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Primal Approximation Spaces by κ-neighborhoods with Applications

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Abstract. Real-world problems often involve imprecision and uncertainty, presenting challenges across various domains such as engineering, artificial intelligence, social sciences, and medical sciences. To bridge this gap, Pawlak introduced classical rough set models, focusing on upper and lower approximations based on equivalence relations. However, these relations restrict the applicability of rough sets in many contexts. This paper explores new approaches to extending rough set theory by introducing "primal approximation spaces". Primal is defined as novel structures designed to generalize rough set approximations beyond the traditional methods. This paper proposes a new technique for generating rough approximations by using κ-neighborhoods and primal which allows for creating diverse supra-topologies and enhancing the flexibility of rough set models. Furthermore, we here introduce bi-primal approximation spaces, a new form of approximation that can be examined through two distinct methods, as a result, revealing their unique characteristics and relationships. The research underlines the practical applications of these new methods by providing a detailed case study that demonstrates their effectiveness in solving decision-making problems. Moreover, this study also compares the primal-based methods with existing approaches based on ideals, illustrating their distinctive advantages and limitations. Overall, this work offers a remarkable advancement in rough set theory by expanding its theoretical framework and practical applicability through the introduction of primal and bi-primal approximation spaces.

2020 Mathematics Subject Classifications: 03E99, 54A05, 54A10, 54E99

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1. Introduction

There are considerable real-life problems involving imprecision, such as those in engineering, artificial intelligence, social science, and medical science. Various mathematical

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modeling techniques have been presented to address these problems, including the theory of probability, fuzzy sets, rough sets, and decision-making theory. These theories have reduced the gap between classical mathematical methods and the inaccuracies of real-world information. However, each of these theories has inherent difficulties and disadvantages, which motivated Pawlak [59] to initiate classical rough set models as modern approach to modeling the vagueness of data from real-life problems. While the core concepts include upper and lower approximations based on equivalence relations, these restrict application domains. As a result, several researchers have employed topological concepts to generalize these approximations and replace equivalence relations with any binary relation (see, for instance, generalized rough sets [21, 24, 37, 42, 64], information systems [65], feature selection [37, 42], topological structures of rough sets [26, 28, 50, 53], and medical applications of topological rough sets [5, 22]).

In 1996, Yao [73] introduced a method that led to several generalizations incorporating various types of relations, including tolerance relations [48, 60, 66], similarity relations [67], and general binary relations [7, 12, 13, 49]. Abd El-Monsef et al. [35] introduced the notion of κ-neighborhood spaces (briefly, κ-NS) to generalize rough set theory through various topologies induced by arbitrary relations. These methods represent an extension of the work [21, 49], which opened the way for more topological applications in rough sets [26, 38, 43, 63] and their applications in many fields [18, 32, 71, 72].

It is worth noting that the concept of a "basic-neighborhood" (introduced in [7]) was studied by El-Gayar et al. [31] and Taher et al. [71, 72] provided a comprehensive analysis, exploring the relationships between the generated topologies and approximations. Furthermore, these papers established new results, comparing their methods with previous techniques such as those in [7, 12, 13, 20, 35, 73]. Additionally, these works demonstrated impressive applications in the medical and economic fields [28, 30, 31].

Mashhour [56] expanded the notion of topology to supra-topology by ignoring the finite intersection condition. A supra-topology is defined on a nonempty set $\mathcal{V}$ as a subclass $\mathcal{S}$ of the power set of $\mathcal{V}$ satisfying two axioms: $\emptyset$, $\mathcal{V} \in \mathcal{S}$, and $\mathcal{S}$ is closed under arbitrary union. This makes supra-topology more elastic in characterizing some real-life problems [51] and establishing examples that illustrate the relationships between topological notions.

The idea of minimal spaces as a generalization of topological spaces was introduced in [55]. These spaces have yielded many valuable results consistent with general topology. Kuratowski [52] offered the concept of an ideal from the filter notion. One may consider an ideal as the dual of a filter. Similarly, one of the classical structures of topology is the grill, which was defined by Choquet [19]. Acharjee et al. [9] presented a primal structure, which is dual to the grill and also generated a primal topology.

Another promising direction for extending rough sets involves the concept of ideals [52], which are foundational in both topology and rough set theory, albeit serving different purposes in each field. In topology, ideals are pivotal for defining closure operations, characterizing convergence, and facilitating compactification. In rough set theory, they play a crucial role in information granulation, shaping lower and upper approximations essential for managing data uncertainty [41, 44, 46, 47]. This framework has been applied to various domains, as demonstrated in [45, 58]. Additionally, numerous studies have leveraged topological structures to expand the applicability of rough sets [3, 6, 23, 27]. For a comprehensive overview of the applications of topological structures and rough sets, we direct readers to the existing literature [1, 2, 4, 50, 70, 74].

So, the current manuscript emphasizes the versatility of primal as mathematical constructs, demonstrating their impact not only in rough set theory but also in other fields, especially topology. The paper delicacy primal as a class of objects within an information system, governed by particular conditions.

The main goal of the present manuscript is to provide a new framework for approximate rough sets using novel structures called "primal." The choice of a primal is carefully guided and prepared by domain experts, emphasizing the practical importance of primal in customizing granulation methods for definite problem domains.

Two different directions to extend Pawlak's philosophy are suggested based on primal. In the first direction, we define the original method of generating rough approximations depending on κ-neighborhoods and primal. The technique represents a novel method to induce different supra-topologies via κ-neighborhoods and primal, which opens the way for more topological structures in rough-set models. The current paradigm loosens the constraints that typically restrict the modeling of a given problem. By relaxing the primary condition imposed on the model, we achieve greater flexibility in describing and addressing practical issues. For instance, eliminating the intersection condition from topological rough models, as proposed in this approach, broadens the scope of problems that can be addressed.

Secondly, a novel type of approximation space, termed bi-primal approximation spaces, was introduced for the first time. This new form of approximation was examined through two distinct methods, with their properties thoroughly investigated and the relationship between these methods explored. The importance of these approaches lies in their foundation on the concept of primal, which serve as topological tools; here, the two primal represent two distinct perspectives rather than a single one. It should be noted that the two notions, "ideals" and "primal," are not dual; rather, they are independent, as illustrated in the current paper. Consequently, the methods introduced represent new tools that are entirely distinct from those that use ideals in their approaches (such as those proposed by M. Hosny et al.[41]). These methods offer solutions to problems that ideals (or other methods) maybe unable to address, as we will illustrate in the practical application presented at the end of the paper. Thus, we can conclude that the methods based on primal provide a different perspective and solutions to decision-making problems that ideals may not be able to solve, while primal may also be limited in solving some problems

that ideals can address.
The current framework offers the flexibility to relax constraints that typically limit the scope of modeling problems. It is widely recognized that loosening key conditions imposed on a model allows for greater adaptability in tackling real-world challenges. For example, in the proposed model, removing the requirement for intersection in topological rough models broadens the range of problems that can be addressed. Specifically, generating a supra-topological space leads to unique outcomes that are especially useful in decision-making processes, such as competitive job selection. In these scenarios, candidates are usually required to meet several criteria. However, applying the concept of supra-topology helps to systematically eliminate redundant or overlapping criteria, allowing for a more focused assessment of each candidate's distinct qualifications. This refined approach reduces the uncertainty involved in making final decisions by concentrating on the most relevant attributes for the role, thereby simplifying the selection process.
The paper concludes with a practical example that productively illustrates the definitions in a clear and comprehensive way.

- **Objectives:**

The main objective of this research is developing a novel framework for approximate rough sets by introducing and employing the concept of "primal." This study aims to extend and refine existing rough set theory by incorporating these new structures, focusing on enhancing the flexibility and applicability of rough set models in various domains. In short, the research seeks to achieve the following:
1. Defining a new method for generating rough approximations using κ-neighborhoods and primals, thereby inducing diverse supra-topologies.
2. Introducing and thoroughly investigating a new type of approximation space called bi-primal approximation spaces, exploring the distinct methods and properties of this approach.
3. Demonstrating the practical applications of these new methods in solving decision-making problems that are challenging or impossible to address by using existing approaches, such as those based on ideals.

- **Motivations:**

The motivation that drives us to doing this research arises from the drawbacks of the existing mathematical methods in effectively dealing with the vagueness and imprecision inherent in real-world problems. While classical rough set theory and its various extensions have made significant strides, they still face challenges in certain applications owing to the restrictive conditions imposed by traditional topological and set-theoretical constructs. The introduction of primal and bi-primal approximation spaces offers a response to these challenges, offering a more versatile and adaptable framework. This approach is motivated by the need to:
1. Overcome the drawbacks of existing rough set models, especially those that rely on rigid conditions such as the finite intersection condition in topologies.
2. Provide new methods which allow for greater flexibility in modeling and analyzing

complex systems, especially in fields where traditional methods fall short.
3. Explore the potential of topological structures like primal in broadening the scope of rough set theory and its applications across various disciplines, including artificial intelligence, engineering, and medical science.

- **Contributions:**

This paper makes several significant contributions to the field of rough set theory and its applications:
1. Introduction of primal: The concept of primal is introduced as a novel structure in rough set theory, providing a new method for generating rough approximations through κ-neighborhoods and primal. This contribution widens the theoretical foundations of rough set models and opens up further avenues for topological applications.
2. Development of Bi-Primal Approximation Spaces: The research presents the first-ever introduction of bi-primal approximation spaces, offering two distinct ways for their construction and examining their properties and relationships. This contribution provides a new perspective on approximation spaces and enriches the toolbox for dealing with uncertainty and imprecision in data.
3. Practical Applications and Case Study: The paper includes a practical example demonstrating the applicability and effectiveness of the proposed methods in solving decision-making problems. This contribution highlights the practical value of the research and its potential impact on various real-world applications.
4. Comparison with Existing Methods: The research provides a detailed comparison between the newly introduced methods and existing approaches based on ideals, illustrating the unique advantages and limitations of each. This contribution helps to clarify the distinctiveness and utility of the primal-based methods in different contexts.

2. Fundamental Ideas

This part exhibits the master ideas about primal, κ-neighborhood, and supra topological spaces cited in [9, 12, 13, 35, 56, 73].

Definition 1. *[9] Let $\mathcal{V} \neq \emptyset$. A class $\mathcal{P} \subseteq 2^{\mathcal{V}}$ is named a primal on $\mathcal{V}$, if it satisfies the next conditions:*

(i) $\mathcal{V} \notin \mathcal{P}$,

(ii) $O \notin \mathcal{P}$ *and* $O \subseteq H \Rightarrow H \notin \mathcal{P}$,

(iii) $H \notin \mathcal{P}$ *and* $O \notin \mathcal{P} \Rightarrow H \cap O \notin \mathcal{P}$.

Lemma 1. *Let $\mathcal{P}$ be a primal on $\mathcal{V}$. $H \notin \mathcal{P}$ and $O \notin \mathcal{P} \Leftrightarrow H \cap O \notin \mathcal{P}$.*

Proof. By Definition 1, the proof is obvious.

Remark 1. *The class $\mathcal{P} = \{\emptyset\}$ does not constitute a primal on any set. For example, consider $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}\}$. In this case, we have $\{\mathfrak{a}\} \cap \{\mathfrak{b}\} = \emptyset \in \mathcal{P}$, despite the fact that $\{\mathfrak{a}\} \notin \mathcal{P}$ and $\{\mathfrak{b}\} \notin \mathcal{P}$.*

Remark 2. *When the universe consists of a single element, such as $\mathcal{V} = \{\mathfrak{a}\}$, it is possible to construct an ideal; however, constructing a primal is not feasible. To construct a primal, the set $\mathcal{V}$ must contain at least two distinct elements.*

Remark 3. *[9] The collection of sets obtained through the intersection (or union) of elements from two primal does not necessarily form a primal on $\mathcal{V}$, as demonstrated in the next example:*

Example 1. *Let $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{a, b, d\}\}$, and $\tilde{\mathcal{P}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ be two primals on $\mathcal{V} = \{a, b, c, d\}$. Then*

(i) $\mathcal{P} \cup \tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$ *is a primal.*

(ii) The family $\nabla = \{A \cap B : A \in \mathcal{P}, B \in \tilde{\mathcal{P}}\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ *is not a primal on* $\mathcal{V}$ *since* $\{a, b, c\} \cap \{a, b, d\} = \{a, b\} \in \nabla$ *but neither* $\{a, b, c\} \in \nabla$ *nor* $\{a, b, d\} \in \nabla$.

