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Solving Geometric Problems by Introducing an Equilateral Triangle

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Abstract. The article focuses on using an equilateral triangle as an auxiliary element in solving geometric (planimetric) problems. This method aims to simplify the problem-solving process and make it more intuitive by introducing an equilateral triangle in specific scenarios. By adding this triangle, the relationships between known and unknown quantities in the problem become more apparent or even obvious. The paper discusses four different scenarios where this method can be applied, illustrating each case with sample problems. The elegance and utility of the method in enhancing the solution process are highlighted. In this context, the applicability of this method in various olympiad problems or more complex geometric problems is demonstrated. This approach often reduces problems to simpler forms, saving time and effort in finding solutions. The paper provides evidence of the method's effectiveness and offers guidance on its application.

2020 Mathematics Subject Classifications: 97G10, 97G30, 97G40

Key Words and Phrases: Planimetric problem, equilateral triangle, isosceles triangle, auxiliary element

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1. Introduction

In the study of geometry, especially in competitive problem-solving settings, understanding and identifying the relationships between known and unknown quantities within a given problem is crucial yet challenging. Such problems often require deeper insight and inventive approaches to unravel the hidden structures and properties of geometric shapes. The introduction of auxiliary elements, such as additional lines or figures, has proven to be an effective strategy for making these underlying relationships more accessible and apparent. By introducing auxiliary constructions, it is often possible to reduce a complex

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problem to one that is simpler and, in some cases, more familiar, allowing the solver to apply known theorems and methods more effectively.

In this paper, we explore the method of introducing an equilateral triangle as an auxiliary element in solving planimetric problems, demonstrating its versatility and utility in various problem scenarios. The equilateral triangle holds particular appeal in geometry due to its inherent symmetry and well-defined properties, which can simplify the problem structure when appropriately incorporated. However, identifying the appropriate placement and alignment of an equilateral triangle is not always straightforward, as this often requires both experience and geometric intuition. When successfully applied, this method not only clarifies complex relationships but also leads to elegant and concise solutions, reducing otherwise intricate problems to a series of elementary steps.

The concept of using equilateral triangles as auxiliary elements is grounded in classical geometry but finds renewed relevance in the context of mathematical competitions and advanced geometry education. Studies have shown that auxiliary constructions, such as the addition of equilateral triangles, foster intuitive problem-solving skills, helping learners grasp abstract geometric principles in a more concrete manner. As such, this method can serve as a pedagogical tool for enhancing students' understanding of geometric concepts and improving their analytical skills. By applying the equilateral triangle method, solvers can often identify pathways to solutions that might not be apparent through conventional methods alone.

The structure of this paper is organized as follows: We begin by discussing the theoretical foundation and conditions under which the equilateral triangle method can be applied. Then, we categorize and illustrate four specific situations in which an equilateral triangle can be introduced to aid in solving a problem. Each category is supported by sample problems, which highlight the method's effectiveness and its potential for simplifying even the most challenging problems. Lastly, we provide insights into how this method may be further utilized in various educational and competitive contexts, with suggestions for future research and applications.

This study aims to contribute to the field of geometric problem-solving by offering a systematic approach to using equilateral triangles as a powerful tool in the solver's toolkit. It is our hope that this method will inspire both educators and students to view geometry problems from a new perspective, encouraging the creative use of auxiliary elements to uncover elegant and efficient solutions.

The books [7, 8] details classical methods in teaching geometry, emphasizing the importance of auxiliary elements and the relationships between geometric shapes. It encourages the development of intuitive skills in problem-solving, especially in planimetric problems, by highlighting the role of symmetry and structures like equilateral and isosceles triangles.

The books [6, 10] details how to incorporate auxiliary elements and position structures effectively. It highlights how even simple shapes like equilateral triangles can serve as powerful aids in geometric problem-solving.

The works [1–4, 9, 12] devoted to the study of geometric methods, including the method additional constructions. In these papers, the solution of some kinds of problems is presented by this method. Also lets mention [5], co-authored by the first and second authors, where, using the circle (divided into equal parts) as an auxiliary element, the solution is found in a quite simple and smart way.

2. The Method

One of the interesting and original ways of solving planimetric problems is the method of introducing an equilateral triangle. Its attractive sides are brevity of reasoning, external and internal beauty.

