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New Inequalities for Differentiable Mappings via Fractional Integral Operators

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Abstract. In this study, explicit bounds for the midpoint type inequalities for functions whose twice differentiable in absolute value raised to positive real powers are (ρ, s) and (ρ, s, m)-convexities are explored through the integral fractional operator. Several estimate for special functions including Euler gamma, incomplete Beta and hypergeometric functions are presented in the study.

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1. Introduction

Fractional calculus – whose impact on both pure and applied sciences substantially increased in the last two decades – captures the attention of many researchers. In addition, this area of interest invariably remains one of the few disciplines that immensely contribute to not only different areas of mathematics, but also to other natural sciences. One of such examples is associated to the existence of many fractional operators in those disciplines, most of which occur through different formulations differential equations. This area has been developed through the contributions of many researchers; for example, Kilbas et al. [9] gave the background of fractional differential equations, Xing et al. [22] extended Hermite-Hadamard inequalities using fractional integrals, Zhang et al.[7] generalized the inequalities for strongly (s,m)-convexities, and Noor and Awan [11] established the inequalities with two different convexities. These operators have been used to understand many problems, such as the propagation of sound, vibrations of strings and waves in liquids. This leads to the establishment of numerous fractional operators – including Riemann Liouville, Caputo and Atangana Baleanu Caputo – as well as studying many vital concepts for investigating these operators. For example, the existence of unique solution of many fractional differential and integral equations –

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for initial and boundary value problems – have been reported through these fractional operators. For further studies, see [10, 13–15, 17].

Fractional calculus, whose broad contents have been rapidly developing in mathematical analysis, plays a vital role in approximation theory. One example of this is the frequent use of integral operators in the study of inequalities. Consider an integrable function $\eta : [k_1, k_2] \to \mathbb{R}$ over $[k_1, k_2]$. This function is belonging to a space $L_1[k_1, k_2]$ representing the set of Lebesgue integrable over the same interval. Using Riemann-Liouville fractional integrals, Sarikaya et al. [19] studied the following new integral inequalities.

Theorem 1. *[19] Let $\eta : [k_1, k_2] \to \mathbb{R}$ be a positive function with $0 \le k_1 < k_2$ and $\eta \in L_1[k_1, k_2]$. If η is a convex function on $[k_1, k_2]$, then the following inequalities for fractional integrals hold:*

$$\eta\left(\frac{k_1+k_2}{2}\right) \le \frac{\Gamma(\rho+1)}{2(k_2-k_1)^\rho}\left[J^\rho_{k_1^+}\eta(k_2) + J^\rho_{k_2^-}\eta(k_1)\right] \le \frac{\eta(k_1)+\eta(k_2)}{2}, \quad \rho > 0. \tag{1}$$

Due to vital roles played by this inequality in both science and engineering [2, 4], many authors improve, extend and generalize inequality (1) through various types of convexities and fractional integrals. For example, Agarwal et al. [1] established new Hermite-Hadamard type inequalities via generalized k-fractional integrals; Almutairi [3] explored fractional inequalities using Euler's beta function; Budak et al. [5] developed Hermite-Hadamard-type inequalities for interval-valued functions; Du and Peng [6] extended the idea to report Hermite-Hadamard type inequalities involving multiplicative Riemann-Liouville fractional integrals.

Noor and Awan [11] established new results related to the left hand side of (1) for twice differentiable s-convex functions using the following lemma.

Lemma 1. *Let $\eta : [k_1, k_2] \to \mathbb{R}$ be a twice differentiable function on (k_1, k_2) with $k_1 < k_2$. Also, let $\eta'' \in L[k_1, k_2]$. Then the following identity holds true:*

$$\begin{aligned}&\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1) + J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right] - \eta\left(\frac{k_1+k_2}{2}\right)\\ &= \frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1 (1-\varpi)^{\rho+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1 + \frac{1-\varpi}{2}k_2\right) + \eta''\left(\frac{1+\varpi}{2}k_2 + \frac{1-\varpi}{2}k_1\right)\right]d\varpi\end{aligned} \tag{2}$$

Even though a classical convexity - which was later replaced by s-convexity [11] - has been previously used to establish integral inequalities [19], the central idea of our study lies in establishing more generalized integral inequalities through two different classes of convexities. Motivated by these two mentioned independent studies, we opt to obtain new bounds for the midpoint inequalities via fractional integral operator. Two generalized convexities (ρ, s) and (ρ, s, m)-convexity are used to establish the new bounds. Some of our findings - which can be reduced to different inequalities through various convexities - are obtained. The other parts of this paper are organized as follows. Preliminary studies are presented in Section 2. Section 3 presents integral inequalities involving (ρ, s) and (ρ, s, m) convex functions via fractional integral operators. Section 4 concludes the study.

