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Properties of Bazilevič Functions Involving q-analogue of the Generalized M-series

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Abstract. With primary motive to unify and extend the various well-known studies, we define a new family of differential operator using the q-analogue of the generalized M-series. The generalized M-series unifies two well-known and extensively used special functions namely *generalized hypergeometric function* and *Mittag-Leffler function*. Making use of the defined operator, we define a new family of analytic functions expressed as a combination two differential characterizations. The combination of differential characterizations involving the operator not only unifies studies of starlike, convex, Bazilevič and α-convex function classes, it extends to new classes. Estimates involving the initial coefficients of the functions, which belong to the defined function class are our main results. Some examples along with graphs have been used to establish the inclusion and closure properties.

2020 Mathematics Subject Classifications: 30C45; 30C50, 30C55

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1. Introduction

Fractional calculus has been a very important tool in the mathematical analysis irrespective of whether the study involves theoretical aspects or applications oriented. It is not a new calculus, in fact it is as old as classical calculus. But its has acquired the interests of several researchers since it fits very well in modelling of problems involving natural

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phenomena. Its applications can be found in various scientific disciplines like biology, chemistry, physics, acoustics, materials science, fluid mechanics and dynamical systems. Mittag-Leffler function is one such special function which cannot be avoided if one has too delve into the field fractional calculus.

Fractional q-calculus is an extension of the classical fractional calculus aimed at the discretization, unification and generalization. It mainly unifies the study of continuous and discrete analysis. Quantum calculus is essentially motivated by the concept of finite difference rescaling. To be precise, it is nothing but a ratio that is similar to one used in Newton's divided difference table. The primary reason for this calculus to be relevant even today is due to the fact the all the concepts of classical calculus cannot be translated to quantum calculus, even the basic chain rule needs adaptation. Also the interest are because quantum computing models involves lots of mathematics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and the theory of relativity. Refer to Kac and Cheung [27] for its definition and basic properties. Refer to [2, 3, 11, 12] for its recent developments.

The *Meijer G-function* and *Fox's H-function* are most generalized function to which nearly all special functions will form to be their special cases. A brief overview of the some special functions which helps in unification with the generalized M-series (see [45, Eq. 1]).

Let Λ, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the unit disc, set of real numbers, set of complex numbers and set of natural numbers respectively. We denote Θ to be the class of functions $\chi(\xi)$ analytic in Λ with the normalization $\chi(0) = \chi'(0) - 1 = 0$ which will result in a series of the form

$$\chi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad (\xi \in \Lambda;\, \varphi_n \in \mathbb{C}). \tag{1}$$

For $\kappa_i \in \mathbb{C}\,(i = 1, \ldots, r)$ and $\sigma_j \in \mathbb{C}\backslash Z_0^- = \{0, -1, \ldots\}\,(j = 1, \ldots, s)$, the *Fox–Wright function* ${}_r\Psi_s$, which is defined by (see ([51, Equation 1.6]), ([53, p. 19]) and ([54, p. 21]))

$${}_r\Psi_s \begin{bmatrix} (\kappa_1, A_1) & \ldots & (\kappa_r, A_r) & \\ (\sigma_1, B_1) & \ldots & (\sigma_s, B_s) & ;\xi \end{bmatrix} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(\kappa_j + A_j n)}{\prod_{j=1}^{s} \Gamma(\sigma_j + B_j n)} \frac{\xi^n}{n!}. \tag{2}$$

where $\mathrm{Re}(A_i) > 0$, $(i = 1, \ldots, r)$ and $\mathrm{Re}(B_j) > 0 \in \mathbb{C}\,(j = 1, \ldots, s)$ with $1 + \mathrm{Re}\left(\sum_{j=1}^{s} B_j - \sum_{i=1}^{r} A_i\right) \geq 0$. For discussion on the convergence of the series (2), refer to Srivastava ([52], Definition 2).

Lin and Srivastava [32, eq. 8] introduced a following generalization of the well-known *Hurwitz–Lerch zeta function* $\phi_{\kappa,\sigma}^{k,\epsilon}(\xi, m, \kappa)$ given by

$$\phi_{\kappa,\sigma}^{k,\epsilon}(\xi, m, \kappa) = \sum_{n=0}^{\infty} \frac{(\kappa)_{kn}}{(\sigma)_{\epsilon n}} \frac{\xi^n}{(n+\kappa)^m},$$

where $\kappa \in \mathbb{C}$; $\sigma, \kappa \in \mathbb{C} \setminus Z_0^-$; $k, \epsilon \in \mathbb{R}^+$; $k < \epsilon$ when $m, \xi \in \mathbb{C}$; $k = \epsilon$ and $m \in \mathbb{C}$ when $|\xi| < 1$; $k = \epsilon$ and $\mathrm{Re}(m - \kappa + \sigma) > 1$ when $|\xi| = 1$. These types of generalizations are very common as it not only unifies studies, it extends various studies. In some cases, such generalizations requires adaptation or deviations and it has proved to be very useful tool in analysis.

The study of generalized M-series is one such generalization which has proved to be an important tool in the studies pertaining to duality theory. The generalized M-series[45, Eq. 1] (also see [55]) defined to unify the studies pertaining to Mittag-Leffler function and Gaussian hypergeometric function, is given by

$$ {}_r\mathcal{M}_s^{\eta,\theta}(\xi) = {}_r\mathcal{M}_s^{\eta,\theta}(\kappa_1, \ldots, \kappa_r; \sigma_1, \ldots, \sigma_s; \xi) = \sum_{n=0}^{\infty} \frac{(\kappa_1)_n \ldots (\kappa_r)_n}{(\sigma_1)_n \ldots (\sigma_s)_n} \frac{\xi^n}{\Gamma(n\eta + \theta)}, \tag{3} $$