The essence of the method of introducing an equilateral triangles lies in the fact that you add some new (auxiliary) elements to the drawing of your problem. As a result, the relationships between the problem data and unknown quantities, which have been hard to see before, become more tangible, or even obvious.

Equilateral triangle can be introduced in the drawing of a given problem as follows:

Situation 1. Equilateral triangle can be constructed in the internal domain of a given problem.

Situation 2. Equilateral triangle can be constructed in the external domain of a given problem.

Situation 3. Equilateral triangle can be constructed so that it includes the drawing of a given problem completely in itself.

Situation 4. Equilateral triangle can be constructed in the internal and external domain of a given problem.

Now, we will consider all the above cases in solving specific problems.

First, we implement Situation 1 in solving the following problem.

Problem 1. In triangle ABC $AB = AC$ and $\angle BAC = 80^\circ$. Let D the interior point of this triangle. If $\angle ABD = \angle BAD = 10^\circ$, then find $\angle BDC$. (Fig.1.)

First, we will show the solution to the problem using the algebraic method, given in work [11, p. 123].

Let us denote $\angle BDC = x$. If we apply the sine theorem to triangles ADC and ABD, respectively, we get:

$$\frac{AD}{\sin(x-90^\circ)} = \frac{AC}{\sin(200^\circ - x)},$$

$$\frac{AD}{\sin 10^\circ} = \frac{AB}{\sin 160^\circ}.$$

Considering the relation $AB = AC$, if we divide the two above equalities term by term, we get:

$$\frac{\sin 10^\circ}{\sin(x-90^\circ)} = \frac{\sin 160^\circ}{\sin(200^\circ - x)}$$

or

$$\sin 160^\circ \cdot \sin(x-90^\circ) = \sin 10^\circ \cdot \sin(200^\circ - x).$$

To find x from the last relation, let us perform the following transformations:

$$\sin 20^\circ \cdot \sin(x-90^\circ) = \sin 10^\circ \cdot \sin(200^\circ - x),$$

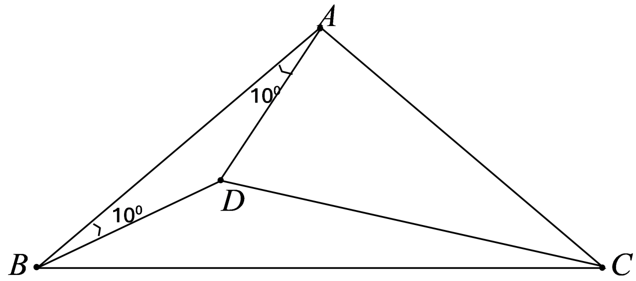

Figure 1: (Isosceles triangle ABC with the given angles)

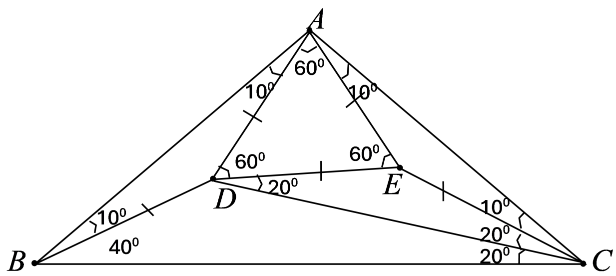

Figure 2: (Triangle ADE is equilateral, constructed inside the triangle ABC)

$$2 \cdot \sin 10^\circ \cos 10^\circ \cdot \sin (x - 90^\circ) = \sin 10^\circ \cdot \sin (200^\circ - x),$$

$$\cos 10^\circ \cdot \sin (x - 90^\circ) = \cos 60^\circ \cdot \sin (200^\circ - x),$$

$$\frac{1}{2} \cdot [\sin (x - 80^\circ) + \sin (100^\circ - x)] = \frac{1}{2} \cdot [\sin (260^\circ - x) + \sin (x - 140^\circ)].$$

Note that $\sin (x - 80^\circ) = \sin (260^\circ - x)$, because

$$(x - 80^\circ) + (260^\circ - x) = 180^\circ.$$

Then, from the last relation, we find

$$\sin (100^\circ - x) = \sin (x - 140^\circ),$$

$$100^\circ - x = x - 140^\circ, \Rightarrow x = 120^\circ.$$

Now, let us solve the given problem using a purely geometric method (by including an equilateral triangle).

