graph is labelled, its vertices are given values from a predetermined set, its edges have a predetermined induced labelling, and its labelling must adhere to specified requirements. The study by Gallian [17] is a great resource on this topic graceful and harmonious labelling are two of the most significant categories. Rosa [22] and Golomb

*Corresponding author.
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Email addresses: atef_1992@yahoo.com (A. Abd El-hay),
satamy@qu.edu.sa (K. A. Alsatami), ashraf.hefnawy68@yahoo.com (A. ELrokh)

[18] independently created graceful labelling in 1966 and 1972, respectively, while Graham and Sloane [19] conducted the first study on harmonic labelling in 1980. Cordial labelling is a third crucial sort of labelling that Cahit [4] established in 1990 and which combines elements of the other two. Unlike graceful and harmonious labelling, which utilize labels $|f(v)-f(w)|$ and ($f(v)+f(w)$ (modulo the number of edges), respectively, cordial labelings simply use labels 0 and 1, as well as the induced label $(f(v)+f(w))(mod 2)$, which naturally equals $|f(v) - f(w)|$. The original concept of cordial graphs is due to Cahit[3]. He showed that each tree is cordial; a complete graph K_n is cordial if and only if $n \leq 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers n and m. The fifth power of graphs G^5 is the graph obtained from the graph G by adding edges that join all vertices u and v with $d(u, v) \leq 5$. More precisely, and with the fifth Power of Paths P_n^5, the fifth power of cycles C_n^5, the fifth power of fans $F_{n+1}^5 = P_1 + P_n^5$, the fifth power of wheels $W_{n+1}^5 = P_1 + C_n^5$, and The fifth power of a lemniscate graph is denoted by $L_{n,m}^5 = C_n^5 \cup C_m^5$ where both cycles C_n^5 and C_m^5 have a common vertex.

As stated in [17], every path P_n is Cordial for every n, a cycle C_n is Cordial if and only if $n \neq 2(mod 4)$, the complete graph K_n is Cordial if and only if $n \leq 3$. P_n^2 is Cordial for every n, P_n^3 is Cordial if and only if $n \neq 4$, P_n^4 is cordial if and only if $m \neq 4, 5$ or 6, all fans F_n are Cordial and the Wheel W_n is Cordial if and only if $n \neq 3(mod 4)$.

In [7, 8], Diab has reported several results concerning the sum and union of the cycles C_n and paths P_m together and with other specific graphs. He introduced the cordiality of the join and the union of pairs of wheels and graphs consisting of a wheel and a path or a cycle he investigated that the second power of cycles is cordial for all $n \geq 3$ if and only if $n = 3$ or even $n > 4$. Moreover, he studies the cordiality of certain combinations of second power of cycles, cycles and paths. Specifically, he discussed the cordiality of the join and union of pairs of second power of cycles and graphs consisting of one second power of cycle with one cycle and one path. In [16] the cordiality of the join and union of graphs involving one path and one cycle, as well as the third power of paths. Elrokh and Rabie [14] proved $P_n^4 + P_m^4$ and $P_n^4 \cup P_m^4$ are cordial for all $n, m \geq 7$, and $C_n^4 + C_m^4$, and $C_n^4 \cup C_m^4$ are cordial for all n, m except $(n, m) = (7, 7)$.

Cordial labelling has significant connections to computer science because arithmetic modulo 2 is a fundamental component of that discipline. Cordial labeling has applications in various fields, such as coding theory, communication networks, X-ray crystallography, and computational and communication paradigms. One interesting application is to produce new cordial families by using a technique called balanced cordial labeling, which can generate cordial graphs from two given graphs by taking one copy of one graph and j copies of another graph and joining them by an edge. For more details about the cordial labeling and types of labeling, the reader can refer to [1, 2, 9–13, 15].

We defined cordial graphs more accurately as follows. Let $G = (V, E)$ be a graph, let $f : V \to \{0, 1\}$ labeling of the vertices, and let $f^* : E \to \{0, 1\}$ be the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w)$ (mod 2). (Thus for any edges $e = vw, f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called cordial if it has a cordial labeling.

The main object of this paper is to extend the above results as follows. In Section 3, we study the cordiality of the fifth power of paths, cycles, fans, wheels, and lemniscate graphs. Section 4 investigates the cordiality for the union of fifth power of paths and cycles. Section 5, proposes an algorithm for determining the cordiality of a given graph. The last section, is the conclusion which summarize the important points of our finding in this paper.

2. Terminology and Notation

We let L_{4r} denote the labeling 0011...0011 (repeated r-times), let $L^{'}_{4r}$ denote the labeling 1100...1100 (repeated r times). The labeling 1001 1001...1001 (repeated r times) and 0110...0110 (repeated r times) are denoted by S_{4r} and S'_{4r}respectively. Let M_{2r} denote the labeling 0101...01, zero-one repeated $r-$times if r is even and 0101...010 if r is odd. Sometimes, we modify labeling by adding symbols at one end or the other (or both). In most cases, we then modify this by adding symbols at one end or the other (or both), thus $L_{4r}101$ denotes the labeling 0011 0011...0011 101 (repeated r-times) when $r{\geq}1$ and 101 when $r{=}\ 0$. Similarly, $1L^{'}_{4r}$ is the labeling 1 1100 1100...1100 (repeated r-times) when $r{\geq}1$ and 1 when $r{=}\ 0$. Similarly, $0{L^{'}}_{4r}1$ is the labeling 0 1100 1100...1100 1 when $r{\geq}1$ and 01 when $r{=}\ 0$. Additional notation that we use the following for a given labeling of the union $G \cup H$, we let v_G and v_H be the numbers of vertices G and H respectively. Also e_G and e_H be the numbers of edges G and H respectively. It is follows that $v_{G\cup H} = v_G + v_H$, and $e_{G\cup H} = e_G + e_H$, for more details see [6]. Moreover, we used the symbol $[L, M]$ for the union of two graphs G and H, where L is the labeling of G and M is the labeling of H.