(iii) The family $\triangle = \{A \cup B : A \in \mathcal{P}, B \in \tilde{\mathcal{P}}\} = 2^{\mathcal{V}}$ *is not a primal on* $\mathcal{V}$*, since* $\mathcal{V} \in \triangle$.

Definition 2. *[56] Let $\mathcal{S}$ be a supra-topology on $\mathcal{V}$. A set $H \subseteq \mathcal{V}$ is called supra-open (resp. supra-closed) if it is a member of $\mathcal{S}$ (resp. its complement belongs to $\mathcal{S}$). The supra-interior points of a set H, denoted by $\mathcal{S}int(H)$, is defined as a union of all supra-open subsets of this set. Also, the supra-closure points of a H, denoted by $\mathcal{S}cl(H)$, is defined as an intersection of all supra-closed supersets of this set.*

Various types of neighborhoods in a set $\mathcal{V}$ based on a binary relation $\mathcal{R}$ were defined. These neighborhoods are determined by different ways in which elements of $\mathcal{V}$ relate to each other according to $\mathcal{R}$. Here's an explanation of the different neighborhoods:

Definition 3. *[12, 13, 35, 49, 73] If $\mathcal{R}$ is a binary relation on $\mathcal{V}$. Then, κ-neighborhoods of $y \in \mathcal{V}$ (briefly, $N_\kappa(y)$), for various choices of κ ($\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$) is defined as follows:*

(i) r-neighborhood: $N_r(y) = \{z \in \mathcal{V} : y\mathcal{R}z\}$.

(ii) l-neighborhood: $N_l(y) = \{z \in \mathcal{V} : z\mathcal{R}y\}$.

(iii) i-neighborhood: $N_i(y) = N_r(y) \cap N_l(y)$.

(iv) u-neighborhood: $N_u(y) = N_r(y) \cup N_l(y)$.

(v) $\langle r\rangle$*-neighborhood:* $N_{\langle r\rangle}(y) = \cap\{N_r(z) : y \in N_r(z)\}$ *provided that there exists* $N_r(z)$ *containing* y*. Otherwise,* $N_{\langle r\rangle}(y) = \emptyset$.

(vi) *$\langle l\rangle$-neighborhood:* $N_{\langle l\rangle}(y) = \cap\{N_l(z) : y{\in}N_l(z)\}$ *provided that there exists* $N_l(z)$ *containing* y*. Otherwise,* $N_{\langle l\rangle}(y) = \emptyset$*.*

(vii) *$\langle i\rangle$-neighborhood:* $N_{\langle i\rangle}(y) = N_{\langle r\rangle}(y) \cap N_{\langle l\rangle}(y)$*.*

(viii) *$\langle u\rangle$-neighborhood:* $N_{\langle u\rangle}(y) = N_{\langle r\rangle}(y) \cup N_{\langle l\rangle}(y)$*.*

Henceforward, κ, $\forall\ \kappa{\in}\{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, will be treated, unless otherwise noted.

Definition 4. *[35] Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$, and ζ_κ: $\mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be a mapping which designates for each y in $\mathcal{V}$ its κ-neighborhood in $2^{\mathcal{V}}$. Then $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ is named a κ-neighborhood space (κ-NS).*

Proposition 1. *[6, 22, 31, 43, 71, 72] Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS and $\lambda \in \mathcal{V}$. Then,*

(i) $\lambda \in N_\kappa(\lambda)$*, i. e.* $N_\kappa(\lambda) \neq \emptyset$*, for each* κ*, if* $\mathcal{R}$ *is a reflexive relation.*

(ii) $N_{\langle\kappa\rangle}(\lambda) \subseteq N_\kappa(\lambda)$*,* $\kappa{\in}\{r, l, i, u\}$*, if* $\mathcal{R}$ *is a reflexive relation.*

(iii) $N_r(\lambda) = N_l(\lambda) = N_i(\lambda) = N_u(\lambda)$ *and* $N_{\langle r\rangle}(\lambda) = N_{\langle l\rangle}(\lambda) = N_{\langle i\rangle}(\lambda) = N_{\langle u\rangle}(\lambda)$*, if* $\mathcal{R}$ *is a symmetric relation.*

(iv) $N_{\langle\kappa\rangle}(\lambda) = N_\kappa(\lambda)$*,* $\kappa{\in}\{r, l, i, u\}$*, if* $\mathcal{R}$ *is a preorder (reflexive and transitive) relation.*

Theorem 1. *[35] Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and let $H \subseteq \mathcal{V}$. Then, for each κ, the class $\tau_\kappa = \{H \subseteq \mathcal{V} : \forall y \in H, N_\kappa(y) \subseteq H\}$ is a topology on $\mathcal{V}$.*

Definition 5. *[35] Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS. A set $H \subseteq \mathcal{V}$ is called a τ_κ-open set if $H \in \tau_\kappa$, and its complement is called a τ_κ-closed set. The family Υ_κ of all τ_κ-closed sets is defined as*

$$\Upsilon_\kappa = \{E \subseteq \mathcal{V} : E^c \in \tau_\kappa\},$$

where E^c is the complement of E.

Definition 6. *[35] Let τ_κ be a topology on $\mathcal{V}$ generated by κ-NS. Then the κ-lower, κ-upper approximations, κ-boundary and κ-accuracy of a subset $H \subseteq \mathcal{V}$ are defined respectively for each κ as:*

(i) $\tau_\kappa L(H) = \tau_\kappa int(H)$*, where* $\tau_\kappa int(H)$ *represents interior of* H *w.r.t.* τ_κ*.*

(ii) $\tau_\kappa U(H) = \tau_\kappa cl(H)$*, where* $\tau_\kappa cl(H)$ *represents closure of* H *w.r.t.* τ_κ*.*

(iii) $\tau_\kappa \mathcal{B}(H) = \tau_\kappa U(H) - \tau_\kappa L(H)$*.*

(iv) $\tau_\kappa \sigma(H) = \frac{|\tau_\kappa L(H)|}{|\tau_\kappa U(H)|}$*, where* $|\tau_\kappa U(H)| {\neq} 0$*.*

In [41] (which were corrected by R. Hosny et al. in [44]), Hosny used the method of [35] to define a new technique for generating different topologies via κ-NS and ideals, as illustrated in the following theorem.

Theorem 2. *[41, 44] Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and let $H \subseteq \mathcal{V}$. If $\mathcal{I}$ is an ideal on $\mathcal{V}$, then for each κ, the class*

$$\tau_\kappa^{\mathcal{I}} = \{H \subseteq \mathcal{V} : \forall y \in H, N_\kappa(y) - H \in \mathcal{I}\}$$

is a topology on $\mathcal{V}$.

Definition 7. *[41, 44] Let $\tau_\kappa^{\mathcal{I}}$ be a topology on $\mathcal{V}$ generated by κ-NS and ideal $\mathcal{I}$. Then, for every κ, the κ-lower, κ-upper approximations, and κ-accuracy of a set H are respectively given as:*

(i) $\tau_\kappa^{\mathcal{I}} L(H) = \tau_\kappa^{\mathcal{I}} int(H)$, *where* $\tau_\kappa^{\mathcal{I}} int(H)$ *represents interior of H w.r.t.* $\tau_\kappa^{\mathcal{I}}$.

(ii) $\tau_\kappa^{\mathcal{I}} U(H) = \tau_\kappa^{\mathcal{I}} cl(H)$, *where* $\tau_\kappa^{\mathcal{I}} cl(H)$ *represents closure of H w.r.t.* $\tau_\kappa^{\mathcal{I}}$.

(iii) $\tau_\kappa^{\mathcal{I}} \sigma(H) = \frac{|\tau_\kappa^{\mathcal{I}} L(H)|}{|\tau_\kappa^{\mathcal{I}} U(H)|}$, *where* $|\tau_\kappa^{\mathcal{I}} U(H)| \neq 0$.

In what follows, we explore some of the previous generations of rough sets.

Definition 8. $\mathcal{Y}$*ao* **approach** *[73] Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$. For each $\mathcal{G} \subseteq \mathcal{W}$, the $\mathcal{Y}$-lower (resp. $\mathcal{Y}$-upper) approximation, $\mathcal{Y}$-boundary region and $\mathcal{Y}$-accuracy of the approximations are defined as follows:*

$\underline{\mathcal{Y}}_r(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_r(\lambda) \subseteq \mathcal{G}\}$;

$\overline{\mathcal{Y}}_r(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_r(\lambda) \cap \mathcal{G} \neq \emptyset\}$;

$\mathcal{B}_{\mathcal{Y}}(\mathcal{G}) = \overline{\mathcal{Y}}_r(\mathcal{G}) - \underline{\mathcal{Y}}_r(\mathcal{G})$; *and*

$\sigma_{\mathcal{Y}}(\mathcal{G}) = \frac{|\underline{\mathcal{Y}}_r(\mathcal{G})|}{|\overline{\mathcal{Y}}_r(\mathcal{G})|}$, $\mid \overline{\mathcal{Y}}_r(\mathcal{G}) \mid \neq 0$.

Definition 9. $\mathcal{A}$*llam* ***et al.*** **approach** *[12, 13] Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$. For each $\mathcal{G} \subseteq \mathcal{V}$, the $\mathcal{A}$-lower (resp. $\mathcal{A}$-upper) approximation, $\mathcal{A}$-boundary region and $\mathcal{A}$-accuracy of the approximations are defined as follows:*

$\underline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_{\langle r \rangle}(\lambda) \subseteq \mathcal{G}\}$;

$\overline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_{\langle r \rangle}(\lambda) \cap \mathcal{G} \neq \emptyset\}$;

$\mathcal{B}_{\mathcal{A}}(\mathcal{G}) = \overline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G}) - \underline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G})$; *and*

$\sigma_{\mathcal{A}}(\mathcal{G}) = \frac{|\underline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G})|}{|\overline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G})|}$, $\mid \overline{\mathcal{A}}_{\langle r \rangle}(\mathcal{G}) \mid \neq 0$.

Definition 10. *[20] Suppose $\mathcal{R}$ is a binary relation on $\mathcal{V}$. For each $\lambda \in \mathcal{V}$, the maximal right neighborhood (shortened by $N_{\mathfrak{m}}$) is defined as follows:*

$$N_{\mathfrak{m}}(\lambda) = \cup_{\lambda \in N_r(\mu)} N_r(\mu).$$

Definition 11. $\mathcal{D}ai$ ***et al.*** **approach** *[20] Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$. For any subset $\mathcal{G} \subseteq \mathcal{V}$, the $\mathcal{D}$-lower (resp. $\mathcal{D}$-upper) approximation, $\mathcal{D}$-boundary region, and $\mathcal{D}$-accuracy of the approximations are defined as follows:*

$\underline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_{\mathfrak{m}}(\lambda) \subseteq \mathcal{G}\}$;

$\overline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G}) = \{\lambda \in \mathcal{V} : N_{\mathfrak{m}}(\lambda) \cap \mathcal{G} \neq \emptyset\}$;

$\mathcal{B}_{\mathcal{D}}(\mathcal{G}) = \overline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G}) - \underline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G})$; *and*

$\sigma_{\mathcal{D}}(\mathcal{G}) = \frac{|\underline{\mathcal{A}}_{\mathfrak{m}}(\mathcal{G})|}{|\overline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G})|}, \mid \overline{\mathcal{D}}_{\mathfrak{m}}(\mathcal{G}) \mid \neq 0.$

3. Approximations and Supra Topologies Generated by Different κ-NS and primal

The aim of this section is to present eight distinct supra-topologies generated from primal and κ-neighborhoods, where $\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. The relationships between these topologies will be examined, along with comparisons to highlight their differences. Additionally, new rough approximations derived from these topologies will be constructed, and some of their properties will be explored. It is evident that the best approximations and the highest accuracy measures are obtained for $\kappa \in \{i, \langle i\rangle\}$. Furthermore, the proposed approximations will be compared with previous ones from the literature [12, 20, 35, 73]. We will prove that primal play a key role in extending these methods.

The following result presents a method for generating κ-supra topology structures from N_κ-neighborhood systems and primal.

Theorem 3. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, $\mathcal{P}$ be a primal on $\mathcal{V}$, and $H \subseteq \mathcal{V}$. Then, for every κ, the class $\tau_\kappa^{\mathcal{P}} = \{H \subseteq \mathcal{V} : \forall z \in H, N_\kappa(z) - H \in \mathcal{P}\}$ is a supra topology on $\mathcal{V}$.*

Proof.

(i) Clearly $\mathcal{V}$ and $\emptyset$ belong to $\tau_\kappa^{\mathcal{P}}$.

