EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5848
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some New Characterizations of Open and Closed Fuzzy Mappings Inspired by Induced Mappings

Sandeep Kaur[1], Fathea M. Osman Birkea[2,*], Alkan Özkan[3], Tareq M. Al-shami[4,5], M. Omran[6]

[1] *Department of Mathematics and Statistics, Central University of Punjab, Bathinda, India*
[2] *Department of Mathematics, Faculty of Science, Northern Border University, Arar, Saudi Arabia*
[3] *Department of Mathematics, Faculty of Arts and Sciences, Iğdır University, Iğdır, Turkey*
[4] *Department of Mathematics, Sana'a University, P.O.Box 1247 Sana'a, Yemen*
[5] *Jadara University Research Center, Jadara University, Irbid, Jordan*
[6] *Department of Physics and Engineering Mathematics, Faculty of Engineering, Tanta University, Egypt*

Abstract. In this paper, we use the induced mappings to obtain some new characterizations of open fuzzy mappings in connection with closures and closed fuzzy mappings in connection with interiors of fuzzy sets. We also investigate another representation of open fuzzy and closed fuzzy mappings under surjective mappings. Furthermore, we prove that images of saturated fuzzy sets under surjective open fuzzy and closed fuzzy mappings are closed fuzzy and open fuzzy sets, respectively. We furnish illustrative examples to elucidate the displayed results.

2020 Mathematics Subject Classifications: 54A40, 03E72, 54C05, 94D05
Key Words and Phrases: Fuzzy set, Fuzzy topological space, Open fuzzy mapping, Closed fuzzy mapping, Induced mapping

1. Introduction

In [30], Zadeh initiated the essential principle of fuzzy units and set up the pillar of fuzzy mathematics. This theory has extreme potential for packages in various directions [24, 25]. After introducing the theory of fuzzy topological spaces (FTSs, in short) in [17], Chang introduced the notion of fuzzy continuous mapping alongside its characterizations. Afterwards, several authors worked on the fuzzification of numerous classical notions related to topology [1, 2, 14, 15, 26] and its generalizations [11, 27, 28]. Interest in topological studies and their practical applications has grown recently, particularly

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5848

Email addresses: sandybrar86@gmail.com (S. Kaur), fathia.birkia@nbu.edu.sa (F.M.O. Birkea), alkan.ozkan@igdir.edu.tr (A. Özkan), tareqalshami83@gmail.com (T.M. Al-shami), Manar.Omran@f-eng.tanta.edu.eg (M. Omran)

after realizing that certain abstract topological concepts can be applied to solve real-life problems. For example:

- The studies in [7, 10, 19] utilized certain extended forms of open subsets, including somewhat dense, somewhere dense, and $\theta\beta$-open subsets, to analyze information systems described by rough approximation operators.
- Various separation axioms were examined and applied in [8, 9, 18] to select the optimal tourism program and determine the appropriate nutrition system for individuals.
- The concepts of compactness and connectedness were exploited in [6, 13] for optimal selection and to investigate the fixed point theorem.

In 1974, Wong [29] described open fuzzy and closed fuzzy mappings, and few characterizations of those mappings were observed in [22], whereas, in 1984 [23], a few additional characterizations of open fuzzy and closed fuzzy mappings were investigated. Open fuzzy mappings offer a logical extension for addressing uncertainties and ambiguities inherent in real-world applications, much as open mappings are vital in classical set theory. One of the key properties of open fuzzy mappings is that they preserve the structure of open fuzzy sets. Open fuzzy mappings can also be characterized using the fuzzy interior operator. Similarly, closed fuzzy mappings also play a significant role in advancing our understanding of uncertainty modeling and reasoning which is characterized in terms of fuzzy closure operator.