Since the angle at the vertex $\angle BAC = 80^\circ$, then inside the triangle ABC can be construct an equilateral triangle. To this end, we construct triangle AEC on side AC , which is equal to triangle ADB .

Connecting points D and E , taking into account $AD = AE$, we get an equilateral triangle ADE . As the $DE = EC$, then the triangle DEC is isosceles and $\angle EDC = \angle ECD = 20^\circ$.

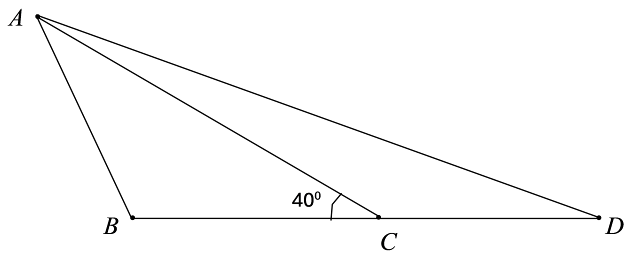

Figure 3: (ABD is a given triangle and triangle ABC is isosceles)

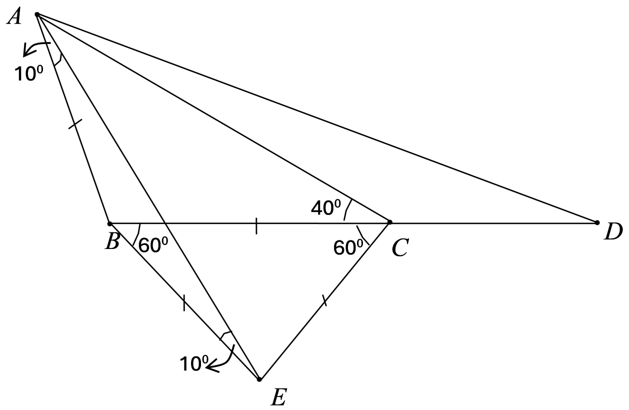

Figure 4: (Triangle BCE is equilateral, constructed outside the triangle ABD)

Since the adjacent angles at the base of triangle ABC are equal to 50° , then in the triangle BDC , we find $\angle BDC = 180^\circ - (40^\circ + 20^\circ) = 120^\circ$. (Fig.2.)

The comparison of these two methods show that, although the introduction of an equilateral triangle may seem difficult when solving the problem using the second method (this may not occur to everyone), with this method the answer is found very quickly, without using any formulas or theorems.

We implement Situation 2 in solving the following problem.

Problem 2. The point C is given on the side BD of the triangle ABD , so that $AB = BC, AC = BD$ and $\angle ACB = 40^\circ$. Find $\angle CAD$. (Fig.3.)

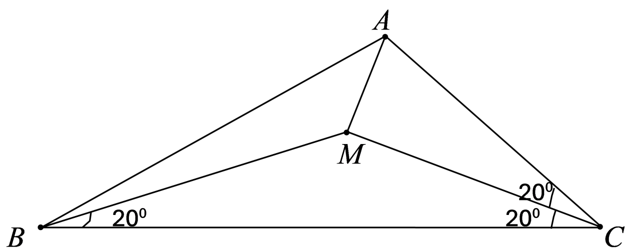

Figure 5: (Triangle ABC with the given angles)

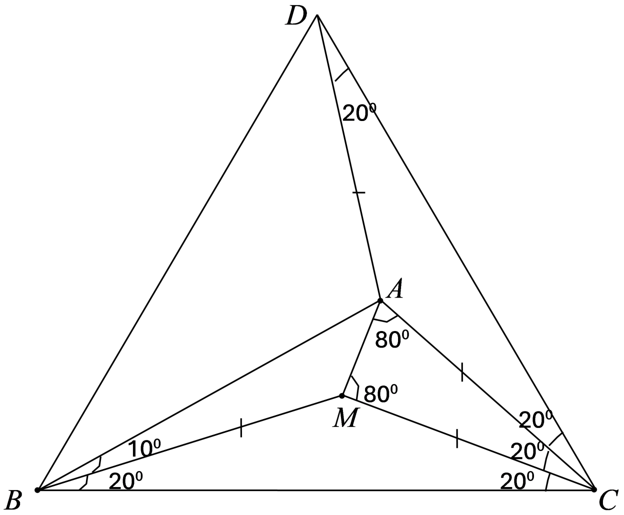

Figure 6: (Given triangle ABC, placed inside the equilateral triangle BDC)