3. Cordial Labeling of fifth power of Some Graphs

In this section we shall prove that the cordiality of the fifth power of paths, cycles, fan wheel and lemniscate graphs. More over, we introduce some illustrate examples for each graphs.

3.1. Cordial Labeling of fifth power of Cycles graphs

Obviously, the order of C_n^5 is n, and the size of C_n^5 is $5n - 14$ for every $n > 7$. In Particular, $C_3^5 \cong C_3, C_4^5 \cong K_4, C_5^5 \cong K_5, C_6^5 \cong K_6$ and $C_7^5 \cong K_7$. In this subsection, we show that the cordiality of C_n^5 if and only if $n = 3$, and $n > 8$.

Lemma 3.1.1. *The fifth power of Cycles graphs C_n^5 is cordial for all $n{>}\ 8$.*
Proof. Let $n{=}\ 4t + i$ ($0 \leq i \leq 3$ and $t \geq 2$), then for a given value of i with $0 \leq i \leq 3$, we use the labeling A_i or $A^{'}_i$ for C_n as shown in Table 1. It is easy to see that C_n^5 for every $n > 8$ is cordial from the last two columns of the Table 1 and thus the Lemma is proved.

Table 1: Labeling of C_n^5.

$n = 4t + i$, $i = 0, 1, 2, 3$	Labeling of C_n^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$A_0 = 0L_{4t}1_20$	$2t + 2$	$2t + 2$	$10t + 3$	$10t + 3$	0	0
$i = 1$	$A_1 = L_{4t}1$	$2t$	$2t + 1$	$10t - 5$	$10t - 4$	-1	-1
	$A_1' = 1_2M_{4t-4}0_3$	$2t + 1$	$2t$	$10t - 4$	$10t - 5$	1	1
$i = 2$	$A_2 = L_{4t}'10$	$2t + 1$	$2t + 1$	$10t - 2$	$10t - 2$	0	0
$i = 3$	$A_3 = L_{4t}100$	$2t + 2$	$2t + 1$	$10t$	$10t + 1$	1	-1
	$A_3' = 11L_{4t}'0$	$2t + 1$	$2t + 2$	$10t + 1$	$10t$	-1	1

Example 3.1.1. The graph C_3^5 is cordial, but the graphs C_n^5 are not cordial for all $4 \leq n \leq 8$.
Solution. Since $C_3^5 \cong C_3$ and C_3 is cordial [3], then C_3^5 is cordial. In case of $4 \leq n \leq 7$, it is easy to verify that $C_4^5 \cong K_4, C_5^5 \cong K_5, C_6^5 \cong K_6$ and $C_7^5 \cong K_7$ are not cordial by cahit [9]. In case of $n = 8$, since $n = 8$, then by investigation all possible labelings of vertices of C_8^5 with $v_0 = v_1 = 4$, we obtained that $|e_0 - e_1| > 1$. Therefore C_8^5 is not cordial.

Theorem 3.1.1. *The fifth power of Cycles graphs C_n^5 is cordial if and only if $n = 3$ and $n > 8$.*
Proof. The proof follows directly from Lemma 3.1.1 and Example 3.1.1.

3.2. Cordial Labeling of fifth power of Paths graphs

Clearly, the order of P_n^5 is n, and the size of P_n^5 is $5n - 15$ for every $n > 7$. In Particular, $P_1^5 \cong P_1, P_2^5 \cong P_2, P_3^5 \cong C_3, P_4^5 \cong K_4, P_5^5 \cong K_5$ and $P_6^5 \cong K_6$. In this subsection, we show that P_n^5 is cordial if and only if $1 \leq n \leq 3$ and $n > 7$.

Lemma 3.2.1. *The fifth power of Paths graphs P_n^5 is cordial for all $n > 8$.*
Proof. Let $n = 4t + i$ ($0 \leq i \leq 3$ and $t \geq 2$), then for a given value of i with $0 \leq i \leq 3$, we use the labeling B_i or B_i' for P_n as shown in Table 2. It is easy to see that P_n^5 for every $n > 8$ is cordial from the last two columns of the Table 2 and thus the Lemma is proved.

Table 2: Labeling of P_n^5.

$n = 4t + i$, $n > 8$, $i = 0, 1, 2, 3$	Labeling of P_n^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$B_0 = 0L_{4t}1_20$	$2t$	$2t$	$10t + 2$	$10t + 3$	0	1
$i = 1$	$B_1 = L_{4t}1$	$2t$	$2t + 1$	$10t - 5$	$10t - 5$	-1	0
$i = 2$	$B_2 = L'_{4t}10$	$2t + 1$	$2t + 1$	$10t - 2$	$10t - 3$	0	1
$i = 3$	$B_3 = L'_{4t}010$	$2t + 2$	$2t + 1$	$10t$	$10t$	1	0
	$B_3' = 010L_{4t}$	$2t + 2$	$2t + 1$	$10t$	$10t$	1	0

Example 3.2.1. The graphs P_1^5 , P_2^5, P_3^5 and P_8^5 are cordial, but the graphs P_n^5 are not cordial for all $4 \le n \le 7$.
solution. Since $P_1^5 \cong P_1, P_2^5 \cong P_2$, $P_3^5 \cong C_3$, P_8^5 the labeling $[0_4 1_4]$ is sufficient for P_8^5 and P_1, P_2 and C_3 are cordial [3], then P_1^5 , P_2^5 and P_3^5 are cordial. In case of $4 \le n \le 7$, it is easy to verify that $P_4^5 \cong K_4, P_5^5 \cong K_5, P_6^5 \cong K_6$ are not cordial by cahit [9]. Since $n = 7$, then by investigation all possible labelings of vertices of P_7^5 with $v_0 = 3, v_1 = 4$ or $v_0 = 4, v_1 = 3$, we obtained that $|e_0 - e_1| > 1$. Therefore P_8^5 is not cordial

Theorem 3.2.1. *The fifth power of Paths graphs P_n^5 are cordial if and only if* $1 \le n \le 3$ *and* $n > 7$.
Proof. The proof follows directly from Lemma 3.2.1 and Example 3.2.1.