ξ, η, $\theta \in \mathbb{C}, \mathrm{Re}(\eta) > 0$ and $(\kappa_i)_n$, $(\sigma_j)_n$ are the well-known Pochhammer symbol. Further the primary condition for the existence of the series (3) is that the denominator terms $\sigma_j' s$, $(j = 1, 2, \ldots s)$ are never zero or negative integer. Whereas if any of the numerator terms $\kappa_j' s$, $(j = 1, 2, \ldots r)$ is zero or negative integer, then the infinite series terminates to be polynomial in ξ. Note that generalized M-series can be represented in terms of the Fox–Wright function as follow:

$$ {}_r\mathcal{M}_s^{\eta,\theta}(\xi) = \varepsilon\, {}_{r+1}\Psi_{s+1} \begin{bmatrix} (\kappa_1, 1) & \ldots & (\kappa_r, 1), (1, 1) & ; \xi \\ (\sigma_1, 1) & \ldots & (\sigma_s, 1)(\theta, \eta) & \end{bmatrix}, \ \varepsilon = \prod_{j=1}^{r} \Gamma(\sigma_j) / \prod_{j=1}^{s} \Gamma(\kappa_j) $$

Hence the convergence of the series (3) is same as that of *Fox–Wright function.*

The q-analogue of the generalized M-series (3) was studied by Shimelis and Suthar [46–48] and is defined by

$$ {}_r\mathcal{M}_s^{\eta,\theta}(q, \xi) = \sum_{n=0}^{\infty} \frac{(q^{\kappa_1}; q)_n \ldots (q^{\kappa_r}; q)_n}{(q^{\sigma_1}; q)_n \ldots (q^{\sigma_s}; q)_n} \left[(-1)^n q^{n(n-1)/2} \right]^{s-r+1} \frac{\xi^n}{\Gamma_q(n\eta + \theta)}, \tag{4} $$

where $\Gamma_q(.)$ is the q-gamma function and $(q^{\kappa_i}; q)_n$, $(q^{\sigma_j}; q)_n$; $\kappa_i, \sigma_j \neq 0, -1, \ldots$ $(i = 1, 2, \ldots r;\ j = 1, 2, \ldots, s)$ are a q-analogue of Pochhammer symbol. Again, the convergence details of (4) can be found in Srivastava ([52], Definition 2).

The primary aim of this study is unification, extension and discertization. For this purpose, we will define a new family of linear operator involving the q-analogue of the generalized M-series. In this study, we aim to analyze the behaviour and geometrical implications when two differential characterization are expressed as convex combination. We will obtain the coefficient inequalities which would in turn help us understand algebraic and geometric properties.

1.1. New family of generalized differential operators and its special cases.

Corresponding to a function $\mathcal{G}_{r,s}^{\eta,\theta}(\kappa_1, \sigma_1; \xi)$ defined by

$$ \mathcal{G}_{r,s}^{\eta,\theta}(\kappa_1, \sigma_1; \xi) := \xi \Gamma(\theta) \left[{}_r\mathcal{M}_{s+1}^{\eta,\theta}(\kappa_1, \ldots, \kappa_r; \sigma_1, \ldots, \sigma_s, 1; \xi) \right]. \tag{5} $$

We now define the following operator $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi : \Lambda \longrightarrow \Lambda$ by

$$D_\lambda^0(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi = \chi(\xi) * \mathcal{G}_{r,\,s}^{\eta,\,\theta}(\kappa_1,\,\sigma_1;\,\xi)$$

$$D_\lambda^1(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi = (1-\lambda)\left(\chi(\xi) * \mathcal{G}_{r,\,s}^{\eta,\,\theta}(\kappa_1,\,\sigma_1;\,\xi)\right) + \lambda\,\xi\left(\chi(\xi) * \mathcal{G}_{r,\,s}^{\eta,\,\theta}(\kappa_1,\,\sigma_1;\,\xi)\right)' \tag{6}$$

$$D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi = D_\lambda^1\left[D_\lambda^{m-1}(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi\right]. \tag{7}$$

Here $*$ denotes the Hadamard product or convolution. If $\chi \in \Theta$, then from (6) and (7) we may easily deduce that

$$D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi = \xi + \sum_{n=2}^{\infty}\left[1+(n-1)\lambda\right]^m \frac{(\kappa_1)_{n-1}\,\ldots\,(\kappa_r)_{n-1}}{(\sigma_1)_{n-1}\,\ldots\,(\sigma_s)_{n-1}}\,\frac{\Gamma(\theta)\,\varphi_n\xi^n}{(n-1)!\Gamma(\eta(n-1)+\theta)} \tag{8}$$

where $\kappa_j \in \mathbb{C}\,(j = 1,\ldots,r)$; $\sigma_j \in \mathbb{C}\setminus Z_0^- = \{0,\,-1,\ldots\}\,(j = 1,\ldots,s)$; $m \in N_0$; $\lambda \geq 0$; $\eta,\,\theta \in \mathbb{C}$ and $\mathrm{Re}(\eta) > 0$. Letting $\eta = 0$ in (8), we get the operator studied by Selvaraj and Karthikeyan [42, Eq. 1.5]. Letting $r = 2$, $s = 1$, $\kappa_1 = \sigma_1$ and $\kappa_2 = 1$ in (8), we get the operator

$$D_\lambda^m(\eta,\,\theta)\chi(\xi) = \xi + \sum_{n=2}^{\infty}\left[1+(n-1)\lambda\right]^m \frac{\Gamma(\theta)\,\varphi_n\xi^n}{\Gamma(\eta(n-1)+\theta)}. \tag{9}$$

The operator $D_\lambda^m(\eta,\,\theta)\chi$ was introduced by Elhaddad et al. [25, Eq 1.6] and was further studied by Mashwan et al. [35, Eq. 16]. For the choice of $m = 0$ in (8), the operator $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi$ reduces to the well-known Dziok- Srivastava operator [22]. The operator recently introduced by Breaz et al. [13, 14] (also see [58]), Cağlar et al [19] and Cang and Liu [17] are closely related to the operator $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi$, in fact we could have obtained the same operators if we had defined the equation (5) in the form

$$\mathcal{G}_{r,\,s}^{\eta,\,\theta}(\kappa_1,\,\sigma_1;\,\xi) := \frac{\Gamma(\eta+\theta)\prod_{j=1}^{r}\Gamma(\sigma_j)}{\prod_{j=1}^{s}\Gamma(\kappa_j)}\left[{}_r\mathcal{M}_s^{\eta,\theta}(\kappa_1,\,\ldots,\kappa_r;\sigma_1,\,\ldots,\sigma_s;\,\xi) - \frac{1}{\Gamma(\theta)}\right].$$

Also many (well known and new) integral and differential operators can be obtained by specializing the parameters involved in $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi$.