2. Preliminaries

Some basic results of different classes of convex functions, Riemann Liouville fractional integral operator and special functions are presented in this section. These preliminary results and definitions can be later used to establish our main results.

Therefore, the definitions of some types of convexities are given as follows.

Definition 1. *[20] A function $\eta : [0, d] \to R_0 = [0, \infty)$ is said to be m-convex on $[0, d]$ for some $m \in (0, 1]$, if*

$$\eta(\varpi k_1 + m(1-\varpi)k_2) \leq \varpi\eta(k_1) + m(1-\varpi)\eta(k_2),$$

for all $k_1, k_2 \in [0, d]$ and $\varpi \in [0, 1]$.

Definition 2. *[8] A function $\eta : [k_1, k_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be s-convex function of the second kind if*

$$\eta((1-\varpi)k_1 + \varpi k_2) \leq (1-\varpi)^s\eta(k_1) + \varpi^s\eta(k_2),$$

for all $k_1, k_2 \in (0, \infty], \varpi \in [0, 1]$ and $s \in (0, 1]$.

Definition 3. *[21] For some $s \in [-1, 1]$ and $\rho \in (0, 1]$, a function $\eta : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (ρ, s)-convex if*

$$\eta(\varpi k_1 + (1-\varpi)k_2) \leq \varpi^{\rho s}\eta(k_1) + (1-\varpi^\rho)^s\,\eta(k_2),$$

for all $k_1, k_2 \in I$ and $\varpi \in (0, 1)$.

Definition 4. *[21] The function $\eta : [0, d] \to R$ is said to be (ρ, s, m)-convex, if we have*

$$\eta(\varpi k_1 + m(1-\varpi)k_2) \leq \varpi^{\rho s}\eta(k_1) + m\,(1-\varpi^\rho)^s\,\eta(k_2),$$

where $k_1, k_2 \in [0, d], \varpi \in (0, 1)$ and for some $s \in [-1, 1], (\rho, m) \in (0, 1]^2$.

Different special cases are considered in the following remark.

Remark 1. *Definition 4 produces the following:*

i. If $s = 1$ in Definition 4, then we get the class of (ρ, m)-convex function.

ii. If $\rho = 1$, then Definition 4 reduces to the definition for (s, m)-convex function.

iii. If $\rho = m = 1$ in Definition 4, then we have the class of extended s-convex function.

iv. If $\rho = s = m = 1$ in Definition 4, then we obtain the classical convex function.

We know present the defintions of Riemann-Liouville integrals as follows.

Definition 5. *[18] Let $\eta \in L_1[k_1, k_2]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{k_1+}\eta$ and $J^{\alpha}_{k_2-}\eta$ of order $\rho > 0$ with $k_1 \geq 0$ are defined by*

$$J^{\rho}_{k_1^+}\eta(x) = \frac{1}{\Gamma(\rho)}\int_{k_1}^{x}(x-\varpi)^{\rho-1}\eta(\varpi)d\varpi, x > k_1$$

and

$$J^{\rho}_{k_2^-}\eta(x) = \frac{1}{\Gamma(\rho)}\int_x^{k_2} (\varpi - x)^{\rho-1}\eta(\varpi)d\varpi, x < k_2$$

respectively. Here, $\Gamma(\rho)$ *is the Gamma function and* $J^0_{k_1+}\eta(x) = J^0_{k_2-}\eta(x) = \eta(x)$.

We further, present the definitions of some special functions as follows.