3.3. Cordial Labeling of fifth power of Wheals graphs

Obviously, the order of W_{n+1}^5 is $n+1$, and the size of W_{n+1}^5 is $6n-14$ for every $n > 7$. In Particular, $W_4^5 \cong K_4, W_5^5 \cong K_5, W_6^5 \cong K_6, W_7^5 \cong K_7$ and $W_8^5 \cong K_8$. In this subsection, we show that W_{n+1}^5 is cordial if and only if $n \ge 3$ except $3 \le n \le 8$.

Theorem 3.3.1. *The fifth power of Wheals graphs* $W_{n+1}^5 = P_1 + C_n^5$ *are cordial if and only if* $n \ge 3$ except $3 \le n \le 8$.
Proof. In case of $3 \le n \le 7$, it is easy to verify that $W_4^5 \cong K_4, W_5^5 \cong K_5, W_6^5 \cong K_6, W_7^5 \cong K_7$ and $W_8^5 \cong K_8$ are not cordial by cahit [3]. Since $n = 8$, then by investigation all possible labelings of vertices of W_9^5 with $v_0 = 5, v_1 = 4$ or $v_0 = 4, v_1 = 5$, we obtained that $|e_0 - e_1| > 1$. Therefore W_9^5 is not cordial. In case of $n \ge 9$, let $n = 4t + i$ ($0 \le i \le 3$ and $t > 2$), then for a given value of i with $0 \le i \le 3$, we use the labeling $[u, v]$ for $W_{n+1}^5 = P_1 + C_n^5$ where u be the labeling of P_1 and $v = A_i$ or $A_i^{'}$ be the labeling of C_n^5 as shown in Tables 1, 3. It is easy to see that W_{n+1}^5 for every $n > 8$ is cordial from the last two columns of the Table w and the Theorem is proved.

Table 3: Labeling of W_{n+1}^5.

$n = 4t + i$, $n \ge 9$ $i = 0, 1, 2, 3$	Labeling of W_{n+1}^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$[0, A_0] = [0, L_{4t} 1_2 0]$	$2t + 3$	$2t + 2$	$12t + 5$	$12t + 5$	1	0
$i = 1$	$[1, A_1'] = [1, 1_2 M_{4t-4} 0_3]$	$2t + 1$	$2t + 1$	$12t - 4$	$12t - 4$	0	0
$i = 2$	$[0, A_2] = [0, L'_{4t} 10]$	$2t + 2$	$2t + 1$	$12t - 1$	$10t - 1$	1	0
$i = 3$	$[0, A_3'] = [0, 11 L'_{4t} 0]$	$2t + 2$	$2t + 2$	$12t + 1$	$12t + 1$	0	0

3.4. Cordial Labeling of fifth power of Fans graphs

Clearly, the order of F_{n+1}^5 is $n+1$, and the size of F_{n+1}^5 is $6n - 15$ for every $n > 7$. In Particular, $F_2^5 \cong P_2$, $F_3^5 \cong C_3$, $F_4^5 \cong K_4, F_5^5 \cong K_5, F_6^5 \cong K_6$ and $F_7^5 \cong K_7$. In this subsection, we show that F_{n+1}^5 is cordial if and only if $n \ge 1$ except $3 \le n \le 7$.

Theorem 3.4.1. *The fifth power of Fans graphs* $F_{n+1}^5 = P_1 + P_n^5$ *is cordial if and only if* $n \ge 1$ except $3 \le n \le 7$.

Proof. Since $F_2^5 \cong P_2$, $F_3^5 \cong C_3$, F_9^5 the labeling $[1; 0_4 1_4]$ is sufficient for F_9^5 and P_2 ,C_3 are cordial [3], then F_2^5 ,F_3^5 and F_8^5 are cordial. In case of $3 \leq n \leq 6$, it is easy to verify that $F_4^5 \cong K_4, F_5^5 \cong K_5, F_6^5 \cong K_6$ and $F_7^5 \cong K_7$ are not cordial by cahit [3]. Since $n = 7$, then by investigation all possible labelings of vertices of P_8^5 with $v_0 = v_1 = 4$, we obtained that $|e_0 - e_1| > 1$. Therefore F_8^5 is not cordial. In case of $n \geq 8$, let $n= 4t + i$ ($0 \leq i \leq 3$ and $t \geq 2$), then for a given value of i with $0 \leq i \leq 3$, we use the labeling $[u, v]$ for $F_{n+1}^5= P_1 + P_n^5$where u be the labeling of P_1 and v=B_i or $B_i^{'}$ be the labeling of P_n^5 as shown in Tables 2, 4. It is easy to see that F_{n+1}^5 for every $n \geq 8$ is cordial from the last two columns of the Table 4 and thus the Theorem is proved.

Table 4: Labeling of F_{n+1}^5.