1.2. q-analogue of the operator $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi$.

The q-analogue of the operator $D_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta)\chi$ will be of the form

$$J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi = \xi + \sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m \frac{(\kappa_1;q)_{n-1}\cdots(\kappa_r;q)_{n-1}}{(q;q)_{n-1}\,(\sigma_1;q)_{n-1}\cdots(\sigma_s;q)_{n-1}}\,\frac{\Gamma_q(\theta)\varphi_n\xi^n}{\Gamma_q(\eta(n-1)+\theta)} \tag{10}$$

For $\kappa_i = q^{c_i}$, $\sigma_j = q^{d_j}$, $c_i \in \mathbb{C}$, $d_j \in \mathbb{C}\setminus Z_0^{-1}$, $(i = 1,\ldots,\,r;\,j = 1,\,\ldots,\,s)$ and $q \to 1-$ in (10), the operator $J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi$ reduces to the operator $D_\lambda^m(c_1,\,d_1;\,\eta,\,\theta)\chi$.

Letting $\eta = 0$ in $J^m_\lambda(\kappa_1, \sigma_1; \eta, \theta; q, \xi)\chi$, we get the operator introduced and studied by Reddy et al [40, Eq. 8]. Further setting $\eta = m = 0$ in (10), we get the operator introduced and studied by Darus in [21, Eq. 3]. The q-Calson-Shaffer operator [43], q-Ruscheweyh derivative operator [30], q-Salagean operators [7] and various other operators involving Mittag-Leffler function are the special cases of the operator $J^m_\lambda(\kappa_1, \sigma_1; \eta, \theta; q, \xi)\xi$.

The study of various subclasses of analytic functions involving with various special functions was spotlighted after De Branges's used it in the proof of Bieberbach conjecture. It should be noted that various convolution properties studied by Ruscheweyh in [41] was a stimulant to study this duality theory. After finding several applications of convolutions in various fields, he posed many questions which led to the development in this duality theory. Refer to [6, 15, 16, 24, 29, 31] for the recent developments pertaining to this duality theory

1.3. Short introduction to geometric function theory.

We call $\mathcal{F}$ (see [18]) to denote the class of functions with normalization $p(0) = 1$ which satisfies $\mathrm{Re}\,(p(\xi)) > 0$, $\xi \in \Lambda$. We denote the subclasses of Θ namely starlike and convex functions which satisfies the following respective differential inclusions

$$\frac{\xi\chi'(\xi)}{\chi(\xi)} \in \mathcal{F} \quad \text{and} \quad \frac{(\xi\chi'(\xi))'}{\chi'(\xi)} \in \mathcal{F}.$$

We denote the class of starlike and convex functions by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Expressing the analytic characterizations of $\mathcal{S}^*$ and $\mathcal{C}$ as a convex combination, Mocanu studied the so-called δ-convex functions defined by

$$(1-\delta)\frac{\xi\chi'(\xi)}{\chi(\xi)} + \delta\frac{(\xi\chi'(\xi))'}{\chi'(\xi)} \in \mathcal{F}, \quad (\chi \in \Theta;\, 0 \le \delta \le 1).$$

Here we will denote the class of δ-convex functions as $\mathcal{MC}(\delta)$.

Ma-Minda [34] obtained the coefficient estimates of a class of function $\Psi \in \mathcal{F}$ which are starlike with respect to 1 and has a series expansion of the form

$$\Psi(\xi) = 1 + \psi_1\xi + \psi_2\xi^2 + \psi_3\xi^3 + \cdots, \qquad (\psi_1 > 0;\, \xi \in \Lambda). \tag{11}$$

Motivated by

$$\mathcal{S}^*(\Psi) := \left\{\chi \in \Theta;\, \frac{\xi\chi'(\xi)}{\chi(\xi)} \prec \Psi(\xi)\right\}$$

and

$$\mathcal{C}(\Psi) := \left\{\chi \in \Theta;\, \frac{(\xi\chi'(\xi))'}{\chi'(\xi)} \prec \Psi(\xi)\right\}.$$

Replacing the Ma-Minda function Ψ in $\mathcal{S}^*(\Psi)$ and $\mathcal{C}(\Psi)$ with some special functions, several authors studied interesting subclasses of starlike and convex functions. Here we will tabulate only a few of them which was studied for class of analytic functions.

Conic Region	$\Psi(\xi)$	Reference
Right-half of the lemniscate of Bernoulli	$\sqrt{1+\xi}$	Sokół[49, 50]
Left-half of the lemniscate of Bernoulli	$\sqrt{2}-\left(\sqrt{2}-1\right)\sqrt{\frac{1-\xi}{1+2\left(\sqrt{2}-1\right)\xi}}$	Mendiratta et al. [36]
Cardioid	$1+\frac{4\xi}{3}+\frac{2\xi^2}{3}$	Sharma et al. [44]
Crescent or Lune shape	$\xi+\sqrt{1+\xi^2}$	Raina and Sokół[39]
Limacon	$1+\sqrt{2}\xi+\frac{\xi^2}{2}$	Cho et al. [20]
Nephroid	$1+\xi-\frac{\xi^3}{3}$	Wani and Swaminathan [56]

Table 1: Study of subclasses of analytic functions impacted by conic regions

1.4. New Subclass of Analytic Functions.

Motivated by [1, 8–10, 28], we define the following.