Definition 6. *[9] For any complex numbers and nonpositive integers* k_1, k_2 *such that* $\mathrm{Re}(k_1) > 0$ *and* $\mathrm{Re}(k_2) > 0$. *The beta function is defined by*

$$B(k_1, k_2) = \int_0^1 \varpi^{k_1-1}(1-\varpi)^{k_2-1}d\varpi = \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)},$$

where Γ *is the Gamma function.*

Definition 7. *[12] For any complex numbers* k_1, k_2 *with* $\mathrm{Re}\,(k_1), \mathrm{Re}\,(k_2) > 0$, *the incomplete beta function is defined as*

$$B_a(k_1, k_2) = \int_0^a \varpi^{k_1-1}(1-\varpi)^{k_2-1}d\varpi, \quad 0 < a < 1.$$

Definition 8. *[9] The integral representation of the hypergeometric function is defined for* $k_1, k_2 \in \mathbb{C}$ *and* $k_3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(k_3) > \mathrm{Re}(k_2) > 0$, *and* $|k_4| < 1$, *as follows*

$${}_2F_1(k_1, k_2, k_3; k_4) = \frac{1}{B(k_2, k_3-k_2)}\int_0^1 \varpi^{k_2-1}(1-\varpi)^{k_3-k_2-1}(1-k_4\varpi)^{-k_1}d\varpi,$$

where $B(.,.)$ *is the beta function.*

3. Main Result

In this section, we first present some generalized midpoint type inequalities for (ρ, s)-convex functions.

Theorem 2. *Suppose that* $\eta : [k_1, k_2] \to \mathbb{R}$ *is a twice differentiable function on* (k_1, k_2) *with* $k_1 < k_2$. *If* $|\eta''|$ *is* (ρ, s) *convex function, where* $(\rho, s) \in (0, 1]^2$ *and* $\eta'' \in L[k_1, k_2]$, *then the following*

$$\begin{aligned}&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^{\rho}}\left[J^{\rho}_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1) + J^{\alpha}_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right] - \eta\left(\frac{k_1+k_2}{2}\right)\right|\\&\leq \frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[\left|\eta''(k_1)\right|\frac{{}_2F_1(1, -\rho s; 3+k_1; -1)}{k_1+2} + \left|\eta''(k_2)\right|\Omega(\rho, s, \varpi)\right.\\&\left.+\left|\eta''(k_1)\right|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)} + \left|\eta''(k_2)\right|\Omega(\rho, s, \varpi)\right].\end{aligned}$$

where

$$\Omega(\rho, s, \varpi) = \int_0^1 (1-\varpi)^{\rho+1}(2^{\rho} - (1+\varpi)^{\rho})^s d\varpi$$

holds.

Proof. We now use identity (2) and (ρ, s)-convexity of $|\eta''|$ to obtain the following

$$\begin{aligned}
&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|\\
&=\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)+\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right]d\varpi\right|\\
&\le\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)d\varpi\right|\\
&+\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)d\varpi\right|\\
&\le\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|d\varpi\\
&+\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|d\varpi\\
&\le\frac{(k_2-k_1)^2}{8(\rho+1)}\left[\int_0^1(1-\varpi)^{\rho+1}\left[\left(\frac{1+\varpi}{2}\right)^{\rho s}|\eta''(k_1)|+\left(1-\left(\frac{1-\varpi}{2}\right)^\rho\right)^s|\eta''(k_2)|\right]d\varpi\right.\\
&\quad\left.+\int_0^1(1-\varpi)^{\rho+1}\left[\left(\frac{1-\varpi}{2}\right)^{\rho s}|\eta''(k_1)|+\left(1-\left(\frac{1+\varpi}{2}\right)^\rho\right)^s|\eta''(k_2)|\right]d\varpi\right]\\
&=\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\int_0^1(1-\varpi)^{\rho+1}(1+\varpi)^{\rho s}d\varpi+|\eta''(k_2)|\int_0^1(1-\varpi)^{\rho+1}(2^\rho-(1-\varpi)^\rho)^s d\varpi\right.\\
&\left.+|\eta''(k_1)|\int_0^1(1-\varpi)^{\rho+1}(1-\varpi)^{\rho s}d\varpi+|\eta''(k_2)|\int_0^1(1-\varpi)^{\rho+1}(2^\rho-(1+\varpi)^\rho)^s d\varpi\right]\\
&=\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\frac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+|\eta''(k_2)|\int_0^1(1-\varpi)^{\rho+1}(2^\rho-(1-\varpi)^\rho)^s d\varpi\right.\\
&\left.+|\eta''(k_1)|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}+|\eta''(k_2)|\int_0^1(1+\varpi)^{\rho+1}(2^\rho-(1+\varpi)^\rho)^s d\varpi\right]\\
&\le\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\frac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+|\eta''(k_2)|\,\Omega(\rho,s,\varpi)\right.\\
&\left.+|\eta''(k_1)|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}+|\eta''(k_2)|\,\Omega(\rho,s,\varpi)\right].
\end{aligned}$$