$n = 4t + i$, $n > 8$ $i = 0, 1, 2, 3$	**Labeling of** F_{n+1}^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
i= 0	[0;B_0= $0L_{4t}1_20$]	$2t + 3$	$2t + 2$	$12t + 4$	$12t + 5$	1	-1
i= 1	[0;B_1=$L_{4t}1$]	$2t + 1$	$2t + 1$	$12t - 5$	$12t - 4$	0	-1
i= 2	[0;B_2=$L^{'}{}_{4t}10$]	$2t + 2$	$2t + 1$	$12t - 1$	$12t - 2$	1	1
i= 3	[1;$B_3^{'}$= $010L_{4t}$]	$2t + 2$	$2t + 2$	$12t + 1$	$12t + 2$	0	-1

3.5. Cordial Labeling of fifth power of Lemniscate graphs

The fifth power of a lemniscate graph is denoted by $L_{n,m}^5$=$C_n^5 \cup C_m^5$ where both cycles have a common point. So, the order of the fifth power of lemniscate is $n + m - 1$, and the size of the fifth power of lemniscate graph $L_{n,m}^5$ equals to $5(n + m) - 28$. In the following section, we study and investigate the cordiality of the fifth power of lemniscate $L_{n,m}^5$ for all $n, m \geq 3$. The labeling of $L_{n,m}^5$ takes the form $[A;B]$ where the labeling A is given to n and the labeling B is given to m. For this purpose, we labeling each fifth power of cycle separately.

Lemma 3.5.1. *If* $3 \leq n \leq 8$, and $m > 8$, *then the fifth power of lemniscate* $L_{n,m}^5$ *is cordial for all* $3 \leq n \leq 8$, and $m > 8$. Considering that $L_{n,m}^5 \approxeq L_{m,n}^5$.

Proof. Let $3 \leq n \leq 8$ and $m= 4t + j$ ($j= 4, 1, 2, 3$ and $t \geq 2$), then using Tables 1, 5 and formulas $v_0 - v_1$and $e_0 - e_1$, we can compute the values shown in the last two columns of Tables 5. Since all of these values are -1, 0 or 1, the lemma is proved.

Table 5: Labeling of $L^5_{n,m}$;$3 \le n \le 8$, and $m > 8$

$3 \le n \le 8$	$m = 4t + j$, $t \ge 2, m > 8$ $i = 4, 1, 2, 3$	Labeling of $L^5_{n,m}$	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
3	$m = 4t+4$	$[0_3; 0L_{4t}1_3]$	$2t+3$	$2t+3$	$10t+5$	$10t+4$	0	1
	$m = 4t+1$	$[010; L_{4t}]$	$2t+2$	$2t+1$	$10t-3$	$10t-3$	1	0
	$m = 4t+2$	$[010; 01_2L'_{4t-4}10]$	$2t+2$	$2t+2$	$10t-1$	$10t$	0	-1
	$m = 4t+3$	$[101; L'_{4t}01]$	$2t+2$	$2t+3$	$10t+2$	$10t+2$	-1	0
4	$m = 4t+4$	$[1_30; 0_2L'_{4t}0_2]$	$2t+4$	$2t+3$	$10t+6$	$10t+6$	1	0
	$m = 4t+1$	$[M_4; 1_2M'_{4t-4}0_3]$	$2t+2$	$2t+2$	$10t-2$	$10t-1$	0	-1
	$m = 4t+2$	$[0_31; 0L_{4t}1]$	$2t+3$	$2t+2$	$10t+1$	$10t+1$	1	0
	$m = 4t+3$	$[0_31; 1L_{4t}1_2]$	$2t+3$	$2t+3$	$10t+4$	$10t+3$	1	0
5	$m = 4t+4$	$[010_3; 0L'_{4t}1_3]$	$2t+4$	$2t+4$	$10t+8$	$10t+8$	0	0
	$m = 4t+1$	$[010_3; 0L'_{4t-4}1_3]$	$2t+3$	$2t+2$	$10t+1$	$10t$	1	1
	$m = 4t+2$	$[010_3; 1_2L'_{4t}1_20]$	$2t+3$	$2t+3$	$10t+3$	$10t+3$	0	0
	$m = 4t+3$	$[0_31_2; L'_{4t}01]$	$2t+4$	$2t+3$	$10t+5$	$10t+6$	1	-1
6	$m = 4t+4$	$[10_21_3; L'_{4t-4}0_210]$	$2t+5$	$2t+4$	$10t+3$	$10t+4$	1	-1
	$m = 4t+1$	$[10_21_3; 1_2M_{4t-4}0_3]$	$2t+3$	$2t+3$	$10t+3$	$10t+3$	0	0
	$m = 4t+2$	$[0L_40; 01_2L_{4t-4}10]$	$2t+4$	$2t+3$	$10t+5$	$10t+6$	1	-1
	$m = 4t+3$	$[1_3010; 0L_{4t}10]$	$2t+4$	$2t+5$	$10t+8$	$10t+8$	-1	0
7	$m = 4t+4$	$[0_21_20_3; L_{4t}1_201]$	$2t+4$	$2t+3$	$10t+14$	$10t+13$	1	1
	$m = 4t+1$	$0_21_20_3; L_{4t}1$	$2t+4$	$2t+3$	$10t+6$	$10t+6$	1	-1
	$m = 4t+2$	$0_21_20_3; L_{4t}1_2$	$2t+4$	$2t+4$	$10t+9$	$10t+8$	0	1
	$m = 4t+3$	$[0_4101; 11L_{4t}1]$	$2t+5$	$2t+4$	$10t+11$	$10t+11$	1	0
8	$m = 4t+4$	$[10_310_21; 101_2L_{4t}]$	$2t+6$	$2t+5$	$10t+16$	$10t+16$	1	0
	$m = 4t+1$	$010_2M'_4; L_{4t}1$	$2t+4$	$2t+4$	$10t+8$	$10t+9$	0	-1
	$m = 4t+2$	$10_310_21; 1_2L_{4t}$	$2t+5$	$2t+4$	$10t+11$	$10t+11$	1	0
	$m = 4t+3$	$[010_2M'_4; 0L_{4t}1_2]$	$2t+5$	$2t+5$	$10t+14$	$10t+13$	0	1

Lemma 3.5.2. *If $n, m \ge 9$, then the fifth power of lemniscate $L^5_{n,m}$ is cordial for all $n \ge 9$ and $m \ge 9$.*

Proof. Let $n = 4r + i$ ($i = 0, 1, 2, 3$ and $r \ge 1$) and $m = 4t + j$ ($j = 0, 1, 2, 3$ and $t \ge 1$), then using Tables 1, 6 and formulas $v_0 - v_1$and $e_0 - e_1$, we can compute the values shown in the last two columns of Table 6. Since all of these values are -1, 0 or 1, the lemma is proved.