Definition 1. *For* $\omega \geq 0$ *and* $\delta \in \mathbb{C}$ *such that* $\operatorname{Re}(\delta) > 0$*, a function* χ *belongs to the class* $\mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1, \sigma_1; \eta,\, \theta;\, \delta;\, \Psi(\xi))$ *if it satisfies*

$$\left\{(1-\delta)\left(\frac{J_{\lambda}^{m}(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)}{\xi}\right)^{\omega}+\delta\frac{\xi^{1-\omega}J_{\lambda}^{m}(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi'(\xi)}{\left[J_{\lambda}^{m}(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)\right]^{1-\omega}}\right\}\prec\Psi(\xi), \tag{12}$$

where $\Psi(\xi) \in \mathcal{F}$ *has a power series representation of the form* (11)*.*

Remark 1. *Now we will discuss few special cases of our class* $\mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1, \sigma_1; \eta,\, \theta;\, \delta;\, \Psi(\xi))$*:*

(i) Letting $r = 2$*,* $s = 1$*,* $\kappa_1 = \sigma_1$*,* $\sigma_2 = q$ *and* $q \to 1-$ *in* (12)*, we get the class* $\mathcal{M}_{\lambda}^{m,\,\omega}(\eta,\, \theta;\, \delta;\, k;\, \rho)$ *which satisfies the condition*

$$(1-\delta)\left(\frac{D_{\lambda}^{m}(\eta,\,\theta)\chi(\xi)}{\xi}\right)^{\omega}+\delta\frac{\xi^{1-\omega}D_{\lambda}^{m}(\eta,\,\theta)\chi'(\xi)}{\left[D_{\lambda}^{m}(\eta,\,\theta)\chi(\xi)\right]^{1-\omega}}\in\mathcal{F}_k(\rho)$$

where $D_{\lambda}^{m}(\eta,\, \theta)\chi$ *is given by* (9) *and* $\mathcal{F}_k(\rho)$ *consist of functions in* $\mathcal{F}$ *satisfying*

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}p(\xi)-\rho}{1-\rho}\right|d\nu\leq k\pi,\quad (\xi=re^{i\nu};\,k\geq 2;\,0\leq\rho<1).$$

The class $\mathcal{M}_{\lambda}^{m,\,\omega}(\eta,\, \theta;\, \delta;\, k;\, \rho)$ *was studied by Ahuja et al. [1, Definition 1].*

(ii) *Setting* $r = 2$, $s = 1$, $\kappa_1 = \sigma_1$, $\sigma_2 = q$, $m = \eta = 0$ *and* $q \to 1-$ *in Definition 1, we get*

$$\mathcal{BS}(\delta;\, \Psi) = \left\{ \chi \in \Theta :\, (1-\delta)\left(\frac{\chi(\xi)}{\xi}\right)^{\omega} + \delta \frac{\xi\chi'(\xi)}{\chi(\xi)} \left(\frac{\chi(\xi)}{\xi}\right)^{\omega} \prec \Psi(\xi) \right\}.$$

(iii) *Letting* $\delta = 1$ *and* $\psi(\xi) = (1+\xi)/(1-\xi)$ *in* $\mathcal{BS}(\delta;\, \Psi)$, *we get the famous Bazilevič class.*

Specializing the function $\Psi(\xi)$ *in the Definition 1, we can obtain the classes studied by various authors.*

We will need the following lemmas to establish our main results.

Lemma 1. *[23, Theorem 1] If* $L(\xi) = 1 + \sum\limits_{r=1}^{\infty} \ell_r \xi^r \in \mathcal{F}$, *and* $\rho \in \mathbb{C}$, *then*

$$|\ell_\varepsilon - \rho \ell_r \ell_{\varepsilon - r}| \leq 2 \max\left\{1; |2\rho - 1|\right\},$$

for all $1 \leq r \leq \varepsilon - 1$.

Note that the above results is generalization of the well-known results of Ma-Minda [34, p. 162] and Livingston [33, Lemma 1].

Lemma 2. *[26] Let* g *be convex in* Λ, *with* $g(0) = d$, $\varrho \neq 0$ *and* $\mathrm{Re}(\varrho) > 0$. *Suppose that* $\Delta(\xi)$ *is analytic* Λ, *which is given by*

$$\Delta(\xi) = d + d_n \xi^n + d_{n+1}\xi^{n+1} + \cdots, \quad \xi \in \Lambda. \tag{13}$$

If

$$\Delta(\xi) + \frac{\xi \Delta'(\xi)}{\varrho} \prec g(\xi),$$

then

$$\Delta(\xi) \prec q(\xi) \prec g(\xi),$$

where

$$q(\xi) = \frac{\varrho}{n\, \xi^{\varrho/n}} \int_0^{\xi} g(t)\, t^{(\varrho/n)-1} dt.$$

The function q *is convex and is the best* $(a,\, n)$*-dominant.*

2. Inclusion Relations and Integral Representations

Theorem 1. *Let the function* $\Psi(\xi)$ *defined as in* (11) *be convex univalent in* Λ. *Let* $\chi \in \mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta,\, \theta;\, \delta;\, \Psi(\xi))$ *with* $\mathrm{Re}(\delta) > 0$ *and* $\omega \neq 0$, *then*

$$\left(\frac{J_\lambda^m(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^{\omega} \prec q(\xi) = \frac{\omega}{\delta} \xi^{\frac{-\omega}{\delta}} \int_0^{\xi} t^{\frac{\omega}{\delta}-1} \Psi(t) dt \prec \Psi(\xi). \tag{14}$$

and for $\omega = 0$, we have

$$J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi) = \xi\, \exp\left\{\int_0^{\xi} \frac{\Psi[w(t)] - 1}{\delta t} dt\right\}, \tag{15}$$

where $w(\xi)$ is the Schwarz function and $q(\xi)$ is the best dominant.