Theorem 3. *Let $\eta:[k_1,k_2]\to\mathbb{R}$ be twice differentiable function on (k_1,k_2) with $k_1<k_2$. If $\eta''\in L[k_1,k_2]$ and $|\eta''|^q$ is (ρ,s)-convex function of second kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}
&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|\\
&\le\frac{(k_2-k_1)^2}{2^3(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left(\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)+\left(\frac{2^{-\rho s}}{\rho s+1}\right)\right)^{\frac{1}{q}}\left[|\eta''(k_1)|^q+|\eta''(k_2)|^q\right]^{\frac{1}{q}}
\end{aligned}$$

Proof. Applying Hölder's inequality, (ρ, s)-convex of $|\eta''|^q$ and Lemma1, we get

$$\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\alpha}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\alpha_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|$$
$$=\left|\frac{(k_2-k_1)^2}{8(\alpha+1)}\int_0^1(1-\varpi)^{\alpha+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)+\eta'\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right]d\varpi\right|$$
$$\leq\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)d\varpi\right|$$
$$+\left|\frac{k_2-k_1}{4}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)d\varpi\right|$$
$$\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left\{\left(\int_0^1(1-\varpi)^{p(\rho+1)}d\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-\varpi)^{p(\rho+1)}d\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left\{\left(|\eta''(k_1)|^q\int_0^1\left(\frac{1+\varpi}{2}\right)^{\rho s}d\varpi+|\eta''(k_2)|^q\int_0^1\left(1-\left(\frac{1-\varpi}{2}\right)^\rho\right)^s d\varpi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\eta''(k_1)|^q\int_0^1\left(\frac{1-\varpi}{2}\right)^{\rho s}d\varpi+|\eta''(k_2)|^q\int_0^1\left(1-\left(\frac{1+\varpi}{2}\right)^\rho\right)^s d\varpi\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left[\left\{|\eta''(k_1)|^q\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)+|\eta''(k_2)|^q\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{|\eta''(k_1)|^q\left(\frac{2^{-\rho s}}{\rho s+1}\right)+|\eta''(k_2)|^q\left(\frac{2^{-\rho s}}{\rho s+1}\right)\right\}^{\frac{1}{q}}\right]$$
$$\leq\frac{(k_2-k_1)^2}{2^3(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left(\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)+\left(\frac{2^{-\rho s}}{\rho s+1}\right)\right)^{\frac{1}{q}}[|\eta''(k_1)|^q+|\eta''(k_2)|^q]^{\frac{1}{q}}$$

Using power-mean inequality, we obtain the following integral inequalities.

Theorem 4. *Let $\eta:[k_1,k_2]\to\mathbb{R}$ be twice differentiable function on (k_1,k_2) with $k_1<k_2$. If $\eta''\in L[k_1,k_2]$ and $|\eta''|^q$ is (ρ,s)-convex function of second kind, then, we have the following inequality for fractional integrals:*

$$\left|\tfrac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\alpha}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\rho_{\left(\frac{k_1+k_2b}{2}\right)^+}\eta(k_2)\right]-\eta\left(\tfrac{k_1+k_2}{2}\right)\right|$$
$$\leq\tfrac{(k_2-k_1)^2}{2^3(\rho+1)}\left(\tfrac{1}{\rho+2}\right)^{1-\frac{1}{q}}\left\{\left(|\eta''(k_1)|^q\,\tfrac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+|\eta''(k_2)|^q\,\Omega(\rho,s,\varpi)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\eta''(k_1)|^q\left(\tfrac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}\right)+|\eta''(k_2)|^q\,\Omega(\rho,s,\varpi)\right)^{\frac{1}{q}}\right\}$$