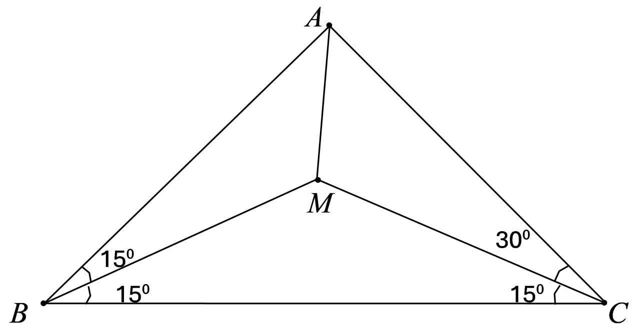

Figure 7: (Triangle ABC with the given angles)

Outside the given triangle on the segment BC we construct an equilateral triangle BCE. Then, as $AB = BE$, we have $\angle BAE = \angle BEA = 10°$. and the triangle ABE is isosceles. In the triangle ACE we find $\angle ACE = 100°, \angle EAC = 30°$ and $\angle AEC = 50°$. Since $EC = AB$, $AC = BD$ and $\angle ACE = \angle ABD = 100°$, then the triangles EAC and ADB are equal. Consequently, $\angle BAD = 50°$, and therefore, $\angle CAD = 10°$. (Fig.4.)

Now, we implement Situation 3 in solving the following problem.

Problem 3. Let M be the interior point of the triangle ABC, $\angle MBC = \angle MCB = 20°$ and $AC = CM$. Find $\angle BAM$.(Fig.5.)

As $AC = CM$, the triangle AMC is isosceles and $\angle MAC = \angle CMA = 80°$.

On the side BC we construct an equilateral triangle DBC. If we join the points A and D, then the triangle CAD will be equal to the triangle CMB, because $BC = CD, AC = CM$ and $\angle MCB = \angle ACD = 20°$. From here, we have $\angle ADC = 20°$ and $AC = AD$. (Fig.6.)

On the other hand, as $BD = BC$ and $AC = AD$, then the triangles BDA and BCA

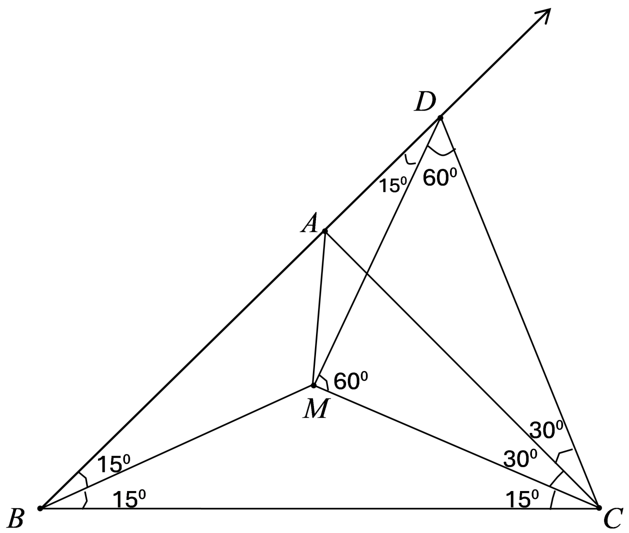

Figure 8: (Triangle CMD is equilateral, while triangles CAD and CMA are isosceles)

are equal. Consequently, $\angle DBA = \angle CBA = 30°$, and therefore,$\angle ABM = 10°$. From the triangle ABC we have $\angle BAM = 30°$.

Finally, we implement Situation 4 with our new method on the problem solved by using the method of auxiliary circle divided into equal parts [1].

Problem 4. The point M is given inside the triangle ABC so that $\angle ABM = \angle CBM = \angle BCM = 15°$ and $\angle ACM = 30°$. Prove that $\angle CAM = 75°$.