Our motive in this study is to find more characterizations of open fuzzy and closed fuzzy mappings in terms of fuzzy closure operator and fuzzy interior operator, respectively. For this, we make use of induced mappings which are essential in the observation of many spaces in topology. As well-known, any type of structure or application of an operator to a mapping can deliver upward thrust to useful induced mappings. In view of an induced mapping $\psi^{\#} : V \to W$ given by $\psi^{\#}(U) = \{w \in W | \psi^{-1}(w) \subseteq U\}$ for any subset U of V introduced Arkhangel'skii in [16], Kaur and Goyal introduced the #-image of a fuzzy set (F-set, in short) to define a useful induced mapping $\psi^{\#}$ in [21] together with its properties. With this induced mapping $\psi^{\#}$, they presented a new sight of reading bluecontinuous fuzzy mappings and their characterizations. This indicates the possibility of describing open fuzzy and closed fuzzy mappings in connection with this induced mapping which further helps us to find their new representations in terms of the fuzzy closure operator and fuzzy interior operator, respectively. To complete the literature review presentation, we draw the readers' attention to the fact that the proposed technique of describing mappings via a soft framework was studied in [12, 20].

In this study, some new characterizations of open fuzzy and closed fuzzy mappings and the use of induced mapping $\psi^{\#}$ are discussed. We describe the open fuzzy mappings in recognition of induced mapping of closed fuzzy sets rather than already proven results of open fuzzy mappings in regards to interiors. A few characterizations of these mappings are also taken into consideration in the use of saturated fuzzy sets.

2. Preliminaries

Throughout this study, $V = \{v_i : i \in I\}$ refers to a space of points and a function μ_E is the membership function for any F-set E. An F-set A in V is represented by a membership function μ_A from V to $[0, 1]$ that associates every v in V with its "Membership grade" $\mu_A(v)$ in $[0, 1]$.

Chang has discussed some basic definitions and related results of the F-set theory, fuzzy topologies, and fuzzy mappings in [17]. Therefore, we recall some other useful definitions related to induced mapping $\psi^{\#}$, open fuzzy mapping, and closed fuzzy mapping which we shall use in the results investigated through this content. We begin with definitions of FTS, interior, and closure of an F-set.

Definition 1. *[17] A family T of F-sets in V is said to be a fuzzy topology if it satisfies following conditions:*

(a) $\phi, V \in T$.

(b) If $A, B \in T$, *then* $A \cap B \in T$.

(c) If $A_i \in T$ *for each* $i \in I$, *then* $\cup A_i \in T$.

Every member of T is called a T-open F-set and the pair (V, T) is known as FTS. *Also an F-set is named T-closed if and only if its complement is a T-open F-set.*

Definition 2. *[26] Let (V, T) be* FTS *and M be any F-set in V. Then:*

(i) *Interior of M is described as the union of all T-open F-sets contained in M, indicated by A^o. Equivalently, M^o is the largest T-open F-set contained in M and $(M^o)^o = M^o$.*

(ii) *Closure of M is described as the intersection of all T-closed F-sets containing M, indicated by $\overline{M}$. Clearly, $\overline{M}$ is the smallest T-closed F-set containing M and $\overline{\overline{M}} = \overline{M}$.*

Theorem 1. *[26] In any* FTS (V, T), $(\overline{E})^c = (E^c)^o$ *and so* $\overline{E^c} = (E^o)^c$, *for any fuzzy subset E of V.*

Definition 3. *[17] Let $\psi : V \longrightarrow W$ be a mapping. Then, for F-subsets H of V and G of W, we have:*

(a) $\psi(H)$ *is an F-subset of W given as* $\psi(H)(w) = \sup\{H(u) : u \in \psi^{-1}(w)\}$ *if* $\psi^{-1}(w) \neq \emptyset$ *and* $\psi(H)(w) = \underline{0}$ *if* $\psi^{-1}(w) = \emptyset$.

(b) $\psi^{-1}(G)$ *is an F-subset of U given as* $\psi^{-1}(G)(u) = G(\psi(u))$ *for every* $u \in U$.

Theorem 2. *[17] Let ψ be a mapping from a set V into a set W. Then for every F-subsets M and N of V and F-subsets K and L of W we have:*

(i) $\psi(M) \subseteq \psi(N)$ *if* $M \subseteq N$.