Proof. Applying Lemma 1, power-mean inequality and the fact that $|\eta''|^q$ is (ρ, s)-convex function, we have

$$\begin{aligned}
&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\alpha_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|\\
=&\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)+\eta'\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right]d\varpi\right|\\
\leq&\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)d\varpi\right|\\
+&\left|\frac{k_2-k_1}{4}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)d\varpi\right|\\
\leq&\frac{(k_2-k_1)^2}{8(\rho+1)}\left\{\left(\int_0^1(1-\varpi)^{(\rho+1)}d\varpi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\varpi)^{(\rho+1)}\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1(1-\varpi)^{(\rho+1)}d\varpi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\varpi)^{(\rho+1)}\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right\}\\
\leq&\frac{(k_2-k_1)^2}{2^3(\rho+1)}\left(\frac{1}{\rho+2}\right)^{1-\frac{1}{q}}\left\{\left(|\eta''(k_1)|^q\,\frac{{}_2F_1(1,-\rho s;3+k_2;-1)}{k_1+2}+|\eta''(k_2)|^q\,\Omega(\rho,s,\varpi)\right)^{\frac{1}{q}}\right.\\
+&\left.\left(|\eta''(k_1)|^q\left(\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\alpha+3)}\right)+|\eta''(k_2)|^q\,\Omega(\rho,s,\varpi)\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Now, we present generalized integral inequalities via Riemann-Liouville operators for mappings whose twice differentiable are (ρ, s, m).

Theorem 5. *Let $\eta : [k_1, k_2] \to \mathbb{R}$ be twice differentiable function on (k_1, k_2) with $k_1 < k_2$. If $\eta'' \in L[k_1, k_2]$ and $|\eta''|$ is (ρ, s, m) convex function, where $(\rho, m) \in (0, 1]^2$, $s \in (-1, 1]$ then, we have the following inequality for fractional integrals:*

$$\begin{aligned}
&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho}\left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|\\
&\leq\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[\left|\eta''(k_1)\right|\frac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\Omega(\rho,s,\varpi)\right.\\
&\left.+\left|\eta''(k_1)\right|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\Omega(\rho,s,\varpi)\right],
\end{aligned}$$

where

$$\Omega(\rho, s, \varpi) = \int_0^1 (1-\varpi)^{\rho+1}(2^\rho - (1+\varpi)^\rho)^s d\varpi.$$

Proof. Using Lemma 1 and the fact that $|\eta''|$ is (ρ, s, m)-convexity, we get the following

$$\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^{\rho}}\left[J^{\rho}_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^{\rho}_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right|$$
$$=\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)+\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right]d\varpi\right|$$
$$\leq\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)d\varpi\right|$$
$$+\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)d\varpi\right|$$
$$\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|d\varpi$$
$$+\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|d\varpi$$
$$\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left[\int_0^1(1-\varpi)^{\rho+1}\left[\left(\frac{1+\varpi}{2}\right)^{\rho s}|\eta''(k_1)|+\left(1-\left(\frac{1-\varpi}{2}\right)^{\rho}\right)^s m\left|\eta''\left(\frac{k_2}{m}\right)\right|\right]d\varpi\right.$$
$$\left.+\int_0^1(1-\varpi)^{\rho+1}\left[\left(\frac{1-\varpi}{2}\right)^{\rho s}|\eta''(k_1)|+m\left(1-\left(\frac{1+\varpi}{2}\right)^{\rho}\right)^s\left|\eta''\left(\frac{k_2}{m}\right)\right|\right]d\varpi\right]$$
$$=\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\int_0^1(1-\varpi)^{\rho+1}(1+\varpi)^{\rho s}d\varpi+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\int_0^1(1-\varpi)^{\rho+1}(2^{\rho}-(1-\varpi)^{\rho})^s d\varpi\right.$$
$$\left.+|\eta''(k_1)|\int_0^1(1-\varpi)^{\rho+1}(1-\varpi)^{\rho s}d\varpi+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\int_0^1(1-\varpi)^{\rho+1}(2^{\rho}-(1+\varpi)^{\rho})^s d\varpi\right]$$
$$=\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\frac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\int_0^1(1-\varpi)^{\rho+1}(2^{\rho}-(1-\varpi)^{\rho})^s d\varpi\right.$$
$$\left.+|\eta''(k_1)|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\int_0^1(1+\varpi)^{\rho+1}(2^{\rho}-(1+\varpi)^{\rho})^s d\varpi\right]$$
$$\leq\frac{(k_2-k_1)^2}{2^{\rho s+3}(\rho+1)}\left[|\eta''(k_1)|\frac{{}_2F_1(1,-\rho s;3+k_1;-1)}{k_1+2}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\Omega(\rho,s,\varpi)\right.$$
$$\left.+|\eta''(k_1)|\frac{\Gamma(\rho s+\rho+2)}{\Gamma(\rho s+\rho+3)}+m\left|\eta''\left(\frac{k_2}{m}\right)\right|\Omega(\rho,s,\varpi)\right].$$