Table 6: Labeling of $L^5_{n,m}$

$n = 4r + i$, $n \ge 9$ $0 \le j \le 3$	$m = 4t + j$ $m \ge 9$ $0 \le j \le 3$	Labeling of $L^5_{n,m}$	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$j = 0$	$[0L_{4r}1_20; 0L_{4t}1_20]$	$2(r+t)+3$	$2(r+t)+4$	$10(r+t)+6$	$10(r+t)+6$	-1	0
	$j = 1$	$[0L_{4r}1_20; 0_3M_{4t-4}1_2]$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-1$	$10(r+t)-2$	0	-1
	$j = 2$	$[0L_{4r}1_20; 01L_{4t}]$	$2(r+t)+2$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)+1$	-1	0
	$j = 3$	$[0L_{4r}1_20; 0L_{4t}10]$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)+4$	$10(r+t)+3$	0	1
$i = 1$	$j = 1$	$[L_{4r}1; 1_2M'_{4t-4}0_3]$	$2(r+t)+1$	$2(r+t)$	$10(r+t)-9$	$10(r+t)-9$	1	0
	$j = 2$	$[L_{4r}1; 10L'_{4t}]$	$2(r+t)+1$	$2(r+t)+1$	$10(r+t)-7$	$10(r+t)-6$	0	-1
	$j = 3$	$[L'_{4r}0; 0L_{4t}10]$	$2(r+t)+2$	$2(r+t)+1$	$10(r+t)-4$	$10(r+t)-4$	1	0
$i = 2$	$j = 2$	$[L'_{4r}10; 01L_{4t}]$	$2(r+t)+1$	$2(r+t)+2$	$10(r+t)-4$	$10(r+t)-4$	-1	0
	$j = 3$	$[L'_{4r}10; 0L_{4t}10]$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-1$	$10(r+t)-2$	0	1
$i = 3$	$j = 3$	$[11L'_{4r}0; L_{4t}100]$	$2(r+t)+2$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)+1$	-1	0

Lemma 3.5.3. *If $3 \le n \le 8$, and $3 \le m \le 8$, then the fifth power of lemniscate $L^5_{n,m}$ is cordial except $(n, m) = (3, 3)$,$(3, 6)$,$(4, 5)$,$(5, 4)$,$(5, 6)$,$(6, 3)$,$(6, 5)$ and $(6, 6)$.*

Proof. Let $3 \leq n \leq 8$, and $3 \leq m \leq 8$ except $(n, m) = (3,3),(3,6),(4,5),(5,4),(5,6),(6,3),(6,5)$ and $(6,6)$ then using Tables 7 and formulas $v_0 - v_1$and $e_0 - e_1$, we can compute the values shown in the last two columns of Table 7. Since all of these values are -1, 0 or 1, then $L^5_{n,m}$ are cordial except $(n, m) = (3,3),(3,6),(4,5),(5,4),(5,6),(6,3),(6,5)$ and $(6,6)$. In case of $(n, m) = (3,3)$, it is easy to verify that $L^5_{3,3} \cong L_{3,3}$ is not cordial by cahit [3]. In case of $(n, m) = (3,6),(4,5),(5,4),(5,6),(6,3)$, and $(6,5)$ the graphs $L^5_{n,m}$ have an even order. If these graphs are cordial, it would have an equal number of vertices that are labeled ones and that are labeled zero.Otherwise $|v_0 - v_1| > 1$ and also fo the case $(n, m) = (6,6)$ the graph $L^5_{6,6}$ has an odd order, it would have the absolute difference of the vertices labeled one from those labeled zero will be one. Otherwise $|v_0 - v_1| > 1$. The set of all different possibilities of labeling of the vertices of $L^5_{6,6}$, follows that $|e_0 - e_1| > 1$, Contradiction. Hence $L^5_{n,m}$,$(n, m) = (3,3),(3,6),(4,5),(5,4),(5,6),(6,3),(6,5)$ and $(6,6)$ are not cordial.

Table 7: Labeling of $L^5_{n,m}$;$3 \leq n \leq 8$, and $m > 8$

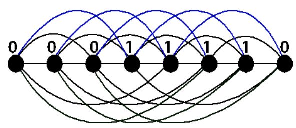
$3 \leq n \leq 8$	$3 \leq m \leq 8$	Labeling of $L^5_{n,m}$	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
3	$m = 4$	[110; 0001]	3	3	4	5	0	-1
	$m = 5$	[111; 10010]	3	4	7	6	-1	1
	$m = 7$	[001; 1101110]	4	5	12	12	-1	0
	$m = 8$	[001; 10110101]	5	5	14	15	0	-1
4	$m = 4$	[1000; 0111]	3	4	6	6	-1	0
	$m = 6$	$[0_31;0_311]$	5	4	11	11	1	0
	$m = 7$	$[01_40;010_210]$	6	6	18	18	0	0
5	$m = 5$	$[010_21; 1_40]$	4	5	10	10	-1	0
	$m = 7$	$[0_2101; 1_201_201]$	5	6	15	16	-1	-1
	$m = 8$	0103; 010214	6	6	18	18	0	0
6	$m = 7$	$[1_3M_4; 10_210_2]$	6	6	18	18	0	0
	$m = 8$	$[M'_411; 10_310_21]$	7	6	20	21	1	-1
7	$m = 7$	$[0_4M'_3;L'_41_2]$	7	6	21	21	1	0
	$m = 8$	$1_21_4; 10_310_21$	7	7	24	23	0	1
8	$m = 8$	$[101_30_21;10_310_2]$	8	7	26	26	1	0

Theorem 3.5.1. *The fifth power of lemniscate* $L^5_{n,m}$ *cordial if and only if* $n, m \geq 3$ *except* $(n, m) = (3,3),(3,6),(4,5),(5,4),(5,6),(6,3),(6,5)$ *and* $(6,6)$.