(ii) Let $H_\alpha \in \tau_\kappa^{\mathcal{P}}$, $\alpha \in \Omega$ and $z \in \cup_{\alpha\in\Omega} H_\alpha$, then there exists $\alpha_0 \in \Omega$ s.t. $z \in H_{\alpha_0}$. Hence, $N_\kappa(z) - H_{\alpha_0} \in \mathcal{P}$. Since $-(\cup_{\alpha\in\Omega} H_\alpha) \subseteq -H_{\alpha_0}$, then $N_\kappa(z) - (\cup_{\alpha\in\Omega} H_\alpha) \in \mathcal{P}$ i.e $\cup_{\alpha\in\Omega} H_\alpha \in \tau_\kappa^{\mathcal{P}}$.
So, $\tau_\kappa^{\mathcal{P}}$ is a supra topology on $\mathcal{V}$.

The next example shows that $\tau_\kappa^{\mathcal{P}}$ need not be a topology.

Example 2. *Let $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, and consider the collection $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}\}$, which forms a primal on $\mathcal{V}$. Let $\mathcal{R} = \{(\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{b}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{c}, \mathfrak{b})\}$ be an arbitrary relation on $\mathcal{V}$. Then, the associated structure is given by:*

$$\tau_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}\}\}.$$

Notably, while $\{\mathfrak{a},\mathfrak{b}\},\{\mathfrak{a},\mathfrak{c}\} \in \tau_r^{\mathcal{P}}$*, their intersection* $\{\mathfrak{a},\mathfrak{b}\} \cap \{\mathfrak{a},\mathfrak{c}\} = \{\mathfrak{a}\} \notin \tau_r^{\mathcal{P}}$*, illustrating that* $\tau_r^{\mathcal{P}}$ *is not necessarily closed under intersections.*

$\tau_\kappa^{\mathcal{P}}$ is a collection of subsets of $\mathcal{V}$ that forms a supra topology. Its structure is determined by the properties of the κ-neighborhoods and the primal $\mathcal{P}$, making it a flexible framework for studying generalized topological spaces.

Remark 4. *The process of obtaining a supra-topological space yields distinctive outcomes. For instance, when applying for a position through a competitive process, various selection criteria are typically required. By employing the concept of supra-topology in the decision-making process, it becomes possible to eliminate shared or redundant conditions. This approach allows for a focus on the unique qualifications of each candidate, thereby reducing uncertainties associated with the final selection.*

Lemma 2. *Let* $\mathcal{V} \neq \emptyset$*. Then the following families are primal on* $\mathcal{V}$

(i) [9] $2^{\mathcal{V}} \setminus \{\mathcal{V}\}$ *(trivial primal).*

(ii) $\mathcal{P}_y = \{M \subseteq \mathcal{V} : y \notin M\}$ *(excluded point primal).*

(iii) $\mathcal{P}_O = \{M \subseteq \mathcal{V} : M \cup O \neq \mathcal{V}\}$.

Proof. Direct to prove.

Remark 5. *(i) If* $\mathcal{P} = \mathcal{P}_y$*, then the supra topology* $\tau_\kappa^{\mathcal{P}}$ *will consist of all subsets* $H \subseteq \mathcal{V}$ *such that for each* $z \in \mathcal{V}$*, the difference* $N_\kappa(z) - H$ *does not contain* y*. This implies that* y *will be excluded from certain neighborhoods that are needed to define the supra topology.*

(ii) The primal $\mathcal{P}_O$ *is dependent on the set* O*, where the choice of* O *directly affects for sets belong to* $\tau_\kappa^{\mathcal{P}}$*. When* O *is an empty set, then* $\mathcal{P}_O$ *includes all proper subsets of* $\mathcal{V}$*, which making* $\tau_\kappa^{\mathcal{P}}$ *potentially larger.*

The κ-supra topology $\tau_\kappa^{\mathcal{P}}$ is finer than the κ-topology τ_κ generated by N_κ-neighbourhood for various choices of κ. The present sort of these topologies are finer than the previous one [35] as it is exhibited in the next result.

Theorem 4. *Let* $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ *be a* κ*-NS, and* $\mathcal{P}$ *be a primal on* $\mathcal{V}$*. Then, for every* κ*,* $\tau_\kappa \subseteq \tau_\kappa^{\mathcal{P}}$*.*

Proof. Let $H \in \tau_\kappa$. Then, $N_\kappa(y) \subseteq H$, $\forall y \in H$ and so $N_\kappa(y) - H = \emptyset \in \mathcal{P}$, $\forall y \in H$. Therefore, $H \in \tau_\kappa^{\mathcal{P}}$. Hence, $\tau_\kappa \subseteq \tau_\kappa^{\mathcal{P}}$.

Remark 6. *According to Theorem 4, the present approaches are consider as extensions to Abd El-Monsef et al.'s results [35]. The converse of Theorem 4 doesn't hold as the next example exhibited.*

Example 3. *Continuation of Example 2. The following structures are obtained:*
$\tau_r = \{\emptyset, \mathcal{V}, \{\mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_l = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}\}$,
$\tau_i = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_u = \{\emptyset, \mathcal{V}\}$,
Additionally, we have:
$\tau_{\langle r\rangle} = \tau_{\langle i\rangle} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_{\langle l\rangle} = \tau_{\langle u\rangle} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}, \mathfrak{b}\}\}$.
For the primal structures, we obtain:
$\tau_r^{\mathcal{P}} = \tau_u^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_l^{\mathcal{P}} = \tau_i^{\mathcal{P}} = \tau_{\langle r\rangle}^{\mathcal{P}} = \tau_{\langle l\rangle}^{\mathcal{P}} = \tau_{\langle i\rangle}^{\mathcal{P}} = \tau_{\langle u\rangle}^{\mathcal{P}} = 2^{\mathcal{V}}$.

These results illustrate the structural differences between various topological and primal frameworks.

Definition 12. *The system $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$ is called a κ-supra topological space, where $\tau_\kappa^{\mathcal{P}}$ is a supra topology on $\mathcal{V}$. A set H of $\mathcal{V}$ is called a $\tau_\kappa^{\mathcal{P}}$open set if $H \in \tau_\kappa^{\mathcal{P}}$, and its complement is termed a $\tau_\kappa^{\mathcal{P}}$-closed set. The family $\Upsilon_\kappa^{\mathcal{P}}$ of all $\tau_\kappa^{\mathcal{P}}$ closed sets is given by $\Upsilon_\kappa^{\mathcal{P}} = \{F \subseteq \mathcal{V} : F^c \in \tau_\kappa^{\mathcal{P}}\}$, where F^c is the complement of F.*

According to Proposition 1, the proof of the next theorem is explicit.

Theorem 5. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and $\mathcal{P}$ be a primal on $\mathcal{V}$. Then,*

(i) $\tau_\kappa^{\mathcal{P}} \subseteq \tau_{\langle\kappa\rangle}^{\mathcal{P}}$, $\kappa \in \{r, l, i, u\}$, *if $\mathcal{R}$ is a reflexive relation.*

(ii) $\tau_r^{\mathcal{P}} = \tau_l^{\mathcal{P}} = \tau_i^{\mathcal{P}} = \tau_u^{\mathcal{P}}$ *and* $\tau_{\langle r\rangle}^{\mathcal{P}} = \tau_{\langle l\rangle}^{\mathcal{P}} = \tau_{\langle i\rangle}^{\mathcal{P}} = \tau_{\langle u\rangle}^{\mathcal{P}}$, *if $\mathcal{R}$ is a symmetric relation.*

(iii) $\tau_\kappa^{\mathcal{P}} = \tau_{\langle\kappa\rangle}^{\mathcal{P}}$, $\kappa \in \{r, l, i, u\}$, *if $\mathcal{R}$ is a preorder relation.*

Remark 7. *$\tau_{\langle\kappa\rangle}^{\mathcal{P}}, \tau_\kappa^{\mathcal{P}}$ are generally incomparable for $\kappa \in \{r, l, i, u\}$ when $\mathcal{R}$ is a transitive relation as illustrated in the next example:*

Example 4. *Let $R = \{(\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{b}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{d}, \mathfrak{d})\}$ be a transitive relation on $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$. If $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}\}$, then $\tau_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{c}\}, \{\mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}\}$, $\tau_{\langle r\rangle}^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}\}$.*

Example 5. *Let $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ and $R = \{(\mathfrak{a}, \mathfrak{a}), (\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{c}, \mathfrak{d})\}$. Hence,*
$\tau_r = \{\emptyset, \mathcal{V}, \{\mathfrak{d}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}\}$,
$\tau_l = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_i = 2^{\mathcal{V}}$,
$\tau_u = \{\emptyset, \mathcal{V}\}$,
$\tau_{\langle r\rangle} = \tau_{\langle u\rangle} = \{\emptyset, \mathcal{V}, \{\mathfrak{c}\}, \{\mathfrak{d}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}\}$,
$\tau_{\langle l\rangle} = \tau_{\langle i\rangle} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{c}\}, \{\mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{d}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}\}$.
If $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}\}$, then

$\tau_u^{\mathcal{P}} = \tau_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b}\}, \{\mathfrak{a},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c},\mathfrak{d}\}\}$,
$\tau_i^{\mathcal{P}} = \tau_l^{\mathcal{P}} = 2^{\mathcal{V}}$,
$\tau_{\langle r\rangle}^{\mathcal{P}} = \tau_{\langle l\rangle}^{\mathcal{P}} = \tau_{\langle i\rangle}^{\mathcal{P}} = \tau_{\langle u\rangle}^{\mathcal{P}} = 2^{\mathcal{V}}$.
If $\tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c},\mathfrak{d}\}\}$, *then*
$\tau_i^{\tilde{\mathcal{P}}} = \tau_l^{\tilde{\mathcal{P}}} = \tau_r^{\tilde{\mathcal{P}}} = 2^{\mathcal{V}}$,
$\tau_u^{\tilde{\mathcal{P}}} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b}\}, \{\mathfrak{a},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c}\}, \{\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c},\mathfrak{d}\}\}$,
$\tau_{\langle r\rangle}^{\mathcal{P}} = \tau_{\langle l\rangle}^{\tilde{\mathcal{P}}} = \tau_{\langle i\rangle}^{\tilde{\mathcal{P}}} = \tau_{\langle u\rangle}^{\tilde{\mathcal{P}}} = 2^{\mathcal{V}}$.

Remark 8. *If* $N_\kappa(z) \in \mathcal{P}\ \forall z \in H$, *then it follows that* $H \in \tau_\kappa^{\mathcal{P}}$. *However, as illustrated in Example 5, the converse does not necessarily hold. Specifically, we observe that* $\{\mathfrak{d}\} \in \tau_{\langle r\rangle}^{\mathcal{P}}$, *while* $N_{\langle r\rangle}(\{\mathfrak{d}\}) = \{\mathfrak{d}\} \notin \mathcal{P}$, *demonstrating a counterexample to the converse implication.*

Proposition 2. *Let* $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ *be a* κ*-NS and* $\mathcal{P}$ *be a primal on* $\mathcal{V}$. *Then the following results hold.*

(i) $\tau_u^{\mathcal{P}} \subseteq \tau_r^{\mathcal{P}} \cap \tau_l^{\mathcal{P}} \subseteq \tau_r^{\mathcal{P}} \cup \tau_l^{\mathcal{P}} \subseteq \tau_i^{\mathcal{P}}$.

(ii) $\tau_{\langle u\rangle}^{\mathcal{P}} \subseteq \tau_{\langle r\rangle}^{\mathcal{P}} \cap \tau_{\langle l\rangle}^{\mathcal{P}} \subseteq \tau_{\langle r\rangle}^{\mathcal{P}} \cup \tau_{\langle l\rangle}^{\mathcal{P}} \subseteq \tau_{\langle i\rangle}^{\mathcal{P}}$.

Proof. Suppose $\varepsilon = r$ or l. Since $N_i(x) \subseteq N_\varepsilon(x) \subseteq N_u(x)$ and $N_{\langle i\rangle}(x) \subseteq N_{\langle \varepsilon\rangle}(x) \subseteq N_{\langle u\rangle}(\mathrm{x})$, for each $x \in \mathcal{V}$ then the proof is evident.

Example 5 shows that the converse of item 1 of Proposition 2 needs not to be true.