Theorem 6. *Let $\eta:[k_1,k_2]\to\mathbb{R}$ be twice differentiable function on (k_1,k_2) with $k_1<k_2$. If $\eta''\in L[k_1,k_2]$ and $|\eta''|^q$ is (ρ,s,m) is convex function, where $(\rho,m)\in(0,1]^2$,*

$s \in (-1, 1]$, *then, we have the following inequality for fractional integrals:*

$$\left| \frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho} \left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1) + J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2) \right] - \eta\left(\frac{k_1+k_2}{2}\right) \right|$$
$$\le \frac{(k_2-k_1)^2}{8(\rho+1)} \left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}} \left[\left\{ |\eta''(k_1)|^q \left(\frac{2-2^{-\rho s}}{\rho s+1}\right) + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \left(\frac{2-2^{-\rho s}}{\rho s+1}\right) \right\}^{\frac{1}{q}} \right.$$
$$\left. + \left\{ |\eta''(k_1)|^q \left(\frac{2^{-\rho s}}{\rho s+1}\right) + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \left(\frac{2^{-\rho s}}{\rho s+1}\right) \right\}^{\frac{1}{q}} \right]$$

Proof. Applying Hölder's inequality, Lemma 1 and the fact that $|\eta''|^q$ is (ρ, s, m)-convex function, we have

$$\left| \frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^\rho} \left[J^\rho_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1) + J^\rho_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2) \right] - \eta\left(\frac{k_1+k_2}{2}\right) \right|$$
$$= \left| \frac{(k_2-k_1)^2}{8(\rho+1)} \int_0^1 (1-\varpi)^{\rho+1} \left[\eta''\left(\frac{1+\varpi}{2}k_1 + \frac{1-\varpi}{2}k_2\right) + \eta'\left(\frac{1-\varpi}{2}k_1 + \frac{1+\varpi}{2}k_2\right) \right] d\varpi \right|$$
$$\le \left| \frac{(k_2-k_1)^2}{8(\rho+1)} \int_0^1 (1-\varpi)^{\rho+1} \eta''\left(\frac{1+\varpi}{2}k_1 + \frac{1-\varpi}{2}k_2\right) d\varpi \right|$$
$$+ \left| \frac{k_2-k_1}{4} \int_0^1 (1-\varpi)^{\rho+1} \eta''\left(\frac{1-\varpi}{2}k_1 + \frac{1+\varpi}{2}k_2\right) d\varpi \right|$$
$$\le \frac{(k_2-k_1)^2}{8(\rho+1)} \left\{ \left(\int_0^1 (1-\varpi)^{p(\rho+1)} d\varpi \right)^{\frac{1}{p}} \left(\int_0^1 \left| \eta''\left(\frac{1+\varpi}{2}k_1 + \frac{1-\varpi}{2}k_2\right) \right|^q d\varpi \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_0^1 (1-\varpi)^{p(\rho+1)} d\varpi \right)^{\frac{1}{p}} \left(\int_0^1 \left| \eta''\left(\frac{1-\varpi}{2}k_1 + \frac{1+\varpi}{2}k_2\right) \right|^q d\varpi \right)^{\frac{1}{q}} \right\}$$
$$\le \frac{(k_2-k_1)^2}{8(\rho+1)} \left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}} \left\{ \left(|\eta''(k_1)|^q \int_0^1 \left(\frac{1+\varpi}{2}\right)^{\rho s} d\varpi + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \int_0^1 \left(1 - \left(\frac{1-\varpi}{2}\right)^\rho\right)^s d\varpi \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(|\eta''(k_1)|^q \int_0^1 \left(\frac{1-\varpi}{2}\right)^{\rho s} d\varpi + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \int_0^1 \left(1 - \left(\frac{1+\varpi}{2}\right)^\rho\right)^s d\varpi \right)^{\frac{1}{q}} \right\}$$
$$\le \frac{(k_2-k_1)^2}{8(\rho+1)} \left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}} \left[\left\{ |\eta''(k_1)|^q \left(\frac{2-2^{-\rho s}}{\rho s+1}\right) + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \left(\frac{2-2^{-\rho s}}{\rho s+1}\right) \right\}^{\frac{1}{q}} \right.$$
$$\left. + \left\{ |\eta''(k_1)|^q \left(\frac{2^{-\rho s}}{\rho s+1}\right) + m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q \left(\frac{2^{-\rho s}}{\rho s+1}\right) \right\}^{\frac{1}{q}} \right].$$