Proof. Let $h(\xi)$ be defined by

$$h(\xi) = \left(\frac{J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^{\omega}, \quad \xi \in \Lambda. \tag{16}$$

Then the function $h(\xi)$ is of the form $h(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$ and is analytic in Λ. Differentiating both sides of (16) and by simplifying, we have

$$(1-\delta)\left(\frac{J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^{\omega} + \delta \frac{\xi^{1-\omega} J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi'(\xi)}{\left[J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)\right]^{1-\omega}} = h(\xi) + \frac{\delta}{\omega}\xi h'(\xi). \tag{17}$$

By hypothesis $\chi \in \mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta,\, \theta;\, \delta;\, \Psi(\xi))$, so from Definition 1 we have

$$h(\xi) + \frac{\delta}{\omega}\xi h'(\xi) \prec \Psi(\xi).$$

Applying Lemma 2 to (17) with $\varrho = \frac{\omega}{\delta}$ and $n = 1$, we get

$$\left(\frac{J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^{\omega} \prec \frac{\omega}{\delta}\xi^{\frac{-\omega}{\delta}} \int_0^{\xi} t^{\frac{\omega}{\delta}-1}\Psi(t)dt \prec \Psi(\xi). \tag{18}$$

Hence the proof (14). Letting $\omega = 0$ in (12), we get

$$\frac{d}{dz}\log\left[\frac{J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right] = \frac{\Psi[w(\xi)] - 1}{\delta\xi}.$$

On integrating the above expression, we get the result (15).

Remark 2. *Note that for the result* (15)*, the necessity of* $\Psi(\xi)$ *to be convex is not required.*

Remark 3. *Now we will discuss the benefits of studying the class involving an operator. Letting $\omega = 1$ in Theorem 1,* (14) *will become*

$$J_{\lambda}^{m}(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi) \prec \frac{1}{\delta}\xi^{1-\frac{1}{\delta}} \int_0^{\xi} t^{\frac{\omega}{\delta}-1}\Psi(t)dt. \tag{19}$$

Now setting $r = 2$, $s = 1$, $\kappa_1 = \sigma_1$, $\sigma_2 = q$, $\eta = 0$ in (19)*, we can have*

$$(1-\lambda)\chi(\xi) + \lambda\xi\chi'(\xi) \prec \frac{1}{\delta}\xi^{1-\frac{1}{\delta}} \int_0^{\xi} t^{\frac{\omega}{\delta}-1}\Psi(t)dt = k(\xi). \tag{20}$$

By Lemma 2, the function $k(\xi)$ is convex provided $\Psi(\xi)$ is convex univalent. Now let us suppose that $k(\xi)$ is convex univalent in Λ. Then we have the following cases: For $\lambda = 0$, we have $\chi(\xi) = k[w(\xi)]$, where $w(\xi)$ is the Schwarz function. For $0 < \lambda \leq 1$, we get

$$\chi(\xi) = \frac{1}{\lambda}\xi^{1-\frac{1}{\lambda}} \int_0^{\xi} u^{\frac{1}{\lambda}-2} k\left[w(u)\right] du,$$

where $k(\xi)$ is given as in (20).

Remark 4. *Theorem 1 is not valid for $\delta = 0$. From (17), it can be easily seen that if $\delta = 0$ we can get*

$$\left(\frac{J_\lambda^m(\kappa_1, \sigma_1; \eta, \theta; q, \xi)\chi(\xi)}{\xi}\right)^{\omega} \prec \Psi(\xi).$$

Corollary 1. *Let $\chi \in \mathcal{BS}(\delta)$ with $\mathrm{Re}(\delta) > 0$, then we have*

$$\left(\frac{\chi(\xi)}{\xi}\right)^{\omega} \prec q(\xi) = \frac{\omega}{\delta}\xi^{\frac{-\omega}{\delta}} \int_0^{\xi} t^{\frac{\omega}{\delta}-1} \left(\frac{1+t}{1-t}\right) dt \prec \frac{1+\xi}{1-\xi}, \quad (\omega \neq 0),$$

$$\chi(\xi) = \xi \, \exp\left\{\int_0^{\xi} \frac{2w(t)}{\delta t\left[1 - w(t)\right]} dt\right\}, \quad (\omega = 0),$$

where $q(\xi)$ is the best dominant and $w(\xi)$ is the Schwarz function.

Proof. Clearly $\Psi(\xi) = \frac{1+\xi}{1-\xi}$ maps Λ onto a convex domain. So letting $r = 2$, $s = 1$, $\kappa_1 = \sigma_1$, $\sigma_2 = q$, $m = \eta = 0$, $\Psi(\xi) = (1+\xi)/(1-\xi)$ in Theorem 1, we can get the assertion of the Corollary.

3. Coefficient Inequalities

We will obtain the bounds for the initial coefficients and solution to the Fekete-Szegő problem for $\chi \in \mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta, \theta; \delta; \Psi(\xi))$.

Theorem 2. *Let $\chi(\xi) \in \mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta, \theta; \delta; \Psi(\xi))$ and ω, δ be chosen such $(\omega + (n-1)\delta) \neq 0$, for $n = 2, 3, 4, \ldots$, then we have*

$$|\varphi_2| \leq \frac{\psi_1}{|(\omega + \delta)\,\Gamma_2|} \tag{21}$$

$$|\varphi_3| \leq \frac{\psi_1}{|(\omega + 2\delta)\Gamma_3|} \max\left\{1; \left|\frac{\psi_2}{\psi_1} - \frac{(\omega - 1)(\omega + 2\delta)\psi_1}{2(\omega + \delta)^2}\right|\right\} \tag{22}$$

and for all $\rho \in \mathbb{C}$

$$\left|\varphi_3 - \rho\varphi_2^2\right| \leq \frac{\psi_1}{|(\omega + 2\delta)\Gamma_3|} \max\left\{1; \left|\frac{\psi_2}{\psi_1} - \frac{(\omega - 1)(\omega + 2\delta)\psi_1}{2(\omega + \delta)^2} - \frac{\rho\psi_1\,(\omega + 2\delta)\,\Gamma_3}{(\omega + \delta)^2\,\Gamma_2^2}\right|\right\}. \tag{23}$$

The inequalities are sharp.