(ii) $\psi^{-1}(K) \subseteq \psi^{-1}(L)$ *if* $K \subseteq L$.

(iii) $M \subseteq \psi^{-1}(\psi(M))$, *equality holds if* ψ *is one-one.*

(iv) $\psi(\psi^{-1}(K)) \subseteq K$, *equality holds if* ψ *is onto.*

(v) $\psi^{-1}(K^c) = (\psi^{-1}(K))^c$.

(vi) $(\psi(M))^c \subseteq \psi(M^c)$.

Definition 4. *[29] Let* ψ *be a mapping from an* FTS (V, T_1) *to an* FTS (W, T_2). *Then,* ψ *is named an open fuzzy (resp., a closed fuzzy) mapping if and only if* $\psi(B)$ *is an open (resp., a closed)* F*-set in* (W, T_2), *for every open (resp., closed)* F*-set* B *in* (V, T_1).

Theorem 3. *[23] Let* ψ *be a mapping from an* FTS (V, T_1) *to an* FTS (W, T_2). *Then, for any* F*-sets* M *in* V *and* N *in* W, *the following statements are equivalent.*

(1) ψ *is open fuzzy mapping.*

(2) $\psi(M^\circ) \subseteq (\psi(M))^\circ$.

(3) $\psi^{-1}(\overline{N}) \subseteq \overline{\psi^{-1}(N)}$.

(4) $(\psi^{-1}(N))^\circ \subseteq \psi^{-1}(N^\circ)$.

Theorem 4. *[22] Let* ψ *be a mapping from an* FTS (V, T_1) *to an* FTS (W, T_2). *Then,* ψ *is a closed fuzzy mapping iff* $\overline{\psi(M)} \subseteq \psi(\overline{M})$ *for every* F*-set* M *in* V.

Kaur and Goyal introduced the # image of an F-set in [21] to define an induced mapping $\psi^\# : Gz(V) \to Gz(W)$ associated to any mapping $\psi : V \to W$ where V and W are crisp sets and $Gz(V)$ and $Gz(W)$ are the collections of all fuzzy subsets of V and W, respectively.

Definition 5. *[21] The # image of an* F*-set* M *in* V *with membership function* $\mu_M(v)$, *written as* $\psi^\#(M)$, *is an* F*-set in* W *with a membership function defined as*

$$\mu_{\psi^\#(M)}(w) = \begin{cases} \inf_{z \in \psi^{-1}(w)} \mu_M(z) & \text{if } \psi^{-1}(w) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Definition 6. *[21] An* F*-set* $M^\#$ *in* V *is defined as* $M^\# = \psi^{-1}(\psi^\#(M))$ *with membership mapping as*

$$\mu_{M^\#}(v) = \mu_{\psi^{-1}(\psi^\#(M))}(v) = \mu_{\psi^\#(M)}(\psi(v))$$

Few properties of $\psi^\#$ mapping are given by the lemma below.

Lemma 1. *[21] Let* $\psi : V \to W$ *be any mapping and* M, N *be fuzzy subsets of* V *and* U *be a fuzzy subset of* W. *Then:*

(a) $(\psi^{\#}(M)) \subseteq (\psi^{\#}(N))$ *if* $M \subseteq N$.

(b) $\psi^{-1}(\psi^{\#}(M)) \subseteq M$ *i.e.* $M^{\#} \subseteq M$.

(c) $U \subseteq \psi^{\#}(\psi^{-1}(U))$ *and equality holds if* ψ *is onto.*

(d) $\psi(M^{\#}) = \psi^{\#}(M) \cap \psi(V)$.

(e) $\psi^{\#}(M \cap N) = \psi^{\#}(M) \cap \psi^{\#}(N)$.

(f) $\psi^{\#}(\psi) = (\psi(V))^c$ *and* $\psi^{\#}(V) = W$.

(g) $\psi(M^{\#}) = \psi^{\#}(M) \cap \psi(M)$.

(h) $\psi^{\#}(\psi^{-1}(\psi^{\#}(M))) = \psi^{\#}(M^{\#}) = \psi^{\#}(M)$.