$$
\begin{aligned}
&\left|\frac{2^{\rho-1}\Gamma(\rho+1)}{(k_2-k_1)^{\rho}}\left[J^{\rho}_{\left(\frac{k_1+k_2}{2}\right)^-}\eta(k_1)+J^{\rho}_{\left(\frac{k_1+k_2}{2}\right)^+}\eta(k_2)\right]-\eta\left(\frac{k_1+k_2}{2}\right)\right| \\
&=\left|\frac{(k_2-k_1)^2}{8(\rho+1)}\int_0^1(1-\varpi)^{\rho+1}\left[\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)+\eta'\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right]d\varpi\right| \\
&\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left[\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|d\varpi\right. \\
&\quad\left.+\int_0^1(1-\varpi)^{\rho+1}\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|d\varpi\right] \\
&\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left\{\left[\left(\int_0^1(1-\varpi)^{p(\rho+1)}d\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|\eta''\left(\frac{1+\varpi}{2}k_1+\frac{1-\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right]\right. \\
&\quad\left.+\left[\left(\int_0^1(1-\varpi)^{p(\rho+1)}d\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|\eta''\left(\frac{1-\varpi}{2}k_1+\frac{1+\varpi}{2}k_2\right)\right|^q d\varpi\right)^{\frac{1}{q}}\right]\right\} \\
&\leq\frac{(k_2-k_1)^2}{8(\rho+1)}\left(\frac{1}{p(\rho+1)+1}\right)^{\frac{1}{p}}\left[\left\{\left|\eta''(k_1)\right|^q\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)+m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q\left(\frac{2-2^{-\rho s}}{\rho s+1}\right)\right\}^{\frac{1}{q}}\right. \\
&\quad\left.+\left\{\left|\eta''(k_1)\right|^q\left(\frac{2^{-\rho s}}{\rho s+1}\right)+m\left|\eta''\left(\frac{k_2}{m}\right)\right|^q\left(\frac{2^{-\rho s}}{\rho s+1}\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

4. Conclusion

Fractional calculus plays a vital role in understanding problems in pure and applied sciences due to its possession of many interesting integral and differential operators. In this study, therefore, we employed a Riemann-Liouville operator to establish some new fractional integral inequalities for mappings whose second derivatives, raised to positive powers, exhibit (ρ, s)-convexity and (ρ, s, m)-convexities. Our study established new inequalities of midpoint type involving fractional operators through generalized classes of convexities. Several estimates of special functions including incomplete Beta, Euler gamma and hyperglycemic functions are reported in this study. Our findings can enhance the techniques by which the properties of convexity along with their generalizations can be thoroughly studied through Fractional Calculus.

The findings of this study can be relevant in different areas of interest where the fractional calculus is extensibility used. Our inequalities can be specifically used to estimate the error bounds for numerical integration by improving the accuracy of quadrature methods. Future studies should include the use of different fractional integral operators, such as the generalized fractional integral operator unifying two existing fractional integral operators [16], together with other classes of convexities to establish different inequalities.

References

[1] Praveen Agarwal, Mohamed Jleli, and Muharrem Tomar. Certain hermite-hadamard type inequalities via generalized k-fractional integrals. *Journal of inequalities and applications*, 2017:1–10, 2017.

[2] Ravi P Agarwal. *Inequalities and applications*, volume 3. World scientific, 1994.

[3] Ohud Bulayhan Almutairi. New fractional integral inequalities via euler's beta function. *Open Mathematics*, 21(1):20230163, 2023.