Proposition 3. *Let* $\mathcal{P}$ *and* $\tilde{\mathcal{P}}$ *be two primals on* κ*-supra topological spaces* $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$ *and* $(\mathcal{V}, \tau_\kappa^{\tilde{\mathcal{P}}})$, *respectively, where* $\mathcal{P} \subseteq \tilde{\mathcal{P}}$. *Then, for any* κ, *it follows that* $\tau_\kappa^{\mathcal{P}} \subseteq \tau_\kappa^{\tilde{\mathcal{P}}}$.

Proof. Direct to prove.

Remark 9. *According to Example 5, the converse of Proposition 3 does not necessarily hold. Consider the primals*

$$\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a},\mathfrak{b}\}, \{\mathfrak{b},\mathfrak{c}\}, \{\mathfrak{a},\mathfrak{c}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}\}, \text{ and}$$

$$\tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c},\mathfrak{d}\}\} \text{ on } \mathcal{V}.$$

Suppose $\kappa = r$. *Then, we have*
$\tau_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b}\}, \{\mathfrak{a},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{b},\mathfrak{d}\}, \{\mathfrak{a},\mathfrak{c},\mathfrak{d}\}, \{\mathfrak{b},\mathfrak{c},\mathfrak{d}\}\}$,
while $\tau_r^{\tilde{\mathcal{P}}} = 2^{\mathcal{V}}$. *Thus, it follows that* $\tau_r^{\tilde{\mathcal{P}}} \nsubseteq \tau_r^{\mathcal{P}}$.

Definition 13. *Let* $\tau_\kappa^{\mathcal{P}}$ *be a* κ*-supra topology generated by* κ*-NS and primal* $\mathcal{P}$. *Then* $\tau_\kappa^{\mathcal{P}} int(H)$, $\tau_\kappa^{\mathcal{P}} cl(H)$ *of a set* H *are assigned respectively for each* κ *as:*

$$\tau_\kappa^{\mathcal{P}} int(H) = \cup\{W \in \tau_\kappa^{\mathcal{P}} : W \subseteq H\},$$

$$\tau_\kappa^{\mathcal{P}} cl(H) = \cap\{F \in \Upsilon_\kappa^{\mathcal{P}} : H \subseteq F\}.$$

The behavior of $\tau_\kappa^{\mathcal{P}} L(.)$ and $\tau_\kappa^{\mathcal{P}} U(.)$ can vary significantly based on the choice of κ and the primal set $\mathcal{P}$. Different selections of κ-neighborhood systems and primal structure can lead to varying levels of granularity in the approximations, thus tailoring the approach to specific contexts or applications. These properties help describe how the lower and upper approximations interact within the κ-supra topology, providing a framework for dealing with uncertainty and approximate reasoning.

In the next, different kinds of rough approximations utilizing the topologies generated from κ-NS and primal $\mathcal{P}$ will be established. In addition, some of their properties will be studied.

Definition 14. *Let $\tau_\kappa^{\mathcal{P}}$ be a κ-supra topology generated by κ-NS and primal $\mathcal{P}$. Then, the $\mathcal{P}_\kappa$-lower, $\mathcal{P}_\kappa$-upper approximations, $\mathcal{P}_\kappa$-boundary and $\mathcal{P}_\kappa$-accuracy of a subset $H \subseteq \mathcal{V}$ are assigned respectively for each κ as:*

(i) $\tau_\kappa^{\mathcal{P}} L(H) = \tau_\kappa^{\mathcal{P}} int(H)$.

(ii) $\tau_\kappa^{\mathcal{P}} U(H) = \tau_\kappa^{\mathcal{P}} cl(H)$.

(iii) $\tau_\kappa^{\mathcal{P}} \mathcal{B}(H) = \tau_\kappa^{\mathcal{P}} U(H) - \tau_\kappa^{\mathcal{P}} L(H)$.

(iv) $\tau_\kappa^{\mathcal{P}} \sigma(H) = \frac{|\tau_\kappa^{\mathcal{P}} L(H)|}{|\tau_\kappa^{\mathcal{P}} U(H)|}$, *where* $|\tau_\kappa^{\mathcal{P}} U(H)| \neq 0$.

It is evident that $0 \leq \tau_\kappa^{\mathcal{P}} \sigma(H) \leq 1$. If $\tau_\kappa^{\mathcal{P}} \sigma$ is close to 1, it implies that the lower and upper bounds are of similar magnitude, suggesting that the estimates are close, leading to higher accuracy. If $\tau_\kappa^{\mathcal{P}} \sigma(H) = 1$, then H is referred to as an $\tau_\kappa^{\mathcal{P}} \sigma$-exact set. Elsewise, H is termed an $\tau_\kappa^{\mathcal{P}} \sigma$-rough set.

Definition 15. *A subset H of κ-supra topological space $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$ is called:*

(i) Totally $\mathcal{P}_\kappa$-definable, if $\tau_\kappa^{\mathcal{P}} L(H) = H = \tau_\kappa^{\mathcal{P}} U(H)$. This means that every element of H can be precisely defined and no ambiguity exists about whether an element belongs to the set H.

(ii) Internally $\mathcal{P}_\kappa$-definable, if $\tau_\kappa^{\mathcal{P}} L(H) = H$ and $\tau_\kappa^{\mathcal{P}} U(H) \neq H$. This means that the set is fully described by its "core" elements.

(iii) Externally $\mathcal{P}_\kappa$-definable, if $\tau_\kappa^{\mathcal{P}} L(H) \neq H$ and $\tau_\kappa^{\mathcal{P}} U(H) = H$. This means that there is some uncertainty about which elements are strictly part of the set, but it is certain that H includes all elements from the upper approximation

(iv) $\mathcal{P}_\kappa$-rough set, if $\tau_\kappa^{\mathcal{P}} L(H) \neq H$ and $\tau_\kappa^{\mathcal{P}} U(H) \neq H$. This means that there are elements that are not fully classified as either definitely in or definitely out of the set.

Remark 10. *According to Example 5, then*

(i) Totally $\mathcal{P}_r$-definable sets are $\{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}, \emptyset$, and $\mathcal{V}$.

(ii) Internally $\mathcal{P}_r$-definable sets are $\{\mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{d}\}$, *and* $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}$.

(iii) Externally $\mathcal{P}_r$-definable sets are $\{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}\}$, *and* $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$.

(iv) There is no any $\mathcal{P}_r$-rough set.

Proposition 4. *Let H be a subset of κ-supra topological space $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$. For each κ, the following statements hold:*

(i) $\tau_\kappa L(H) \subseteq \tau_\kappa^{\mathcal{P}} L(H)$.

(ii) $\tau_\kappa^{\mathcal{P}} U(H) \subseteq \tau_\kappa U(H)$.

(iii) $\tau_\kappa \sigma(H) \leq \tau_\kappa^{\mathcal{P}} \sigma(H)$.

Proof. In view of Theorem 4, the proof is clear.

Proposition 5. *Let H be a subset of κ-supra topological space $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$. For each κ, the following statements hold:*

(i) If $H \in \tau_\kappa^{\mathcal{P}}$, *then* $\tau_\kappa^{\mathcal{P}} L(H) = H$.

(ii) If $H \in \Upsilon_\kappa^{\mathcal{P}}$, *then* $\tau_\kappa^{\mathcal{P}} U(H) = H$.

The next proposition which explains the main properties of $\tau_\kappa^{\mathcal{P}} L(.)$, $\tau_\kappa^{\mathcal{P}} U(.)$ operators are intelligible, by observing that $\tau_\kappa^{\mathcal{P}} int(.), \tau_\kappa^{\mathcal{P}} cl(H)$ achieve all characteristics of the interior and closure operators of the topology, respectively.

Proposition 6. *Let $H, \acute{H}$ be subsets of κ-supra topological space $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$. Then, the following properties hold for each κ.*

(L1) $\tau_\kappa^{\mathcal{P}} L(H) \subseteq H$	(U1) $H \subseteq \tau_\kappa^{\mathcal{P}} U(H)$
(L2) $\tau_\kappa^{\mathcal{P}} L(\emptyset) = \emptyset$	(U2) $\tau_\kappa^{\mathcal{P}} U(\emptyset) = \emptyset$
(L3) $\tau_\kappa^{\mathcal{P}} L(\mathcal{V}) = \mathcal{V}$	(U3) $\tau_\kappa^{\mathcal{P}} U(\mathcal{V}) = \mathcal{V}$
(L4) $If\ H \subseteq \acute{H},\ then\ \tau_\kappa^{\mathcal{P}} L(H) \subseteq \tau_\kappa^{\mathcal{P}} L(\acute{H})$	(U4) $If\ H \subseteq \acute{H},\ then\ \tau_\kappa^{\mathcal{P}} U(H) \subseteq \tau_\kappa^{\mathcal{P}} U(\acute{H})$
(L5) $\tau_\kappa^{\mathcal{P}} L(H \cap \acute{H}) = \tau_\kappa^{\mathcal{P}} L(H) \cap \tau_\kappa^{\mathcal{P}} L(\acute{H})$	(U5) $\tau_\kappa^{\mathcal{P}} U(H \cap \acute{H}) \subseteq \tau_\kappa^{\mathcal{P}} U(H) \cap \tau_\kappa^{\mathcal{P}} U(\acute{H})$
(L6) $\tau_\kappa^{\mathcal{P}} L(H \cup \acute{H}) \supseteq \tau_\kappa^{\mathcal{P}} L(H) \cup \tau_\kappa^{\mathcal{P}} L(\acute{H})$	(U6) $\tau_\kappa^{\mathcal{P}} U(H \cup \acute{H}) = \tau_\kappa^{\mathcal{P}} U(H) \cup \tau_\kappa^{\mathcal{P}} U(\acute{H})$
(L7) $\tau_\kappa^{\mathcal{P}} L(H) = [\tau_\kappa^{\mathcal{P}} U(H^c)]^c$	(U7) $\tau_\kappa^{\mathcal{P}} U(H) = [\tau_\kappa^{\mathcal{P}} L(H^c)]^c$
(L8) $\tau_\kappa^{\mathcal{P}} L(\tau_\kappa^{\mathcal{P}} L(H)) = \tau_\kappa^{\mathcal{P}} L(H)$	(U8) $\tau_\kappa^{\mathcal{P}} U(\tau_\kappa^{\mathcal{P}} U(H)) = \tau_\kappa^{\mathcal{P}} U(H)$.

Remark 11. *Example 5 demonstrates that the converse of* (L1), (L4), (L6) *(resp.* (U1), (U4), (U5)*) for Proposition 6 fails in general.*

*(*L1*) Suppose that $\kappa = r$. Let $H = \{\mathfrak{c}\}$, then $\tau_\kappa^{\mathcal{P}} L(H) = \emptyset$ and so $H \nsubseteq \tau_\kappa^{\mathcal{P}} L(H)$,*

(L4) *Suppose that* $\kappa = r$. *Let* $H = \{\mathfrak{b}, \mathfrak{c}\}, \acute{H} = \{\mathfrak{a}, \mathfrak{b}\}$, *then* $\tau_\kappa^{\mathcal{P}} L(H) = \{\mathfrak{b}\}$, *and* $\tau_\kappa^{\mathcal{P}} L(\acute{H}) = \{\mathfrak{a}, \mathfrak{b}\}$. *Then* $\tau_\kappa^{\mathcal{P}} L(H) \subseteq \tau_\kappa^{\mathcal{P}} L(\acute{H})$, *and* $H \nsubseteq \acute{H}$,

(L6) *Suppose that* $\kappa = r$. *Let* $H = \{\mathfrak{a}, \mathfrak{c}\}, \acute{H} = \{\mathfrak{a}, \mathfrak{d}\}$, *then* $\tau_\kappa^{\mathcal{P}} L(H) = \{\mathfrak{a}\}$, *and* $\tau_\kappa^{\mathcal{P}} L(\acute{H}) = \{\mathfrak{a}, \mathfrak{d}\}$. *So,* $\tau_\kappa^{\mathcal{P}} L(H \cup \acute{H}) = \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$, $\tau_\kappa^{\mathcal{P}} L(H) \cup \tau_\kappa^{\mathcal{P}} L(\acute{H}) = \{\mathfrak{a}, \mathfrak{d}\}$. *Then* $\tau_\kappa^{\mathcal{P}} L(H \cup \acute{H}) \nsubseteq \tau_\kappa^{\mathcal{P}} L(H) \cup \tau_\kappa^{\mathcal{P}} L(\acute{H})$.

Their specific characteristics and relationships may be influenced by the nature of the primal $\mathcal{P}$ and the κ-neighborhoods, giving rise to unique topological structures.