Proof. As $\chi(\xi) \in \mathcal{BS}^{m,\omega}_{\lambda,q}(\kappa_1, \sigma_1; \eta, \theta; \delta; \Psi(\xi))$, by (12), we have

$$(1-\delta)\left(\frac{J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^\omega + \delta\frac{\xi^{1-\omega} J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi'(\xi)}{\left[J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)\right]^{1-\omega}} = \Psi\left[w(\xi)\right]. \quad (24)$$

Equivalently, for an arbitrary function ϑ of the form $\vartheta(\xi) = 1 + \sum_{k=1}^{\infty} \vartheta_n \xi^n \in \mathcal{F}$, the function $w(\xi)$ can be written in the form by

$$\vartheta(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)}, \quad \xi \in \Lambda.$$

The right side of (24) will be of the form

$$\Psi[w(\xi)] = 1 + \frac{\vartheta_1\psi_1}{2}\xi + \frac{\psi_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{\psi_2}{\psi_1}\right)\right]\xi^2 + \cdots. \quad (25)$$

The left hand side of (24) will be of the form

$$(1-\delta)\left(\frac{J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)}{\xi}\right)^\omega + \delta\frac{\xi^{1-\omega} J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi'(\xi)}{\left[J^m_\lambda(\kappa_1,\, \sigma_1;\, \eta,\, \theta; q, \xi)\chi(\xi)\right]^{1-\omega}}$$
$$= 1 + (\omega+\delta)\,\varphi_2\Gamma_2\xi + (\omega+2\delta)\left[\varphi_3\Gamma_3 + \frac{(\omega-1)\varphi_2^2\Gamma_2^2}{2}\right]\xi^2 + \cdots. \quad (26)$$

From (26) and (25), we obtain

$$\varphi_2 = \frac{\vartheta_1\psi_1}{2\,(\omega+\delta)\,\Gamma_2} \quad (27)$$

and

$$\varphi_3 = \frac{\psi_1}{2(\omega+2\delta)\Gamma_3}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{\psi_2}{\psi_1} + \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}\right)\right]. \quad (28)$$

Equation (21) can be obtained by applying $|\vartheta_1| \le 2$ ([38, p. 41]) in (27). Using Lemma 1 in (28), we get (22).

Now to prove (23), we consider

$$\left|\varphi_3 - \rho\varphi_2^2\right| = \left|\frac{\psi_1}{2(\omega+2\delta)\Gamma_3}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{\psi_2}{\psi_1} + \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}\right)\right] - \frac{\rho\vartheta_1^2\psi_1^2}{4\,(\omega+\delta)^2\,\Gamma_2^2}\right|$$

$$= \left|\frac{\psi_1}{2(\omega+2\delta)\Gamma_3}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{\psi_2}{\psi_1} + \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2} + \frac{\rho\psi_1\,(\omega+2\delta)\,\Gamma_3}{(\omega+\delta)^2\,\Gamma_2^2}\right)\right]\right|.$$

$$= \left|\frac{\psi_1}{2(\omega+2\delta)\Gamma_3}\left[\vartheta_2 - \frac{\vartheta_1^2}{2} + \frac{\vartheta_1^2}{2}\left(\frac{\psi_2}{\psi_1} - \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2} - \frac{\rho\psi_1\,(\omega+2\delta)\,\Gamma_3}{(\omega+\delta)^2\,\Gamma_2^2}\right)\right]\right|.$$

$$\leq \frac{\psi_1}{2\left|(\omega+2\delta)\Gamma_3\right|}\left[2+\frac{|\vartheta_1|^2}{2}\left(\left|\frac{\psi_2}{\psi_1}-\frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}-\frac{\rho\psi_1\,(\omega+2\delta)\,\Gamma_3}{(\omega+\delta)^2\,\Gamma_2^2}\right|-1\right)\right]. \quad (29)$$

Denoting

$$\mathfrak{B}:=\left|\frac{\psi_2}{\psi_1}-\frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}-\frac{\rho\psi_1\,(\omega+2\delta)\,\Gamma_3}{(\omega+\delta)^2\,\Gamma_2^2}\right|,$$

if $\mathfrak{B}\leq 1$, from (29) we obtain

$$\left|\varphi_3-\rho\varphi_2^2\right|\leq \frac{\psi_1}{\left|(\omega+2\delta)\Gamma_3\right|}. \quad (30)$$

Further, if $\mathfrak{B}\geq 1$ from (29) we deduce

$$\left|\varphi_3-\rho\varphi_2^2\right|\leq \frac{\psi_1}{\left|(\omega+2\delta)\Gamma_3\right|}\left(\left|\frac{\psi_2}{\psi_1}-\frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}-\frac{\rho\psi_1\,(\omega+2\delta)\,\Gamma_3}{(\omega+\delta)^2\,\Gamma_2^2}\right|\right). \quad (31)$$

An examination of the proof shows that the equality for (30) holds if $\vartheta_1=0$, $\vartheta_2=2$. Equivalently, by Lemma 1 we have $\Psi(\xi^2)=\Psi_2(\xi)=\dfrac{1+\xi^2}{1-\xi^2}$. Therefore, the extremal function of the class $\mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$ is given by

$$(1-\delta)\left(\frac{J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)}{\xi}\right)^\omega+\delta\frac{\xi^{1-\omega}J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi'(\xi)}{\left[J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)\right]^{1-\omega}}=\Psi_2(\xi^2).$$

Similarly, the equality for (31) holds if $\vartheta_2=2$. Equivalently, by Lemma 1 we have $\Psi(\xi)=\Psi_1(\xi)=\dfrac{1+\xi}{1-\xi}$. Therefore, the extremal function in $\mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$ is given by

$$(1-\delta)\left(\frac{J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)}{\xi}\right)^\omega+\delta\frac{\xi^{1-\omega}J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi'(\xi)}{\left[J_\lambda^m(\kappa_1,\,\sigma_1;\,\eta,\,\theta;q,\xi)\chi(\xi)\right]^{1-\omega}}=\Psi_1(\xi),$$

and the proof of the theorem is complete.