[4] Drumi D Bainov and Pavel S Simeonov. *Integral inequalities and applications*, volume 57. Springer Science & Business Media, 2013.

[5] Hüseyin Budak, Tuba Tunç, and Mehmet Sarikaya. Fractional hermite-hadamard-type inequalities for interval-valued functions. *Proceedings of the American Mathematical Society*, 148(2):705–718, 2020.

[6] Tingsong Du and Yu Peng. Hermite–hadamard type inequalities for multiplicative riemann–liouville fractional integrals. *Journal of Computational and Applied Mathematics*, 440:115582, 2024.

[7] Ghulam Farid, Young Chel Kwun, Hafsa Yasmeen, Abdullah Akkurt, and Shin Min Kang. Inequalities for generalized riemann–liouville fractional integrals of generalized strongly convex functions. *Advances in Difference Equations*, 2021:1–25, 2021.

[8] Henryk Hudzik and Lech Maligranda. Some remarks on s-convex functions. *Aequationes mathematicae*, 48:100–111, 1994.

[9] Anatoliĭ Aleksandrovich Kilbas, Hari M Srivastava, and Juan J Trujillo. *Theory and applications of fractional differential equations*, volume 204. elsevier, 2006.

[10] Mohamed Metwali and Vishnu Narayan Mishra. On the measure of noncompactness in l_p(R+) and applications to a product of n-integral equations. *Turkish Journal of Mathematics*, 47(1):372–386, 2023.

[11] Muhammad Aslam Noor and Muhammad Uzair Awan. Some integral inequalities for two kinds of convexities via fractional integrals. *Trans. J. Math. Mech*, 5(2):129–136, 2013.

[12] Emin Özçağ, Inci Ege, and Haşmet Gürçay. An extension of the incomplete beta function for negative integers. *Journal of mathematical analysis and applications*, 338(2):984–992, 2008.

[13] Vijai Kumar Pathak, Lakshmi Narayan Mishra, and Vishnu Narayan Mishra. On the solvability of a class of nonlinear functional integral equations involving erdélyi–kober fractional operator. *Mathematical Methods in the Applied Sciences*, 46(13):14340–14352, 2023.

[14] Supriya Kumar Paul and Lakshmi Narayan Mishra. Approximation of solutions through the fibonacci wavelets and measure of noncompactness to nonlinear volterra-fredholm fractional integral equations. *Korean Journal of Mathematics*, 32(1):137–162, 2024.

[15] Supriya Kumar Paul, Lakshmi Narayan Mishra, and Vishnu Narayan Mishra. Approximate numerical solutions of fractional integral equations using laguerre and touchard polynomials. *Palestine Journal of Mathematics*, 12(3), 2023.

[16] Supriya Kumar Paul, Lakshmi Narayan Mishra, and Vishnu Narayan Mishra. Re-

sults on integral inequalities for a generalized fractional integral operator unifying two existing fractional integral operators. *Nonlinear Analysis: Modelling and Control*, 29(6):1080–1105, 2024.

[17] Supriya Kumar Paul, Lakshmi Narayan Mishra, Vishnu Narayan Mishra, and Dumitru Baleanu. Analysis of mixed type nonlinear volterra–fredholm integral equations involving the erdélyi–kober fractional operator. *Journal of King Saud University-Science*, 35(10):102949, 2023.

[18] Stefan G Samko. Fractional integrals and derivatives. *Theory and applications*, 1993.

[19] Mehmet Zeki Sarikaya, Erhan Set, Hatice Yaldiz, and Nagihan Başak. Hermite–hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57(9-10):2403–2407, 2013.

[20] Gheorghe Toader. Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization*, volume 329, page 338. University of Cluj-Napoca Cluj-Napoca, Romania, 1984.

[21] Bo-Yan Xi, Dan-Dan Gao, and Feng Qi. Integral inequalities of hermite–hadamard type for (α, s)-convex and (α, s, m)-convex functions. *Italian Journal of Pure and Applied Mathematics*, (44):499–510, 2020.

[22] Yi Xing, Chaoqun Jiang, and Jianmiao Ruan. Hermite-hadamard type inequalities for riemann-liouville fractional integrals via strongly h-convex functions. *Authorea Preprints*, 2024.