Proposition 7. *Let* $(\mathcal{V}, \mathcal{R}, \mathcal{P})$ *be a primal approximation space. If* $\mathcal{R}$ *is a preorder relation, then for any* $w \in \mathcal{V}$: $\tau_\kappa^{\mathcal{P}} L(N\kappa(w)) = N_\kappa(w)$, *for each* $\kappa \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle, u, \langle u\rangle\}$.

Proof. According to [6], since $\mathcal{R}$ is preorder, it follows that $N_\kappa(w) \in \tau_\kappa$. Therefore, $N_\kappa(w) \in \tau_\kappa^{\mathcal{P}}$, which implies that $\tau_\kappa^{\mathcal{P}} L(N_\kappa(w)) = N_\kappa(w)$ for each $\kappa \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle, u, \langle u\rangle\}$.

Remark 12. *According to Example 5, it is observed that the preorder conditions in Proposition 7 are indeed strict. Let* $w = \mathfrak{a}$ *and* $\kappa = r$. *Then* $N_r(\mathfrak{a}) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, *which implies that* $\tau_r^{\mathcal{P}} L(N_r(\mathfrak{a})) = \{\mathfrak{a}, \mathfrak{b}\}$. *Therefore,* $\tau_r^{\mathcal{P}} L(N_r(\mathfrak{a})) \neq N_r(\mathfrak{a})$.

The next propositions are straightforward, and their proofs are omitted for brevity.

Proposition 8. *Consider* $\tau_\kappa^{\mathcal{P}}$ *is a* κ*-supra topology generated by* κ*-NS and primal* $\mathcal{P}$. *If* $H \subseteq \mathcal{V}$ *and* $z \in \mathcal{V}$, *then the subsequent items hold:*

(i) $\tau_u^{\mathcal{P}} L(H) \subseteq \tau_r^{\mathcal{P}} L(H) \subseteq \tau_i^{\mathcal{P}} L(H)$ *and* $\tau_u^{\mathcal{P}} L(H) \subseteq \tau_l^{\mathcal{P}} L(H) \subseteq \tau_i^{\mathcal{P}} L(H)$.

(ii) $\tau_i^{\mathcal{P}} U(H) \subseteq \tau_r^{\mathcal{P}} U(H) \subseteq \tau_u^{\mathcal{P}} U(H)$ *and* $\tau_i^{\mathcal{P}} U(H) \subseteq \tau_l^{\mathcal{P}} U(H) \subseteq \tau_u^{\mathcal{P}} U(H)$.

(iii) $\tau_u^{\mathcal{P}} \sigma(H) \leq \tau_r^{\mathcal{P}} \sigma(H) \leq \tau_i^{\mathcal{P}} \sigma(H)$ *and* $\tau_u^{\mathcal{P}} \sigma(H) \leq \tau_l^{\mathcal{P}} \sigma(H) \leq \tau_i^{\mathcal{P}} \sigma(H)$.

(iv) $\tau_{\langle u\rangle}^{\mathcal{P}} L(H) \subseteq \tau_{\langle r\rangle}^{\mathcal{P}} L(H) \subseteq \tau_{\langle i\rangle}^{\mathcal{P}} L(H)$ *and* $\tau_{\langle u\rangle}^{\mathcal{P}} L(H) \subseteq \tau_{\langle l\rangle}^{\mathcal{P}} L(H) \subseteq \tau_{\langle i\rangle}^{\mathcal{P}} L(H)$.

(v) $\tau_{\langle i\rangle}^{\mathcal{P}} U(H) \subseteq \tau_{\langle r\rangle}^{\mathcal{P}} U(H) \subseteq \tau_{\langle u\rangle}^{\mathcal{P}} U(H)$ *and* $\tau_{\langle i\rangle}^{\mathcal{P}} U(H) \subseteq \tau_{\langle l\rangle}^{\mathcal{P}} U(H) \subseteq \tau_{\langle u\rangle}^{\mathcal{P}} U(H)$.

(vi) $\tau_{\langle u\rangle}^{\mathcal{P}} \sigma(H) \leq \tau_{\langle r\rangle}^{\mathcal{P}} \sigma(H) \leq \tau_{\langle i\rangle}^{\mathcal{P}} \sigma(H)$ *and* $\tau_{\langle u\rangle}^{\mathcal{P}} \sigma(H) \leq \tau_{\langle l\rangle}^{\mathcal{P}} \sigma(H) \leq \tau_{\langle i\rangle}^{\mathcal{P}} \sigma(H)$.

Theorem 6. *Let* $\mathcal{P}$ *and* $\tilde{\mathcal{P}}$ *be two primals on two* κ*-supra topological spaces* $(\mathcal{V}, \tau_\kappa^{\mathcal{P}})$ *and* $(\mathcal{V}, \tau_\kappa^{\tilde{\mathcal{P}}})$, *respectively, with* $\mathcal{P} \subseteq \tilde{\mathcal{P}}$. *If* $H \subseteq \mathcal{V}$, *then the following statements hold:*

(i) $\tau_\kappa^{\mathcal{P}} L(H) \subseteq \tau_\kappa^{\tilde{\mathcal{P}}} L(H)$,

(ii) $\tau_\kappa^{\mathcal{P}} U(H) \supseteq \tau_\kappa^{\tilde{\mathcal{P}}} U(H)$,

(iii) $\tau_\kappa^{\mathcal{P}} \sigma(H) \leq \tau_\kappa^{\tilde{\mathcal{P}}} \sigma(H)$.

Remark 13. *Example 5 demonstrates that the converse of Theorem 6 fails in general.*

(i) Suppose that $\kappa = r$*. Let* $H = \{\mathfrak{c}\}$*, then* $\tau_\kappa^{\mathcal{P}} L(H) = \emptyset$*,* $\tau_\kappa^{\tilde{\mathcal{P}}} L(H) = \{\mathfrak{c}\}$ *and so* $\tau_\kappa^{\tilde{\mathcal{P}}} L(H) \nsubseteq \tau_\kappa^{\mathcal{P}} L(H)$*,*

(ii) Suppose that $\kappa = r$*. Let* $H = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}$*, then* $\tau_\kappa^{\mathcal{P}} U(H) = \mathcal{V}, \tau_\kappa^{\tilde{\mathcal{P}}} U(H) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}$*. Hence,* $\tau_\kappa^{\mathcal{P}} U(H) \nsubseteq \tau_\kappa^{\tilde{\mathcal{P}}} U(H)$ *,*

(iii) Suppose that $\kappa = r$*. Let* $H = \{\mathfrak{c}\}$*, then* $\tau_\kappa^{\mathcal{P}} \sigma(H) = 0, \tau_\kappa^{\tilde{\mathcal{P}}} \sigma(H) = 1$ *Hence,* $\tau_\kappa^{\tilde{\mathcal{P}}} \sigma(H) \nleq \tau_\kappa^{\mathcal{P}} \sigma(H)$*.*

Example 6. *Let* $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$*, and consider the primal* $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}\}$*. If* $\mathcal{R}$ *is a binary relation on* $\mathcal{V}$ *defined as* $\mathcal{R} = \{(\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{b}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{c}, \mathfrak{b})\}$*, then the corresponding neighborhoods are determined as follows:*

Right neighborhoods	***Minimal neighborhoods***	***Maximal neighborhoods***
$N_r(\mathfrak{a}) = \{\mathfrak{b}, \mathfrak{c}\}$	$N_{\langle r\rangle}(\mathfrak{a}) = \emptyset$	$N_{\mathfrak{m}}(\mathfrak{a}) = \emptyset$
$N_r(\mathfrak{b}) = \{\mathfrak{b}, \mathfrak{c}\}$	$N_{\langle r\rangle}(\mathfrak{b}) = \{\mathfrak{b}\}$	$N_{\mathfrak{m}}(\mathfrak{b}) = \{\mathfrak{b}, \mathfrak{c}\}$
$N_r(\mathfrak{c}) = \{\mathfrak{b}\}$	$N_{\langle r\rangle}(\mathfrak{c}) = \{\mathfrak{b}, \mathfrak{c}\}$	$N_{\mathfrak{m}}(\mathfrak{c}) = \{\mathfrak{b}, \mathfrak{c}\}$

Hence, Table 1 explains comparative studies between the previous approaches and the proposed technique.

Table 1: (Comparisons of the boundary regions and the accuracy measures among different methods, whenever $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{c}\}\}$

	Yao		Allam		Dai		Abd El-Monsef		Current Method	
$\mathcal{G} \subseteq \mathcal{V}$	$\mathcal{B}_{\mathcal{Y}}(\mathcal{G})$	$\sigma_{\mathcal{Y}}(\mathcal{G})$	$\mathcal{B}_{\mathcal{A}}(\mathcal{G})$	$\sigma_{\mathcal{A}}(\mathcal{G})$	$\mathcal{B}_{\mathcal{D}}(\mathcal{G})$	$\sigma_{\mathcal{D}}(\mathcal{G})$	$\tau_\kappa \mathcal{B}(\mathcal{G})$	$\tau_\kappa \sigma(\mathcal{G})$	$\tau_\kappa^{\mathcal{P}} \mathcal{B}(\mathcal{G})$	$\tau_\kappa^{\mathcal{P}} \sigma(\mathcal{G})$
$\{\mathfrak{a}\}$	$\emptyset$	undef.	$\emptyset$	undef.	$\emptyset$	undef.	$\{\mathfrak{a}\}$	0	$\{\mathfrak{a}\}$	0
$\{\mathfrak{b}\}$	$\{\mathfrak{a}, \mathfrak{b}\}$	1/3	$\{\mathfrak{c}\}$	1	$\emptyset$	1/2	$\emptyset$	0	$\emptyset$	1
$\{\mathfrak{c}\}$	$\{\mathfrak{a}, \mathfrak{b}\}$	0	$\emptyset$	1	$\emptyset$	1/2	$\mathcal{V}$	0	$\emptyset$	1
$\{\mathfrak{a}, \mathfrak{b}\}$	$\{\mathfrak{a}, \mathfrak{b}\}$	1/3	$\{\mathfrak{c}\}$	1	$\{\mathfrak{b}, \mathfrak{c}\}$	1/2	$\emptyset$	0	$\emptyset$	1
$\{\mathfrak{a}, \mathfrak{c}\}$	$\{\mathfrak{a}, \mathfrak{b}\}$	0	$\emptyset$	1	$\{\mathfrak{b}, \mathfrak{c}\}$	0.5	$\emptyset$	0	$\emptyset$	1
$\{\mathfrak{b}, \mathfrak{c}\}$	$\emptyset$	1	$\emptyset$	1.5	$\{\mathfrak{a}\}$	3/2	$\{\mathfrak{a}\}$	2/3	$\{\mathfrak{a}\}$	2/3
$\mathcal{W}$	$\emptyset$	1	$\emptyset$	1.5	$\emptyset$	3/2	$\emptyset$	1	$\emptyset$	1
$\emptyset$	$\emptyset$	1	$\emptyset$	undef.	$\emptyset$	undef.	$\emptyset$	1	$\emptyset$	1

Note: "undef." refers to "undefined quantity" in Table 1

4. Bi-primal approximation spaces

This section introduces a novel type of approximation space referred to as "Bi-primal approximation spaces." These spaces are distinguished by the presence of two distinct primal sets, each offering a unique framework for approximating and estimating the properties of sets. The integration of these two primals allows for multiple analytical perspectives, giving a more flexible and comprehensive approach to approximation theory. The importance of this methodology lies in its banking on primal sets, which serve as essential

topological tools. By incorporating two primals, the framework enriches the analysis, transcending a singular viewpoint. Additionally, the constructed approximations of two primals are more accurate than those generated by a single primal, without affecting the basic properties of these approximations.
A key component of this structure is the primal set $\mathcal{P}$, where the specific selection of these primals directly influences the form of the supra topology. The union of the primal sets further enhances the flexibility of the structure, enabling the development of various kinds of supra topologies based on the chosen primal.