Proof. The proof follows directly from Lemma 3.5.1 ,..., 3.5.3.

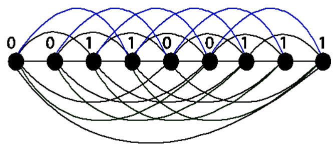
Example 3.1

The cordial graph of P^5_8,C^5_9,F^5_9,W^5_{10}, and $L^5_{8,8}$are illustrated in Figures (1, ...,5).



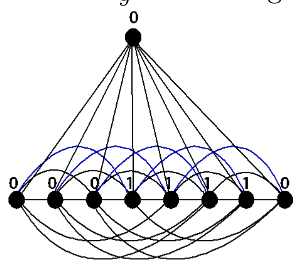
$v_0 = 4$,$v_1 = 4$,$e_0 = 12$,$e_1 = 13$,$v_0 - v_1 = 0$,$e_0 - e_1 = -1$

Figure 1. P^5_8 is cordial graph.



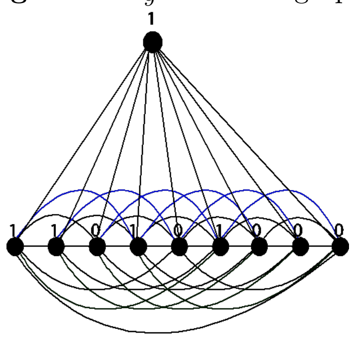
$v_0 = 4, v_1 = 5, e_0 = 15, e_1 = 16, v_0 - v_1 = -1, e_0 - e_1 = -1$

Figure 2. C_9^5 is cordial graph.



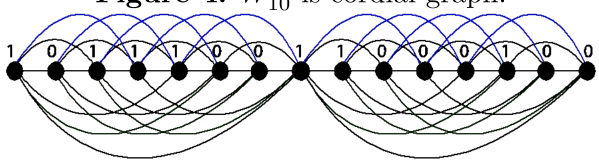
$v_0 = 5, v_1 = 4, e_0 = 16, e_1 = 17, v_0 - v_1 = 1, e_0 - e_1 = -1$

Figure 3. F_9^5 is cordial graph.



$v_0 = 5, v_1 = 5, e_0 = 20, e_1 = 20, v_0 - v_1 = 0, e_0 - e_1 = 0$

Figure 4. W_{10}^5 is cordial graph.



$v_0 = 8, v_1 = 7, e_0 = 26, e_1 = 26, v_0 - v_1 = 1, e_0 - e_1 = 0$

Figure 5. $L_{8,8}^5$ is cordial graph.

4. The Cordial Labeling for Union of fifth power of Paths and Cycles

In this section, we will study the cordiality of the union of two fifth power of paths, and a similar study will be performed of two fifth power of cycles. We end this section by studying the cordiality of the union fifth power of paths with cycles.

Theorem 4.1. *The union of* $P_n^5 \cup P_m^5$, *admits a cordial labeling for every* $n, m > 7$.

Proof. Let $n = 4r + i$ ($i = 0, 1, 2, 3$ and $r \geq 1$) and $m = 4t + j$ ($j = 0, 1, 2, 3$ and $t \geq 1$), then throughout the Table 2 and Table 8. Using Table 8 and formulas $v_0 - v_1$and $e_0 - e_1$, we can compute the values shown in the last two columns of Table 8. Since all of these values are -1, 0 or 1, the Theorem is proved.

Table 8: Labeling of $P_n^5 \cup P_m^5$

$n = 4r + i$, $n \geq 8$ $0 \leq i \leq 3$	$m = 4t + j$ $m \geq 8$ $0 \leq j \leq 3$	Labeling of P_n^5	Labeling of P_m^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$j = 0$	$0L_{4r}1_20$	$1L'_{4t}M_3$	$2(r+t)+4$	$2(r+t)+4$	$10(r+t)+5$	$10(r+t)+5$	0	0
	$j = 1$	$0L_{4r}1_20$	$L_{4t}1$	$2(r+t)+2$	$2(r+t)+3$	$10(r+t)-3$	$10(r+t)-2$	-1	-1
	$j = 2$	$0L_{4r}1_20$	$L'_{4t}10$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)$	$10(r+t)$	0	0
	$j = 3$	$0L_{4r}1_20$	$0L_{4t}10$	$2(r+t)+4$	$2(r+t)+3$	$10(r+t)+2$	$10(r+t)+3$	1	-1
$i = 1$	$j = 1$	$L_{4r}1$	$0L_{4t}$	$2(r+t)+1$	$2(r+t)+1$	$10(r+t)-10$	$10(r+t)-10$	0	0
	$j = 2$	$L_{4r}1$	$L'_{4t}10$	$2(r+t)+1$	$2(r+t)+2$	$10(r+t)-7$	$10(r+t)-8$	-1	1
	$j = 3$	$L_{4r}1$	$L'_{4t}M_3$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-5$	$10(r+t)-5$	0	0
$i = 2$	$j = 2$	$L'_{4r}10$	$L_{4t}10$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-5$	$10(r+t)-5$	0	0
	$j = 3$	$L'_{4r}10$	$L'_{4t}M_3$	$2(r+t)+3$	$2(r+t)+2$	$10(r+t)-2$	$10(r+t)-3$	1	1
$i = 3$	$j = 3$	$0L_{4r}10$	$L_{4t}101$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)$	$10(r+t)$	0	0
$n = 8$	$m = 8$	101_30_21	$1_20_310_2$	8	8	25	25	0	0
	$j = 0, m > 8$	0_41_4	$1L'_{4t}M_3$	$2t+6$	$2t+6$	$10t+6$	$10t+6$	0	0
	$j = 1$	0_41_4	$L_{4t}1$	$2t+4$	$2t+5$	$10t+7$	$10t+8$	-1	-1
	$j = 2$	0_41_4	$L'_{4t}10$	$2t+5$	$2t+5$	$10t+10$	$10t+10$	0	0
	$j = 3$	0_41_4	$L'_{4t}M_3$	$2t+6$	$2t+5$	$10t+12$	$10t+13$	1	-1

Theorem 4.2. *The union of* $C_n^5 \cup C_m^5$, $n, m > 7$ *admits a cordial labeling for every* $n, m > 7$.