(i) $\psi^{-1}(\psi^{\#}(\psi^{-1}(U))) = \psi^{-1}(U)$.

3. Characterizations of Open Fuzzy Mappings

In this section, we study some new representations of open fuzzy mappings which provides valuable insights into the properties and behavior of these mappings. We begin with a characterization of the open fuzzy mapping using induced mapping $\psi^{\#}$ for which we make use of the following lemma on the properties of the #-image of an F-set M in V.

Lemma 2. *[21] Let $\psi : V \to W$ be any mapping, M and U be any fuzzy subsets of V and W, respectively. Then:*

(a) $\psi^{-1}(U) \subseteq M$ *iff* $U \subseteq \psi^{\#}(M)$.

(b) $\psi^{\#}(M^c) = (\psi(M))^c$ *and so* $\psi^{\#}(M) = (\psi(M^c))^c$ *and* $\psi(M) = (\psi^{\#}(M^c))^c$.

Theorem 5. *Let $\psi : V \to W$ where V, W be spaces of points. Then, ψ is an open fuzzy mapping if and only if*

$$\overline{\psi^{\#}(M)} \subseteq \psi^{\#}(\overline{M}) \text{ for every subset } M \text{ of } V.$$

Proof. Let ψ be an open fuzzy mapping and M be a subset of V. By Lemma 2 and Theorem 1, we get

$$\overline{\psi^{\#}(M)} = \overline{(\psi(M^c)^c}$$
$$\text{and } \overline{(\psi(M^c))^c} = ((\psi(M^c))^\circ)^c$$

Since ψ is an open fuzzy mapping, then by Theorem 3 (2), $((\psi(M^c))^\circ)^c \subseteq (\psi((M^c)^\circ)^c$. Therefore, $\overline{\psi^{\#}(M)} \subseteq (\psi((M^c)^\circ))^c = ((\psi(\overline{M})^c))^c$. Again, by Lemma 2 (b), $(\psi(\overline{M})^c)^c = \psi^{\#}(\overline{M})$. Hence, $\overline{\psi^{\#}(M)} \subseteq \psi^{\#}(\overline{M})$.

Conversely, let G be any open fuzzy subset of V. Then by assumption, $\overline{\psi^{\#}(G^c)} \subseteq \psi^{\#}(\overline{G^c})$ we obtain $\overline{\psi^{\#}(G^c)} \subseteq \psi^{\#}(G^c)$ since G^c is a closed fuzzy subset of V. This further implies that $\psi^{\#}(G^c)$ is a closed fuzzy subset of W and hence, $(\psi^{\#}(G^c))^c$ is an open fuzzy set. Therefore, by Lemma 2 (b), $(\psi^{\#}(G^c))^c = \psi(G)$ is an open fuzzy subset of W. Hence, ψ is an open fuzzy mapping.

In the next theorem, we discuss another characterization of the open fuzzy mapping in terms of closed fuzzy subsets.

Theorem 6. *Let ψ be a mapping from V to W. Then, ψ is an open fuzzy mapping if and only if for each closed fuzzy subset G of V, $\psi^{\#}(G)$ is a closed fuzzy subset of W.*

Proof. The necessity of the condition follows from Theorem 5.
Conversely, let O be an open fuzzy subset of V. Then by assumption, $\psi^{\#}(O^c)$ is a closed fuzzy set in W. But by Lemma 2 (b), $\psi^{\#}(O^c) = \psi(O)^c$. Therefore, $\psi(O)^c$ is a closed fuzzy set in W, and hence, $\psi(O)$ is an open fuzzy set in W, which proves ψ is an open fuzzy mapping.

blueAs an instance of the above theorem, we have the following example which shows that if for any closed fuzzy subset G of V, $\psi^{\#}(G)$ is not a closed fuzzy subset of W then ψ is not an open fuzzy mapping.