Definition 16. *The quadrable $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ is called a Bi-primal approximation space where $\mathcal{R}$ is a binary relation on $\mathcal{V}$, and $\mathcal{P}, \tilde{\mathcal{P}}$ are two primals on $\mathcal{V}$.*

Theorem 7. *Let $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ be a Bi-primal approximation space. Then, $\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}} = \tau_\kappa^{\mathcal{P}} \cup \tau_\kappa^{\tilde{\mathcal{P}}}$, $\forall\ \kappa$.*

Proof. Since $\mathcal{P} \subseteq \mathcal{P} \cup \tilde{\mathcal{P}}$, and $\tilde{\mathcal{P}} \subseteq \mathcal{P} \cup \tilde{\mathcal{P}}$, then by Proposition 3 $\tau_\kappa^{\mathcal{P}} \cup \tau_\kappa^{\tilde{\mathcal{P}}} \subseteq \tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}$. Let $H \in \tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}$. Then $N_\kappa(z) - H \in \mathcal{P} \cup \tilde{\mathcal{P}}$, $\forall z \in H$. Hence, $N_\kappa(z) - H \in \mathcal{P}$, $\forall z \in H$ or $N_\kappa(z) - H \in \tilde{\mathcal{P}}$, $\forall z \in H$. So $\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}} \subseteq \tau_\kappa^{\mathcal{P}} \cup \tau_\kappa^{\tilde{\mathcal{P}}}$. Consequently, $\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}} = \tau_\kappa^{\mathcal{P}} \cup \tau_\kappa^{\tilde{\mathcal{P}}}$.

Definition 17. *Let $\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}$ be a κ-supra topology generated by κ-NS and primals $\mathcal{P}, \tilde{\mathcal{P}}$. Then the lower and upper approximations, $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$, $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$ of a set H are assigned respectively for each κ as:*

$$L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \cup\{W \in \tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}} : W \subseteq H\},$$

$$U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \cap\{F \in \Upsilon_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}} : H \subseteq F\}.$$

Using Theorem 7, Definition 17 can be restated as follows:

Definition 18. *Let $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ be a Bi-primal approximation space. If $H \subseteq \mathcal{V}$, then the lower and upper approximations, $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$, $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$ are defined, respectively, as:*

$$L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \tau_\kappa^{\mathcal{P}} L(H) \cup \tau_\kappa^{\tilde{\mathcal{P}}} L(H)$$

$$U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \tau_\kappa^{\mathcal{P}} U(H) \cap \tau_\kappa^{\tilde{\mathcal{P}}} U(H)$$

Example 7. *Let $R = \{(\mathfrak{a}, \mathfrak{a}), (\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}, \mathfrak{c}), (\mathfrak{b}, \mathfrak{c}), (\mathfrak{c}, \mathfrak{d}), (\mathfrak{d}, \mathfrak{c})\}$ be an arbitrary relation on $\mathcal{V} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$.*

If $\mathcal{P} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}\}$, then
$\tau_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}\}$.

If $\tilde{\mathcal{P}} = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{c}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}\}$, then
$\tau_r^{\tilde{\mathcal{P}}} = \{\emptyset, \mathcal{V}, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}, \{\mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}\}$.

If $\mathcal{P} \cup \tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}, \{\mathfrak{c}, \mathfrak{d}\}\}$*, then* $\tau_r^{\mathcal{P}\cup\tilde{\mathcal{P}}} = 2^{\mathcal{V}}$.

Remark 14. *The present operators introduced in Definition 18 can be viewed as a real generalization of the operators offered in Definition 14. Because the two Definitions are coincide, if one of the following conditions is held :*

(i) $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ *or* $\mathcal{P} \supseteq \tilde{\mathcal{P}}$.

(ii) $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ *and* $\mathcal{P} \supseteq \tilde{\mathcal{P}}$ *i.e* $\mathcal{P} = \tilde{\mathcal{P}}$.

If $\tau_\kappa^{\mathcal{P}} L(H)$ (resp. $\tau_\kappa^{\mathcal{P}} U(H)$) and $\tau_\kappa^{\tilde{\mathcal{P}}} L(H)$ (resp. $\tau_\kappa^{\tilde{\mathcal{P}}} U(H)$) preserve certain structural properties of H (see Proposition 6), then $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$ (resp. $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$) might retain those properties.
According to Definition 18 and Proposition 6, the proof of the next proposition is simple.

Proposition 9. *Let* $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ *be a Bi-primal approximation space. If* $H, \acute{H}$ *are subsets of* $\mathcal{V}$*, then the following properties hold:*

(L1) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq H$

(L2) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\emptyset) = \emptyset$

(L3) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\mathcal{V}) = \mathcal{V}$

(L4) $If\ H \subseteq \acute{H},\ then\ L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(L5) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cap \acute{H}) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cap L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(L6) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cup \acute{H}) \supseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cup L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(L7) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = [U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H^c)]^c$

(L8) $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$

(U1) $H \subseteq U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$

(U2) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\emptyset) = \emptyset$

(U3) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\mathcal{V}) = \mathcal{V}$

(U4) $If\ H \subseteq \acute{H},\ then\ U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(U5) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cap \acute{H}) \subseteq U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cap U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(U6) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cup \acute{H}) \supseteq U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cup U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$

(U7) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = [L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H^c)]^c$

(U8) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)) \supseteq U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$.

Remark 15. *Example 5 demonstrates that the converse of* (L1)*,* (L4)*,* (L5)*,* (L6) *(resp.* (U1)*,* (U4)*,* (U5)*,* (U6)*) for Proposition 9 fails in general.*

*(*L1*) Suppose that* $\kappa = u$*. Let* $H = \{\mathfrak{a}, \mathfrak{c}\}$*, then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \{\mathfrak{a}\}$ *and so* $H \nsubseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$*,*

*(*L4*) Suppose that* $\kappa = u$*. Let* $H = \{\mathfrak{c}\}, \acute{H} = \{\mathfrak{b}\}$*, then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \emptyset$*, and* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H}) = \{\mathfrak{b}\}$*. Then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$*, and* $H \nsubseteq \acute{H}$*,*

*(*L5*) Suppose that* $\kappa = u$*. Let* $H = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}, \acute{H} = \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$*, then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$*,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H}) = \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$*,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cap L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H}) = \{\mathfrak{a}, \mathfrak{c}\}$ *and* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cap \acute{H}) = \{\mathfrak{a}\}$*. Hence,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cap \acute{H}) \neq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cap L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$*,*

*(*L6*) Suppose that* $\kappa = u$*. Let* $H = \{\mathfrak{b}\}, \acute{H} = \{\mathfrak{c}\}$*, then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \{\mathfrak{b}\}$*, and* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H}) = \emptyset$*. So,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cup \acute{H}) = \{\mathfrak{b}, \mathfrak{c}\}$*,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cup L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H}) = \{b\}$*. Then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H \cup \acute{H}) \neq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \cup L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(\acute{H})$*.*

Proposition 10. *Let* $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ *be a Bi-primal approximation space. If* $\mathcal{R}$ *is a preorder relation, then for any* $w \in \mathcal{V}$*:* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(N_\kappa(w)) = N_\kappa(w)$*, for each* $\kappa \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle, u, \langle u\rangle\}$.

Proof. According to Proposition 7, the proof is obvious.

Remark 16. *According to Example 5 illustrates that the conditions of reflexivity and transitivity in Proposition 7 are indeed strict. Let* $w = \mathfrak{b}$*. Then* $N_u(w) = \{\mathfrak{a}, \mathfrak{c}\}$*,* $L_u^{\mathcal{P}\cup\tilde{\mathcal{P}}}(N_u(w)) = \{\mathfrak{a}\}$*. Hence,* $N_u(w) \nsubseteq L_u^{\mathcal{P}\cup\tilde{\mathcal{P}}}(N_u(w))$.

Proposition 11. *Let* $(\mathcal{V}, \mathcal{R}, \mathcal{P}, \tilde{\mathcal{P}})$ *be a Bi-primal approximation space. If* H *is a subset of* $\mathcal{V}$*, then*

(i) $\tau_\kappa^{\mathcal{P}} L(H) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$ *and* $\tau_\kappa^{\tilde{\mathcal{P}}} L(H) \subseteq L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$.

(ii) $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq \tau_\kappa^{\mathcal{P}} U(H)$ *and* $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \subseteq \tau_\kappa^{\tilde{\mathcal{P}}} U(H)$.

(iii) $\tau_\kappa^{\mathcal{P}} \sigma(H) \leq \sigma_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$ *and* $\tau_\kappa^{\tilde{\mathcal{P}}} \sigma(H) \leq \sigma_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$.

Proof. Direct to prove.

Remark 17. *Example 5 demonstrates that the converse of Proposition 11 fails in general.*

(i) Suppose that $\kappa = u$*. Let* $H = \{\mathfrak{b}, \mathfrak{c}\}$*, then* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \{\mathfrak{b}, \mathfrak{c}\}, \tau_\kappa^{\mathcal{P}} L(H) = \{\mathfrak{b}\}$*. Hence,* $L_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \nsubseteq \tau_\kappa^{\mathcal{P}} L(H)$.

(ii) Suppose that $\kappa = u$*. Let* $H = \{\mathfrak{d}\}$*, then* $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = \{\mathfrak{d}\}, \tau_\kappa^{\mathcal{P}} U(H) = \{\mathfrak{c}, \mathfrak{d}\}$*. Hence,* $U_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) \nsupseteq \tau_\kappa^{\mathcal{P}} U(H)$.

(iii) Suppose that $\kappa = u$*. Let* $H = \{\mathfrak{d}\}$*, then* $\tau_\kappa^{\mathcal{P}} \sigma(H) = \frac{1}{2}, \sigma_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H) = 1$*. Hence,* $\tau_\kappa^{\mathcal{P}} \sigma(H) \ngeq \sigma_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}(H)$.

5. Application

The present section is devoted to providing an interesting real-life application to demonstrate the importance of Primal approximation spaces in decision-making problems and to identify their differences from other methods, such as Ideal rough sets.

5.1. Experimental Outcomes and Medical Decision Table

This section provides an analysis of the computed results derived from examinations administered to five students across four subjects, using the current approach. The data, originally presented in [11], contained inaccuracies which are addressed and corrected in this analysis. The evaluation focuses on student performance in biology, chemistry, mathematics, and physics. The five students, denoted as $\mathcal{V} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$, were assessed in these academic disciplines. The results of the assessments are classified into

Fig. 1: Graphical representation for students' levels

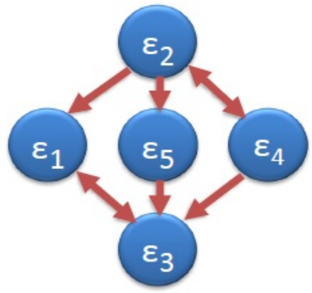

Fig. 2: Graphical representation for right neighborhoods

Figure 1: Graphical representation.

five distinct levels or ranks, illustrated in Figure 1 (**Fig. 1**).
The hierarchical arrangement of these ranks is as follows: excellent > very good > good > fair > failed, where the symbol > denotes "greater than". The relational structure defining associations among students is established as follows: $w\mathcal{R}z$ if and only if student w possesses at least two subjects with a rank superior to that of the corresponding subjects of student z. For instance, $\epsilon_4\mathcal{R}\epsilon_3$ is valid due to the higher ranks of student ϵ_4 in biology, mathematics, and physics compared to those of student ϵ_3 in the same subjects. However, the pair (ϵ_3, ϵ_4) does not belong to $\mathcal{R}$ as student ϵ_3 only has one subject with a rank higher than that of student ϵ_4. The construction of the approximation space for the student's information system begins with the conversion of Table 2 into the binary relation $\mathcal{R} = \{(\epsilon_1, \epsilon_3), (\epsilon_2, \epsilon_1), (\epsilon_2, \epsilon_4), (\epsilon_2, \epsilon_5), (\epsilon_3, \epsilon_1), (\epsilon_4, \epsilon_2), (\epsilon_4, \epsilon_3), (\epsilon_5, \epsilon_3)\}$.

Table 2: The information system pertaining to students' academic standings in each subject

	Biology	Chemistry	Mathematics	Physics
ϵ_1	Good	Fair	Excellent	Excellent
ϵ_2	V. Good	Good	Excellent	Fair
ϵ_3	V. Good	Good	Failed	Good
ϵ_4	Excellent	Fair	V. Good	Excellent
ϵ_5	V. Good	Fair	V. Good	Excellent

The graphical representation shown in Figure 1 (**Fig. 1**) is instrumental for analyzing the relative performance and ranking of students, allowing educators to effectively target interventions or recognitions. Additionally, the κ-neighborhoods for each κ can be depicted, as illustrated in Figure 1 (**Fig. 2**), which specifically shows the right neighborhoods.