Proof. Let $n = 4r + i$ ($0 \leq i \leq 3$ and $r \geq 1$) and $m = 4t + j$ ($0 \leq j \leq 3$ and $t \geq 1$), then throughout the Table 1 and Table 9. Using Table 9 and formulas $v_0 - v_1$ and $e_0 - e_1$, we can compute the values shown in the last two columns of Table 9. Since all of these values are -1, 0 or 1, the Theorem is proved.

Table 9: Labeling of $C_n^5 \cup C_m^5$

$n = 4r + i$, $n \geq 8$ $0 \leq i \leq 3$	$m = 4t + j$ $m \geq 8$ $0 \leq j \leq 3$	Labeling of C_n^5	Labeling of C_m^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0, r > 1$	$j = 0, t > 1$	$0L_{4r}1_20$	$0L_{4t}1_20$	$2(r+t)+4$	$2(r+t)+4$	$10(r+t)+6$	$10(r+t)+6$	0	0
	$j = 1, t > 1$	$0L_{4r}1_20$	$0_3M_{4t-4}1_2$	$2(r+t)+3$	$2(r+t)+2$	$10(r+t)-1$	$10(r+t)-2$	1	-1
	$j = 2, t > 1$	$0L_{4r}1_20$	$01L_{4t}$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)+1$	0	0
	$j = 3, t > 1$	$0L_{4r}1_20$	$0L_{4t}10$	$2(r+t)+4$	$2(r+t)+3$	$10(r+t)+4$	$10(r+t)+3$	1	1
$i = 1$	$j = 1$	$L_{4r}1$	$1_2M'_{4t-4}0_3$	$2(r+t)+1$	$2(r+t)+1$	$10(r+t)-9$	$10(r+t)-9$	0	0
	$j = 2$	$L_{4r}1$	$10L'_{4t}$	$2(r+t)+1$	$2(r+t)+2$	$10(r+t)-7$	$10(r+t)-6$	-1	-1
	$j = 3$	$L_{4r}1$	$L_{4t}10$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-4$	$10(r+t)-4$	0	0
$i = 2$	$j = 2$	$L'_{4r}10$	$01L_{4t}$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-4$	$10(r+t)-4$	0	0
	$j = 3$	$L'_{4r}10$	$0L_{4t}10$	$2(r+t)+3$	$2(r+t)+2$	$10(r+t)-1$	$10(r+t)-2$	1	1
$i = 3$	$j = 3$	$L'_{4r}M_3$	$0L_{4t}10$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)+1$	0	0
$n = 8$	$m = 8$	101_30_21	$1_20_310_2$	8	8	26	26	0	0
	$j = 0, m > 8$	0_41_4	$0_4L'_{4t-4}1_4$	$2t+6$	$2t+6$	$10t+7$	$10t+7$	0	0
	$j = 1$	0_41_4	$1_2M_{4t}0_3$	$2t+5$	$2t+4$	$10t+8$	$10t+9$	1	-1
	$j = 2$	0_41_4	$031L_{4t-4}1_2$	$2t+5$	$2t+5$	$10t+11$	$10t+11$	0	0
	$j = 3$	0_41_4	$L'_{4t}M_3$	$2t+6$	$2t+5$	$10t+13$	$10t+14$	1	-1

Theorem 4.3. *The union of $P_n^5 \cup C_m^5$, $n, m > 7$ admits a cordial labeling for every $n, m > 7$.*

Proof. Let $n = 4r + i$ ($i = 0, 1, 2, 3$ and $r \geq 1$) and $m = 4t + j$ ($j = 0, 1, 2, 3$ and $t \geq 1$), then throughout the Tables 1, 2 and Table 10. Using Table 10 and formulas $v_0 - v_1$ and $e_0 - e_1$, we can compute the values shown in the last two columns of Table 10. Since all of these values are -1, 0 or 1, the Theorem is proved.