Corollary 2. *If* $\chi(\xi)\in\mathcal{BS}(\delta;\,\Psi)$*(see Remark1 (2)) and* ω*,* δ *be chosen such* $(\omega+(n-1)\delta)\neq 0$*, for* $n=2,3,4,\ldots$*, then we have*

$$|\varphi_2|\leq\frac{\psi_1}{|(\omega+\delta)|},$$

$$|\varphi_3|\leq\frac{\psi_1}{|(\omega+2\delta)|}\max\left\{1;\,\left|\frac{\psi_2}{\psi_1}-\frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}\right|\right\}$$

and for all $\rho\in\mathbb{C}$

$$\left|\varphi_3-\rho\varphi_2^2\right| \leq \frac{\psi_1}{|(\omega+2\delta)|}\max\left\{1;\left|\frac{\psi_2}{\psi_1}-\frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2}-\frac{\rho\psi_1\,(\omega+2\delta)}{(\omega+\delta)^2}\right|\right\}.$$

The inequalities are sharp.

Corollary 3. *If $\chi(\xi)\in\mathcal{BS}$(see Remark1 (3)), then we have*

$$|\varphi_2| \leq \frac{2}{|(\omega+1)|}, \qquad |\varphi_3| \leq \frac{2}{|(\omega+2)|}\max\left\{1;\left|1-\frac{(\omega-1)(\omega+2)}{(\omega+1)^2}\right|\right\}$$

and for all $\rho\in\mathbb{C}$

$$\left|\varphi_3-\rho\varphi_2^2\right| \leq \frac{2}{|(\omega+2)|}\max\left\{1;\left|1-\frac{(\omega-1)(\omega+2)}{(\omega+1)^2}-\frac{2\rho\,(\omega+2)}{(\omega+1)^2}\right|\right\}.$$

The inequalities are sharp.

Corollary 4. *[57, Theorem 3.1.] If $\chi(\xi)=\xi+\varphi_2\xi^2+\varphi_3\xi^3+\cdots\in\mathcal{S}^*(\psi)$, then for all $\rho\in\mathbb{C}$ we have*

$$\left|\varphi_3-\rho\varphi_2^2\right| \leq \frac{\psi_1}{2}\max\left\{1;\left|L_1+\frac{\psi_2}{\psi_1}-2\rho\psi_1\right|\right\}.$$

The inequality is sharp for the function χ_* given by

$$\chi_*(\xi)=\begin{cases}\xi\exp\displaystyle\int_0^\xi\frac{\Psi(t)-1}{t}dt, & \textit{if}\quad \left|\psi_1+\dfrac{\psi_2}{\psi_1}-2\rho\psi_1\right|\geq 1,\\[2ex] \xi\exp\displaystyle\int_0^\xi\frac{\Psi(t^2)-1}{t}dt, & \textit{if}\quad \left|\psi_1+\dfrac{\psi_2}{\psi_1}-2\rho\psi_1\right|\leq 1.\end{cases} \tag{32}$$

Proof. In Theorem 2, taking $\omega=0$, $\delta=1$, $r=2$, $s=1$, $\kappa_1=\sigma_1$, $\kappa_2=q$, $m=\eta=0$ and $q\to 1-$, we get the inequality

$$\left|\varphi_3-\rho\varphi_2^2\right| \leq \begin{cases}\dfrac{\psi_1}{2}, & \text{if}\quad \left|\psi_1+\dfrac{\psi_2}{\psi_1}-2\rho\psi_1\right|\leq 1,\\[2ex] \dfrac{\psi_1}{2}\left|\psi_1+\dfrac{\psi_2}{\psi_1}-2\rho\psi_1\right|, & \text{if}\quad \left|\psi_1+\dfrac{\psi_2}{\psi_1}-2\rho\psi_1\right|\geq 1.\end{cases}$$

Finally, following a similar technique to that for the sharpness of Theorem 3.1 of [57], we obtain (32).

Letting $\omega=0$, $\delta=1$, $r=2$, $s=1$, $\kappa_1=\sigma_1$, $\sigma_2=q$, $m=\eta=0$, $\Psi(\xi)=(1+\xi)/(1-\xi)$ and $q\to 1-$ in Theorem 2, we get

Corollary 5. *If $\chi(\xi)\in\mathcal{S}^*$, then the bounds of the initial coefficients of χ are given by*

$$|\varphi_2|\leq 2, \qquad |\varphi_3|\leq 3.$$

and the Fekete-Szegö inequality for $\rho\in\mathbb{C}$ is given by

$$\left|\varphi_3-\rho\varphi_2^2\right| \leq \max\left\{1,\ |4\rho-3|\right\}.$$

4. Coefficient Estimates of $\chi^{-1}(\xi)$

The inverse χ^{-1}, defined by $\chi^{-1}(\chi(\xi)) = \xi$, $\xi \in \Lambda$ and $\chi(\chi^{-1}(t)) = t$, $(|t| < r;\ r \geq 1/4)$ where

$$g(t) = \chi^{-1}(t) = t - \varphi_2 t^2 + (2\varphi_2^2 - \varphi_3)t^3 - \left(5\varphi_2^2 - 5\varphi_2\varphi_3 + \varphi_4\right) t^4 + \cdots . \tag{33}$$

The coefficient inequalities of the inverse functions $\mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta,\ \theta;\ \delta;\ \Psi(\xi))$ are valid only for the functions which are univalent.