Thus, the generated topologies are:

Table 3: Neighborhoods of each student

	$N_r(.)$	$N_l(.)$	$N_i(.)$	$N_u(.)$	$N_{\langle r\rangle}(.)$	$N_{\langle l\rangle}(.)$	$N_{\langle i\rangle}(.)$	$N_{\langle u\rangle}(.)$
ϵ_1	$\{\epsilon_3\}$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_3\}$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_1\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$
ϵ_2	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_4\}$	$\{\epsilon_4\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_2\}$	$\{\epsilon_2\}$	$\{\epsilon_2,\epsilon_3\}$
ϵ_3	$\{\epsilon_1\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_3\}$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_3\}$	$\{\epsilon_2,\epsilon_3\}$
ϵ_4	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_2\}$	$\{\epsilon_2\}$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_4\}$	$\{\epsilon_4\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$
ϵ_5	$\{\epsilon_3\}$	$\{\epsilon_2\}$	$\emptyset$	$\{\epsilon_2,\epsilon_3\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$

$\tau_r=\{\emptyset,\mathcal{V},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_3,\epsilon_5\}\}$.
$\tau_l=\{\emptyset,\mathcal{V},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_4,\epsilon_5\}\}$.
$\tau_i=\{\emptyset,\mathcal{V},\{\epsilon_5\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}\}$.
$\tau_u=\{\emptyset,\mathcal{V}\}$.
$\tau_{\langle r\rangle}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle l\rangle}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_3,\epsilon_4\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle i\rangle}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_3,\epsilon_4\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,$
$\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,$
$\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle u\rangle}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.

Let $\mathcal{P}=2^{\mathcal{V}}\backslash\{\mathcal{V},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\}\}$, and $\tilde{\mathcal{P}}=2^{\mathcal{V}}\backslash\{\mathcal{V},\{\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_3,$ $\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$ be two primals on $\mathcal{V}$. Then,
$\tau_r^{\mathcal{P}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},$
$\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},$
$\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_l^{\mathcal{P}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},$
$\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},$
$\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_i^{\mathcal{P}}=2^{\mathcal{V}}$.
$\tau_u^{\mathcal{P}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},$
$\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},$
$\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle r\rangle}^{\mathcal{P}}=\tau_{\langle l\rangle}^{\mathcal{P}}=\tau_{\langle i\rangle}^{\mathcal{P}}=\tau_{\langle u\rangle}^{\mathcal{P}}=2^{\mathcal{V}}$.
$\tau_r^{\tilde{\mathcal{P}}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_5\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},$
$\{\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_l^{\tilde{\mathcal{P}}}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},\{\epsilon_4,\epsilon_5\},$
$\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},$

$\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_i^{\tilde{\mathcal{P}}}=2^{\mathcal{V}}$.
$\tau_u^{\tilde{\mathcal{P}}}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_2,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},$
$\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},$
$\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle r\rangle}^{\tilde{\mathcal{P}}}=\tau_{\langle l\rangle}^{\tilde{\mathcal{P}}}=\tau_{\langle i\rangle}^{\tilde{\mathcal{P}}}=\tau_{\langle u\rangle}^{\tilde{\mathcal{P}}}=2^{\mathcal{V}}$.
Note that: $\tilde{\mathcal{P}}$ is not ideal.
Let $\mathcal{I}=\{\emptyset,\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_3\}\}$ be an ideal on $\mathcal{V}$. Then,
$\tau_r^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4\},$
$\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_l^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_4\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_i^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_2,\epsilon_4\},\{\epsilon_3,\epsilon_4\},\{\epsilon_3,\epsilon_5\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,$
$\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,$
$\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_u^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_4\},\{\epsilon_5\},\{\epsilon_1,\epsilon_4\},\{\epsilon_1,\epsilon_5\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,$
$\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle r\rangle}^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle l\rangle}^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_3,\epsilon_4\},\{\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},$
$\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle i\rangle}^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_1\},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_4\},\{\epsilon_1,\epsilon_2\},\{\epsilon_1,\epsilon_3\},\{\epsilon_1,\epsilon_4\},\{\epsilon_2,\epsilon_3\},\{\epsilon_2,\epsilon_4\},\{\epsilon_3,\epsilon_4\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,$
$\epsilon_2,\epsilon_3\},\{\epsilon_1,\epsilon_2,\epsilon_4\},\{\epsilon_1,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},\{\epsilon_1,\epsilon_2,$
$\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.
$\tau_{\langle u\rangle}^{\mathcal{I}}=\{\emptyset,\mathcal{V},\{\epsilon_2\},\{\epsilon_3\},\{\epsilon_2,\epsilon_3\},\{\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_4,\epsilon_5\},\{\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\},\{\epsilon_1,\epsilon_2,\epsilon_4,$
$\epsilon_5\},\{\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5\}\}$.

Suppose that $H=\{\epsilon_1,\epsilon_5\}$. Then,

Table 4: Lower, upper approximation, and accuracy degree with τ_κ

κ	r	l	i	u	$\langle r\rangle$	$\langle l\rangle$	$\langle i\rangle$	$\langle u\rangle$
$\tau_\kappa L(H)$	$\emptyset$	$\emptyset$	$\{\epsilon_5\}$	$\emptyset$	$\{\epsilon_1\}$	$\emptyset$	$\{\epsilon_1\}$	$\emptyset$
$\tau_\kappa U(H)$	$\mathcal{V}$	$\{\epsilon_1,\epsilon_3,\epsilon_5\}$	$\{\epsilon_1,\epsilon_3,\epsilon_5\}$	$\mathcal{V}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$
$\tau_\kappa\sigma(H)$	0	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	0

Table 5: Lower, upper approximation, and accuracy degree with $\tau_\kappa^{\mathcal{P}}$

κ	r	l	i	u	$\langle r\rangle$	$\langle l\rangle$	$\langle i\rangle$	$\langle u\rangle$
$\tau_\kappa^{\mathcal{P}}L(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\mathcal{P}}U(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\mathcal{P}}\sigma(H)$	1	1	1	1	1	1	1	1

Table 6: Lower, upper approximation, and accuracy degree with $\tau_\kappa^{\tilde{\mathcal{P}}}$

κ	r	l	i	u	$\langle r\rangle$	$\langle l\rangle$	$\langle i\rangle$	$\langle u\rangle$
$\tau_\kappa^{\tilde{\mathcal{P}}}L(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\emptyset$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\tilde{\mathcal{P}}}U(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\tilde{\mathcal{P}}}\sigma(H)$	1	$\frac{1}{2}$	1	0	1	1	1	1

Table 7: Lower, upper approximation, and accuracy degree with $\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}$

κ	r	l	i	u	$\langle r\rangle$	$\langle l\rangle$	$\langle i\rangle$	$\langle u\rangle$
$\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}L(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}U(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\mathcal{P}\cup\tilde{\mathcal{P}}}\sigma(H)$	1	1	1	1	1	1	1	1

6. Conclusions and Future Works

In this study, we have offered a novel framework for approximate rough sets by introducing the concept of "primal" and exploring bi-primal approximation spaces. The findings indicate that these new structures effectively address the drawbacks of traditional rough set models, especially those arising from rigid topological conditions. By employing κ-neighborhoods and primal, we have established methods for generating diverse supra-topologies that not only enhance flexibility but also allow for a broader range of real-world applications.

The approach of supra-topological structures used to develop new models of rough set theory in this manuscript is more adjustable than traditional topological structures. This flexibility allows for a broader scope in describing various phenomena, as it removes the need for an intersection condition that may not be suitable in certain contexts. For instance, in the provided application, we demonstrated how the generated supra-topologies aided in more accurate decision-making by avoiding common levels, thereby providing more precise decisions.

The introduction of Bi-primal approximation spaces underlines a significant advancement in rough set theory. Our investigation into the distinct methods associated with these spaces reveals their potential for offering solutions to complex decision-making problems. Notably, these approaches enable the modeling of scenarios where existing methods based on ideals may fall short. For instance, the elimination of the finite intersection condition in topological rough models not only widens the applicability of the framework but also facilitates a more nuanced understanding of the relationships between various attributes

Table 8: Lower, upper approximation, and accuracy degree with $\tau_\kappa^{\mathcal{I}}$

κ	r	l	i	u	$\langle r\rangle$	$\langle l\rangle$	$\langle i\rangle$	$\langle u\rangle$
$\tau_\kappa^{\mathcal{I}}L(H)$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1\}$	$\{\epsilon_5\}$	$\{\epsilon_1\}$	$\emptyset$
$\tau_\kappa^{\mathcal{I}}U(H)$	$\{\epsilon_1,\epsilon_2,\epsilon_5\}$	$\{\epsilon_1,\epsilon_3,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5\}$	$\{\epsilon_1,\epsilon_4,\epsilon_5\}$	$\{\epsilon_1,\epsilon_3,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$	$\{\epsilon_1,\epsilon_5\}$
$\tau_\kappa^{\mathcal{I}}\sigma(H)$	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0

involved in decision-making processes. Additionally, we illustrated that the constructed approximations of two primals are more accurate than those generated by a single primal, without affecting the basic properties of these approximations.

Moreover, our practical example demonstrates the efficacy of the proposed methods in real-world applications, underlining the ability of primals to streamline the evaluation of candidates against multiple criteria. This refined approach reduces uncertainty by focusing on the most pertinent attributes, thereby simplifying the selection process in competitive scenarios.

The implications of this research extend beyond rough set theory; they provide new aspects for query into the interplay between topological constructs and the handling of imprecision in diverse fields such as artificial intelligence, engineering, and medical science. Future research may build upon these foundations by exploring additional applications of primal and bi-primal spaces, further enriching the theoretical landscape of rough set modeling.

In summary, the contributions of this paper underscore the importance of flexible mathematical frameworks in addressing the complexities of real-life problems. By relaxing traditional constraints and introducing innovative concepts, we pave the way for more adaptive and effective solutions in the realm of decision-making.

Promising directions for future research include the following:

1. **Investigating practical applications:** Applying these newly developed approximations to various real-world scenarios, as demonstrated in [23, 30].
2. **Incorporating novel neighborhoods:** Advancing current techniques by utilizing new neighborhood structures (maximal neighborhoods [20], basic-neighborhoods [7, 31, 71, 72], initial-neighborhoods [62], and adhesion neighborhoods [17, 25]).
3. **Expanding with near open sets:** Building on the current results by introducing near open sets, as discussed in [29, 36, 39, 40].
4. **Extending to broader fields:** Widening the scope of this study to encompass rough fuzzy methods and other related fields, in alignment with [8, 33, 34, 54, 68, 69], soft topological spaces [10, 14, 15, 61] and decision-making problems in [16, 57].

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Conflict of interest: The authors declare that there is no conflict of interest.

References

[1] E. A. Abo-Tabl. Topological approaches to generalized definitions of rough multiset approximations.

[2] E. A. Abo-Tabl. A comparison of two kinds of definitions of rough approximations based on a similarity relation. *Inform. Sci.*, 181(12):2587–2596, 2011.

[3] E. A. Abo-Tabl. Rough sets and topological spaces based on similarity. *Int. J. Mach. Learn. Cybern.*, 4:451–458, 2013.

[4] E. A. Abo-Tabl and M. K. El-Bably. Rough topological structure based on reflexivity with some applications. *AIMS Mathematics.*, 7(6):9911–9999, 2022.

[5] R. Abu-Gdairi, A. A. El-Atik, and M. K. El-Bably. Topological visualization and graph analysis of rough sets via neighborhoods: A medical application using human heart data. *AIMS Math.*, 8(11):26945–26967, 2023.

[6] R. Abu-Gdairi and M. K. El-Bably. The accurate diagnosis for covid-19 variants using nearly initial-rough sets. *Heliyon.*, 10(10), 2024. https://doi.org/10.1016/j.heliyon.2024.e31288.

[7] R. Abu-Gdairi, M. A. El-Gayar, M. K. El-Bably, and K. K. Fleifel. Two different views for generalized rough sets with applications . *Math.*, 9, 2021. https://doi.org/10.3390/math9182275.

[8] R. Abu-Gdairi, A. A. Nasef, M. A. El-Gayar, and M. K. El-Bably. On fuzzy point applications of fuzzy topological spaces. *Int. J. Fuzz. Log. Intell, Syst.*, 23(2):162–172, 2023.

[9] S. Acharjee, M. Özkoç, and F. Y. Issaka. Primal topological spaces. *arXiv preprint arXiv:2209.12676*, pages 1–9, 2022. https://doi.org/10.48550/arXiv.2209.12676.

[10] J. Al-Mufarrij and S. Saleh. New results on soft generalized topological spaces. *J. Math. Comput. Sci.*, 32(01):43–53, 2023.

[11] T. M. Al-Shami and M. Hosny. Improvement of approximation spaces using maximal left neighborhoods and ideals. *IEEE Access*, 10:79379–79393, 2022.

[12] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl. *New approach for basic rough set concepts, In: Rough sets, fuzzy sets, data mining, and granular computing.* Berlin, Heidelberg, Springer, 2005.