Table 10: Labeling of $P_n^5 \cup C_m^5$

$n = 4r + i$, $n \geq 8$ $0 \leq i \leq 3$	$m = 4t + j$ $m \geq 8$ $0 \leq j \leq 3$	Labeling of P_n^5	Labeling of C_m^5	v_0	v_1	e_0	e_1	$v_0 - v_1$	$e_0 - e_1$
$i = 0$	$j = 0$	$0L_{4r}1_20$	$0L_{4t}1_20$	$2(r+t)+4$	$2(r+t)+4$	$10(r+t)+5$	$10(r+t)+6$	0	-1
	$j = 1$	$0L_{4r}1_20$	$0_3M_{4t-4}1_2$	$2(r+t)+3$	$2(r+t)+2$	$10(r+t)-2$	$10(r+t)-2$	1	0
	$j = 2$	$0L_{4r}1_20$	$01L_{4t}$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)$	$10(r+t)+1$	0	-1
	$j = 3$	$0L_{4r}1_20$	$0L_{4t}10$	$2(r+t)+4$	$2(r+t)+3$	$10(r+t)+3$	$10(r+t)+3$	1	0
$i = 1$	$j = 0$	$L_{4r}1$	$0L_{4t}1_20$	$2(r+t)+2$	$2(r+t)+3$	$10(r+t)-2$	$10(r+t)-2$	-1	0
	$j = 1$	$L_{4r}1$	$1_2M'_{4t-4}0_3$	$2(r+t)+1$	$2(r+t)+1$	$10(r+t)-9$	$10(r+t)-10$	0	1
	$j = 2$	$L_{4r}1$	$10L'_{4t}$	$2(r+t)+1$	$2(r+t)+2$	$10(r+t)-7$	$10(r+t)-7$	-1	0
	$j = 3$	$L_{4r}1$	$L_{4t}10$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-4$	$10(r+t)-5$	0	1
$i = 2$	$j = 0$	$L'_{4r}10$	$0L_{4t}110$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)$	0	1
	$j = 1$	$L'_{4r}10$	$L_{4t}1$	$2(r+t)+1$	$2(r+t)+1$	$10(r+t)-7$	$10(r+t)-7$	0	0
	$j = 2$	$L'_{4r}10$	$01L_{4t}$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-4$	$10(r+t)-5$	0	1
	$j = 3$	$L'_{4r}10$	$L_{4t}101$	$2(r+t)+2$	$2(r+t)+3$	$10(r+t)-2$	$10(r+t)-2$	-1	0
$i = 3$	$j = 0$	$0L_{4r}10$	$0L_{4t}110$	$2(r+t)+4$	$2(r+t)+3$	$10(r+t)+3$	$10(r+t)+3$	1	0
	$j = 1$	$0L_{4r}10$	$L_{4t}1$	$2(r+t)+2$	$2(r+t)+2$	$10(r+t)-5$	$10(r+t)-4$	0	-1
	$j = 2$	$0L_{4r}10$	$L'_{4t}10$	$2(r+t)+3$	$2(r+t)+2$	$10(r+t)-2$	$10(r+t)-2$	1	0
	$j = 3$	$L'_{4r}M_3$	$0L_{4t}10$	$2(r+t)+3$	$2(r+t)+3$	$10(r+t)+1$	$10(r+t)$	0	1
$n = 8$	$m = 8$	101_30_21	$1_20_310_2$	8	8	25	26	0	-1
	$j = 0, m > 8$	0_41_4	$1L'_{4t}M_3$	$2t+6$	$2t+6$	$10t+6$	$10t+7$	0	-1
	$j = 1$	0_41_4	$1_2M_{4t}0_3$	$2t+5$	$2t+4$	$10t+8$	$10t+8$	1	0
	$j = 2$	0_41_4	$L'_{4t}10$	$2t+5$	$2t+5$	$10t+10$	$10t+11$	0	-1
	$j = 3$	0_41_4	$L'_{4t}M_3$	$2t+6$	$2t+5$	$10t+13$	$10t+13$	1	0
$i = 0, n > 8$	$m = 8$	$0_4L'_{4r-4}1_4$	0_41_4	$2r+6$	$2r+6$	$10r+7$	$10r+6$	0	1
$j = 1$		$1_2M_{4r}0_3$	0_41_4	$2r+5$	$2r+4$	$10r+8$	$10r+8$	1	0
$j = 2$		$0_31L_{4r-4}1_2$	0_41_4	$2r+5$	$2r+5$	$10r+11$	$10r+10$	0	1
$j = 3$		$L'_{4r}M_3$	0_41_4	$2r+6$	$2r+5$	$10r+13$	$10r+13$	1	0

5. Algorithm

In this section, we propose an algorithm for calculating the cordial labeling for any graph G . This algorithm provides a framework for attempting to find a cordial labeling for a given graph. It's important to note that not all graphs will admit a cordial labeling, and the specific method of assigning labels to vertices in step 3 can vary based on the graph's structure and properties. We assume that labeling each vertex and edge takes constant time, then the time complexity of Algorithm 1 is $O(n)$, since each vertex and edge is visited once.

Algorithm1: Cordial Labeling of a Graph **Input:** A graph G(V, E)
Output: A cordial labeling of G if it exists
Begin **Step 1.** Initialize two counters, v(0) and v(1), to 0.
Step 2. Initialize two counters, e(0) and e(1), to 0.
Step 3. For each vertex v in V:
a. Assign a label f(v) ∈ {0, 1} to the vertex v.
b. If f(v) == 0, increment v(0); else increment v(1).
Step 4. For each edge e(u, v) in E:
a. Assign a label f(e) = $\|f(u) - f(v)\|$ to the edge e.
b. If f(e) == 0, increment e(0); else increment e(1).
Step 5. Check the cordiality condition:
a. If $\|v(0) - v(1) \leq 1$ and $\|e(0) - e(1) \leq 1$:
i. The labeling is cordial.
ii. Output the labeling of vertices and edges.
b. Else:
i. The graph G does not admit a cordial labeling.
ii. Output that no cordial labeling exists.
End Algorithm

6. Conclusion

We proved that each fifth power of path P_n^5,admits cordial labeling if and only if $1 \leq n \leq 3$ and $n > 7$. Each fifth power of cycle C_n^5, admits cordial labeling if and only if $n = 3$ and $n > 8$. Each fifth power of Fan F_{n+1}^5, admits a cordial labeling if and only if $n \geq 1$ except $3 \leq n \leq 6$. The fifth power of Wheel graph W_{n+1} admits cordial labeling if and only if $n \geq 3$ except $3 \leq n \leq 8$. Moreover, we proved that the fifth power of lemniscate $L_{n,m}^5$, admits a cordial labeling if and only if $n, m \geq 3$ except $(n, m) = (3,3)$,$(3,6)$,$(4,5)$,$(5,4)$,$(5,6)$,$(6,3)$,$(6,5)$ *and* $(6,6)$. Also, we investigated the cordiality for the union of fifth power of paths and cycles. We proposed an algorithm for determining the cordiality of a given graph. In the future, we will apply cordial labeling to other types of graphs.

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Data Availability Statement

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The author declares no conflict of interest.

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