Theorem 3. *Let $\chi \in \mathcal{BS}_{\lambda,q}^{m,\omega}(\kappa_1, \sigma_1; \eta,\ \theta;\ \delta;\ \Psi(\xi))$ and let χ^{-1} be the inverse of χ defined by*

$$\chi^{-1}(t) = t + \sum_{k=2}^{\infty} b_k t^k, \quad (|t| < r;\ r \geq 1/4),$$

then we have

$$|b_2| \leq \frac{\psi_1}{|(\omega + \delta)\,\Gamma_2|}$$

and

$$|b_3| \leq \frac{\psi_1}{|(\omega + 2\delta)\Gamma_3|} \max\left\{1;\ \left|\frac{\psi_2}{\psi_1} - \frac{(\omega - 1)(\omega + 2\delta)\psi_1}{2(\omega + \delta)^2} - \frac{2\psi_1\,(\omega + 2\delta)\,\Gamma_3}{(\omega + \delta)^2\,\Gamma_2^2}\right|\right\}$$

Also, for all $\tau \in \mathbb{C}$

$$\left|b_3 - \tau b_2^2\right| \leq \frac{\psi_1}{|(\omega + 2\delta)\Gamma_3|} \max\left\{1;\ \left|\frac{\psi_2}{\psi_1} - \frac{(\omega - 1)(\omega + 2\delta)\psi_1}{2(\omega + \delta)^2} - \frac{(\tau - 2)\psi_1\,(\omega + 2\delta)\,\Gamma_3}{(\omega + \delta)^2\,\Gamma_2^2}\right|\right\},$$

where ω and δ be chosen such that $\omega + \delta \neq 0$, $\omega + 2\delta \neq 0$.

Proof. From $\chi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n$ and (33), we have

$$b_2 = -\varphi_2 \quad \text{and} \quad b_3 = 2\varphi_2^2 - \varphi_3.$$

The estimate for $|b_2| = |\varphi_2|$ can be got by taking modulus of (27). Letting $\rho = 2$ in (23), we get $|b_3|$. To find the Fekete-Szegő inequality for the inverse function, consider

$$\left|b_3 - \tau b_2^2\right| = \left|2\varphi_2^2 - \varphi_3 - \tau\varphi_2^2\right| = \left|\varphi_3 - (\tau - 2)\varphi_2^2\right|.$$

Changing $\rho = (\tau - 2)$ in the (23), we get the desired result.

5. Logarithmic Coefficients for Functions Belonging to $\mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$

Logarithmic coefficients took the spotlight when Milin in [37] studied its properties which would imply the bounds of the Taylor coefficients of univalent functions. For detailed study, refer to [4, 5].

If χ is analytic in Λ, with $\frac{\chi(\xi)}{\xi} \neq 0$ for all $\xi \in \Lambda$, then the well-known logarithmic coefficients $\phi_n := \phi_n(\chi), n \in \mathbb{N}$, of χ are given by

$$\log \frac{\chi(\xi)}{\xi} = 2\sum_{n=1}^{\infty} \phi_n \xi^n, \xi \in \Lambda, \quad \log 1 = 0. \tag{34}$$

Now we will add additional criterion to the class $\mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$, so that logarithmic coefficients of $\mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$ is well-defined. That is, we let $\mathcal{TS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi)) = \mathcal{BS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi)) \cap \left\{\chi \text{ is analytic in } \Lambda : \frac{\chi(\xi)}{\xi} \neq 0,\, \xi \in \Lambda\right\}$. Note that for all functions $\mathcal{TS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$, the relation (34) is well-defined.

Theorem 4. *If* $\chi(\xi) \in \mathcal{TS}_{\lambda,q}^{m,\,\omega}(\kappa_1,\sigma_1;\eta,\,\theta;\,\delta;\,\Psi(\xi))$ *with the logarithmic coefficients given by (34), then we have*

$$|\phi_1| \leq \frac{\psi_1}{2\,|(\omega+\delta)\,\Gamma_2|}, \tag{35}$$

$$|\phi_2| \leq \frac{\psi_1}{2\,|(\omega+2\delta)\Gamma_3|} \max\left\{1; \left|\frac{\psi_2}{\psi_1} - \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2} - \frac{\psi_1\,(\omega+2\delta)\,\Gamma_3}{2\,(\omega+\delta)^2\,\Gamma_2^2}\right|\right\}, \tag{36}$$

and

$$\left|\phi_2 - \mu\phi_1^2\right| \leq \frac{\psi_1}{2\,|(\omega+2\delta)\Gamma_3|} \max\left\{1; \left|\frac{\psi_2}{\psi_1} - \frac{(\omega-1)(\omega+2\delta)\psi_1}{2(\omega+\delta)^2} - \frac{(1+\mu)\psi_1\,(\omega+2\delta)\,\Gamma_3}{2\,(\omega+\delta)^2\,\Gamma_2^2}\right|\right\}. \tag{37}$$

Proof. From $\chi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n$ and equating the first two coefficients of relation (34), we get

$$\phi_1 = \frac{\varphi_2}{2},\ \phi_2 = \frac{1}{2}\left(\varphi_3 - \frac{\varphi_2^2}{2}\right).$$

Using (27)) and (28) in the above equation and applying Lemma 1, we obtain (35) and (36). To obtain (37), consider

$$\left|\phi_2 - \mu\phi_1^2\right| = \frac{1}{2}\left[\varphi_3 - \frac{(1+\mu)}{2}\varphi_2^2\right].$$

Changing $\rho = \frac{1+\mu}{2}$ in (23), we get the desired result.

6. Conclusions

We have defined an operator which is most generalized and whose definition is not straightforward. Using the defined operator, we defined a subclass of analytic functions whose analytic characterization is associated with the class of Bazilevič functions. Though one has to be content with the parameters involved, but it helps in specializing most of the subclass of the univalent function theory. Some subordination properties and bounds of the initial coefficient are our main results.

Further the questions arises regarding the inclusions and radius problems. In detail,

(i) The functions belongs to the classes $\mathcal{BS}^{m,\omega}_{\lambda,q}(\kappa_1, \sigma_1; \eta, \theta; \delta; \Psi(\xi))$ need not be univalent, so for what radius of the disc $|z| < r$ and for what values of the parameters would the functions in $\mathcal{BS}^{m,\omega}_{\lambda,q}(\kappa_1, \sigma_1; \eta, \theta; \delta; \Psi(\xi))$ be univalent.

(ii) Theorem 1 is not valid if the ordinary derivatives is replaced with a quantum derivative. Are there any equivalent condition for which Theorem 1 remains valid if the ordinary derivatives is replaced with a quantum derivative.

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