[13] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl. New approach for basic rough set concepts, in: Rough sets, fuzzy sets, data mining, and granular computing. *Acta Math. Acad. Paedagog. Nyiregyháziensis*, 22(3):285–304, 2006.

[14] Z. A. Ameen and B. A. Asaad. Methods of generating soft topologies and soft separation axioms. *Eur. J. Pure Appl. Math.*, 17(2):1168–1182, 2023.

[15] B. A. Asaad. Results on soft extremally disconnectedness of soft topological spaces. *J. Math. Comput. Sci.*, 17(4):448–464, 2017.

[16] B. A. Asaad, S. Y. Musa, and Ameen Z. A. Fuzzy bipolar hypersoft sets: a novel approach for decision-making applications. *Math. Comput. Appl.*, 29(4):50, 2024.

[17] M. Atef, A. M. Khalil, S.-G. Li, A. A. Azzam, and A. A. Atik. Comparison of six types of rough approximations based on j-neighborhood space and j-adhesion neighborhood space. *J. Intell. Fuzzy Syst.*, 39:4515–4531, 2020.

[18] M. Atef, S. Nada, and A. S. Nawar. Covering soft rough sets and its topological properties with application. *Soft Comput.*, 27(8):4451–4461, 2023.

[19] G. Choquet. Sur les notions de filtre et de grille. *Comptes Rendus Acad. Sci. Paris*, 224:171–173, 1947. https://doi.org/10.1155/2023/4723233.

[20] J. H. Dai, S. C. Gao, and G. J. Zheng. Generalized rough set models determined by multiple neighborhoods generated from a similarity relation. *Soft Comput.*, 22:2081–2094, 2018.

[21] M. K. El-Bably. *Generalized approximation spaces. M. Sc. Thesis*, Tanta University, Egypt, 2008.

[22] M. K. El-Bably and E. A. Abo-Tabl. A topological reduction for predicting of a lung cancer disease based on generalized rough sets. *J. Intell. Fuzzy Syst.*, 41:335–346, 2021.

[23] M. K. El-Bably, R. Abu-Gdairi, and M. A. El-Gayar. Medical diagnosis for the problem of chikungunya disease using soft rough sets. *AIMS Math.*, 8:9082–9105, 2023.

[24] M. K. El-Bably, R. Abu-Gdairi, K. K. Fleifel, and M. A. El-Gayar. Exploring β-basic rough sets and their applications in medicine. *Eur. J. Pure Appl. Math.*, 17(4):3743–3771, 2024.

[25] M. K. El-Bably, T. M. Al-shami, A. S. Nawar, and A. Mhemdi. Corrigendum to "comparison of six types of rough approximations based on j-neighborhood space and j-adhesion neighborhood space. *J. Intell. Fuzzy Syst.*, 41:7353–7361, 2021.

[26] M. K. El-Bably, M. I. Ali, and E. A. Abo-Tabl. New topological approaches to generalized soft rough approximations with medical applications. *J. Math.*, 2021:2559495, 2021.

[27] M. K. El-Bably and A. A. El Atik. Soft β-rough sets and their application to determine covid-19. *Turk. J. Math.*, 45(3):1133–1148, 2021.

[28] M. K. El-Bably and M. El-Sayed. Three methods to generalize pawlak approximations via simply open concepts with economic applications. *Soft Comput.*, 26:4685–4700, 2022.

[29] M. K. El-Bably and K. K. Fleifel. Some topological structures by relations. *J. Comput. Theor. Nanosci.*, 14(8):4104–4113, 2017.

[30] M. A. El-Gayar and R. Abu-Gdairi. Extension of topological structures using lattices and rough sets. *AIMS Math.*, 9(3):7552–7569, 2024.

[31] M. A. El-Gayar, R. Abu-Gdairi, M. K. El-Bably, and D. I. Taher. Economic decision-making using rough topological structures. *J. Math.*, 2023(4723233):10, 2023. https://doi.org/10.1155/2023/4723233.

[32] M. A. El-Gayar and A. El Atik. Topological models of rough sets and decision making of covid-19. *Complexity.*, 2022, 2022. https://doi.org/10.1155/2022/2989236.

[33] M. E. Abd El-Monsef, M. A. El-Gayar, and R. M. Aqeel. On relationships between revised rough fuzzy approximation operators and fuzzy topological spaces. *Int. J. Granul. Comput. Rough Sets Intell. Syst.*, 3:257–271, 2014.

[34] M. E. Abd El-Monsef, M. A. El-Gayar, and R. M. Aqeel. A comparison of three types of rough fuzzy sets based on two universal sets. *Int. J. Mach. Learn. Cybern.*,

8:343–353, 2017.

[35] M. E. Abd El-Monsef, O. A. Embaby, and M. K. El-Bably. Comparison between rough set approximations based on different topologies. *Int. J. Granul. Comput., Rough Sets Intell. Syst.*, 3:292–305, 2014.

[36] M. E. Abd El-Monsef, A. M. Kozae, and M. K. El-Bably. Generalized covering approximation space and near concepts with some applications. *Appl. Comput. Inform.*, 12(1):51–69, 2016.

[37] M. Abd Elaziz, H. M. Abu-Donia, R. A. Hosny, S. L. Hazae, and R. A. Ibrahim. Improved evolutionary-based feature selection technique using extension of knowledge based on the rough approximations. *Inf. Sci.*, 594:76–94, 2022.

[38] O. A. Embaby and N. Toumi. A method for improving rough set approximation accuracy in terms of j-neighborhood spaces. *Delta J. Sci.*, 38(1):43–52, 2017.

[39] R. B. Esmaeel and M. O. Mustafa. On nano topological spaces with grill-generalized open and closed sets. In *AIP Conf. Proc.*, page 2414, AIP Conf. Proc., 2023. AIP Conf. Proc. https://doi.org/10.1063/5.0117062.

[40] R. B. Esmaeel and N. M. Shahadhuh. On nano topological spaces with grill-generalized open and closed sets on grill-semi-p-separation axioms. In *AIP Conf. Proc.*, page 2414, AIP Conf. Proc., 2023. AIP Conf. Proc. https://doi.org/10.1063/5.0117064.

[41] M. Hosny. Idealization of j-approximation spaces. *Filomat*, 34(2):287—-301, 2020.

[42] R. A. Hosny, M. Abdelaziz, and R. A. Ibrahim. Enhanced feature selection based on integration containment neighborhoods rough set approximations and binary honey badger optimization. *Comput. Intell. Neurosci.*, 2022, Article ID 3991870:17 pages, 2022.

[43] R. A. Hosny, R. Abu-Gdairi, and M. K. El-Bably. Enhancing dengue fever diagnosis with generalized rough sets: Utilizing initial-neighborhoods and ideals. *Alex. Eng. J.*, 94:68–79, 2024.

[44] R. A. Hosny, M. K. El-Bably, and A. S. Nawar. Some modifications and improvements to idealization of j-approximation spaces. *J. Adv. Stud. Topol.*, 1–2:1–7, 2022.

[45] R. A. Hosny, N. M. Madbouly, and M. K. El-Bably. Essential corrigenda to" generalized approximation spaces generation from $\mathbb{I}_j$-neighborhoods and ideals with application to chikungunya disease". *arXiv preprint arXiv:2412.05284*, pages 1–12, 2024. https://doi.org/10.48550/arXiv.2412.05284.

[46] A. Kandil, M. M. Yakout, and A. Zakaria. Generalized rough sets via ideals. *Ann. Fuzzy Math. Inform.*, 5(3):525–532, 2013.

[47] A. Kandil, M. M. Yakout, and A. Zakaria. New approaches of rough sets via ideals. In *Handbook of research on generalized and hybrid set structures and applications for soft computing*, pages 247–264. IGI Global, 2016.

[48] A. A. Abo Khadra and M. K. El-Bably. Topological approach to tolerance space. *Alex. Eng. J.*, 47(6):575–580, 2008.

[49] A. A. Abo Khadra, B. M. Taher, and M. K. El-Bably. Generalization of pawlak approximation space. In *Proceeding of the International Conference on Mathematics: Trends and Developments, The Egyptian Mathematical Society*, volume 3, pages 335–

346, 2007.

[50] M. Kondo and W. A. Dudek. Topological structures of rough sets induced by equivalence relations. *J. Adv. Comput. Intell. Intell. Inform.*, 10(5):621–624, 2006.

[51] A. M. Kozae, M. Shokry, and M. Zidan. Supra topologies for digital plane. *AASCIT Commun.*, 3(1):1–10, 2016.

[52] K. Kuratowski. *Topology: Volume I.* Dordrecht, New York, 1966.

[53] E. F. Lashin, A. M. Kozae, A. A. Abo Khadra, and T. Medhat. Rough set theory for topological spaces. *Int. J. Approx. Reason.*, 40:35–43, 2005.

[54] H. Lu, A. M. Khalil, W.d. Alharbi, and M. A. El-Gayar. A new type of generalized picture fuzzy soft set and its application in decision making. *J. Intell. Fuzzy Syst.*, 40:12459–12475, 2021.

[55] H. Maki, J. Umehara, and T. Noiri. Every topological space is pret$\frac{1}{2}$. *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, 17:33–42, 1996.

[56] A. S. Mashhour. On supratopological spaces. *Indian J. Pure Appl. Math.*, 14:502–510, 1983.

[57] S. Y. Musa and B. A. Asaad. Bipolar m-parametrized n-soft sets: a gateway to informed decision-making. *J. Math. Comput. Sci.*, 36(1):121–141, 2025.

[58] A. S. Nawar, M.A. El-Gayar, M. K. El-Bably, and R. A. Hosny. $\theta\beta$-ideal approximation spaces and their applications. *AIMS Mathematics.*, 7(2):2479–2497, 2022.

[59] Z. Pawlak. Rough sets. *Int. J. Inform. Comput. Sci.*, 11:341–356, 1982.

[60] K. Qin, J. Yang, and Z. Pei. Generalized rough sets based on reflexive and transitive relations. *Inf. Sci.*, 178:4138–4141, 2008.

[61] S. Saleh, L. R. Flaih, and K. F. Jasim. Some applications of soft∂-closed sets in soft closure spaces. *Commun. Math. Appl.*, 14(2):481–492, 2023.

[62] M. El Sayed, M. A. El Safty, and M. K. El-Bably. Topological approach for decision-making of covid-19 infection via a nano-topology model. *AIMS Math.*, 6:7872–7894, 2021.

[63] T. K. Sheeja and S. Kuriakose. Rough topology on approximation spaces. *Int. J. Adv. Res. Comput. Sci.*, 8(9), 2017. https://doi.org/10.26483/ijarcs.v8i9.5028.

[64] P. K. Singh and S. Tiwari. Topological structures in rough set theory a survey. *Hacet. J. Math. Stat.*, 49(4):1270–1294, 2020.

[65] A. Skowron. On topology in information system. *Bull. Pol. Acad. Sci. Math.*, 36:477–480, 1988.

[66] A. Skowron and J. Stepaniuk. Tolerance approximation spaces. *Fundam. Inform.*, 27(2):245–253, 1996.

[67] R. Slowinski and D. Vanderpooten. A generalized definition of rough approximations based on similarity. *IEEE Trans. Knowl. Data Eng.*, 12(2):331–336, 2000.

[68] I. M. Taha. Some new separation axioms in fuzzy soft topological spaces. *Filomat.*, 35(6):1775–1783, 2021.

[69] I. M. Taha. Some new results on fuzzy soft r-minimal spaces. *AIMS Mathematics.*, 7(7):12458–12470, 2022.

[70] I. M. Taha. Some properties of (r, s)-generalized fuzzy semi-closed sets and some applications. *J. Math. Comput. Sci.*, 27(2):164–175, 2022.

[71] D. I. Taher, R. Abu-Gdairi, M. K. El-Bably, and M. A. El-Gayar. Correction: Decision-making in diagnosing heart failure problems using basic rough sets. *AIMS Math.*, 9(12):34270–34271, 2024.

[72] D. I. Taher, R. Abu-Gdairi, M. K. El-Bably, and M. A. El-Gayar. Decision-making in diagnosing heart failure problems using basic rough sets. *AIMS Math.*, 9(8):21816–21847, 2024.

[73] Y. Y. Yao. Two views of the theory of rough sets in finite universes. *Int. J. Approx. Reason.*, 15(4):291–317, 1996.

[74] W. Zhu. Topological approaches to covering rough sets. *Inf. Sci.*, 177(6):1499–1508, 2007.