Let the circle centered at M with a radius $BM = MC$ intersect the continuation of AB at the point D. Then, as $DM = BM$, the triangle DBM is isosceles and $\angle DBM = \angle BDM = 15°$. (Fig.7.)

Connecting points D and C , taking into account $\angle DBM = \angle CMB = 150°$ we get an equilateral triangle DMC . In the triangle ACD, we find $\angle CAD = 180° - (30° + 75°) = 75°$. Hence, $CA = CD$. On the other hand, as $CD = CM$, we have $CA = CM$, i.e. the triangle CAM is isosceles. Consequently, $\angle CAM = 75°$. (Fig.8.)

This example shows that many problems, even very difficult ones, can ben solved using method of introducing an equilateral triangle.

3. Concluding Remarks

This study provides an in-depth examination of using the equilateral triangle as an auxiliary element in solving geometric (planimetric) problems. Especially in planimetric problems, the use of an equilateral triangle shows potential for simplifying the problem-solving process, enhancing intuition, and achieving more efficient solutions. Four distinct cases for applying this method were explored, each illustrated with example problem solutions. These examples demonstrate that the method is applicable to a wide range of problems and contributes significantly to solution processes.

The method draws on classical geometry principles and concepts of symmetry. Auxiliary elements like the equilateral triangle reveal features of the problem that may otherwise remain unnoticed, guiding problem-solvers towards solutions. In this context, the method contributes to developing intuitive thinking skills in students and problem-solvers. Additionally, it allows problems to be simplified to more fundamental forms, enabling the effective application of existing geometric knowledge.

Research in the literature and findings from this study indicate that using auxiliary elements such as the equilateral triangle in geometry education facilitates learning and deepens mathematical understanding. The method encourages students to approach geometry problems more creatively and analytically while adding an aesthetic dimension to the solution process.

In the future, it will be important to examine the impact of this method on a broader range of geometry problems and test its applicability across various educational levels. For example, studies could explore the method's effectiveness with different age groups, its influence on students' mathematical thinking skills, or its potential for accelerating problem-solving processes. Such research could help integrate this method into educational curricula on a larger scale, making geometry education more effective.

In conclusion, using the equilateral triangle as an auxiliary element in planimetric problems emerges as a powerful tool for solving classical geometry problems. By simplifying problem-solving processes and offering an elegant solution path, this method holds valuable potential for developing innovative approaches in geometry teaching.

References

[1] Henry Africk. *Elementary College Geometry.* CUNY Academic Works, 2021.

[2] S. Aliyev. *Geometry.* Tansel Publications, Adapazari, 1996.

[3] S. Aliyev, Sh. Aghazade, and G. Abdullayeva. Using area method in secondary school geometry. *European Journal of Pure and Applied Mathematics*, 14(2):601–607, 2021.

[4] S. Aliyev, Sh. Hamidova, and G. Abdullayeva. Some applications of ptolemy's theorem in secondary school mathematics. *European Journal of Pure and Applied Mathematics*, 13(1):180–184, 2020.

[5] S. Aliyev, M. Heydarova, and Sh. Aghazade. Solving geometry problems by alternative methods in mathematics education. *European Journal of Pure and Applied Mathematics*, 16(2):1110–1117, 2023.

[6] A. P. Kiselev. *Kiselev's Geometry, Book 1, Planimetry.* Sumizdat, California, 2006.

[7] Ron Larson, Laurie Boswell, and Lee Stiff. *Geometry.* McDougal Littell, 2004.

[8] D. Pedoe. *Geometry: A Comprehensive Course.* Dover Publications, New York, 1988.

[9] Y. P. Ponarin. *Elementary Geometry, Volume 1.* MTSNMO, Moscow, 2008. *(in Russian).*

[10] Michael Serra. *Discovering Geometry An Investigative Approach.* Key Curriculum Press, 2003.

[11] I. F. Sharigin. *Elementary Geometry, Volume 1.* Drofa, Moscow, 1999. *(in Russian).*

[12] Hung-Hsi Wu. *Teaching Geometry in Grade 8 and High School According to the Common Core Standards*. 2013.