Example 1. *Let $V = [0, 1]$, consider the F-sets M, N, P with membership functions given by*

$$\mu_M(v) = \begin{cases} 2v & \text{if } 0 \leq v \leq 1/2 \\ 1/2 & \text{if } 1/2 < v \leq 1 \end{cases}$$

$$\mu_N(v) = \begin{cases} 1/2 & \text{if } 0 \leq v < 1/4 \\ 2v & \text{if } 1/4 \leq v \leq 1/2 \\ 0 & \text{if } 1/2 < v \leq 1 \end{cases}$$

$$\mu_P(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq 1/2 \\ 0 & \text{if } 1/2 < v \leq 1 \end{cases}$$

Let $T_1 = \{\emptyset, V, M\}$ be a fuzzy topology on V and $\psi : (V, T_1) \to V(T_1)$ be a fuzzy mapping defined by

$$\psi(v) = \begin{cases} v & \text{if } 0 \leq v \leq 1/2 \\ 1 - v & \text{if } 1/2 < v \leq 1 \end{cases}$$

Clearly, $\psi(M) = N$ and $\psi(V) = P$. Since M and V are open fuzzy subsets of V but their fuzzy images, N and P are not open fuzzy subsets of V, ψ is not an open fuzzy mapping. On the other hand, let Q be a closed fuzzy set in (V, T_1) defined by a membership function

$$\mu_Q(v) = \begin{cases} 1 - 2v & \text{if } 0 \leq v \leq 1/2 \\ 1/2 & \text{if } 1/2 < v \leq 1 \end{cases}$$

then #-image of Q, $\psi^{\#}(Q)$ will be an F-set defined by the membership function

$$\mu_Q(v) = \begin{cases} 1-2v & \text{if } 0 \leq v \leq 1/2 \\ 1-v & \text{if } 1/2 < v \leq 1 \end{cases}$$

which is not a closed fuzzy subset of V.

The following theorem gives another characterization of open fuzzy mappings for a surjective mapping $\psi : V \to W$ in connection with the closed fuzzy subsets of V.

Theorem 7. *Let $\psi : V \to W$ be a surjective mapping. Then ψ is an open fuzzy mapping if and only if for every closed fuzzy subset G of V, $\psi(G^{\#})$ is a closed fuzzy set in W.*

Proof. Assume $\psi(G^{\#})$ is a closed fuzzy subset of W for every closed fuzzy subset G of V. Now, since ψ is surjective mapping, $\psi^{-1}(v) \neq \emptyset$ for every $v \in V$ then, $\mu_{\psi(G^{\#})}(v) = \sup_{z\in\psi^{-1}(v)} \mu_{G^{\#}}(z) = \sup_{z\in\psi^{-1}(v)} \mu_{\psi^{-1}(\psi^{\#}(G))}(z) = \sup_{z\in\psi^{-1}(v)} \mu_{\psi^{\#}(G)}(\psi(z)) = \mu_{\psi^{\#}(G)}(v)$ which gives $\psi^{\#}(G)$ is a closed fuzzy subset of W. Hence, by Theorem 6, ψ is an open fuzzy mapping. The Converse part is obvious by using the same argument that $\psi(E^{\#}) = \psi^{\#}(E)$ and Theorem 6.

In the final result about open fuzzy mappings, we see that images of saturated closed fuzzy sets under surjective open fuzzy mappings are closed fuzzy sets. Firstly, we recall the definition of a saturated fuzzy set and a remark to use in the final corollary.

Definition 7. *[21] A fuzzy subset M of V is said to be a saturated fuzzy subset of V if $M = \psi^{-1}(U)$ for a fuzzy subset U of W i.e. $\mu_M(v) = \mu_{\psi^{-1}(U)}(v)$ for each $v \in V$.*

Remark 1. *For any mapping $\psi : V \to W$, $M^{\#}$ is saturated fuzzy for each fuzzy subset M of V, since, $\psi^{-1}(\psi(M^{\#})) = \psi^{-1}(\psi(\psi^{-1}(\psi^{\#}(M)))) = \psi^{-1}(\psi^{\#}(M)) = M^{\#}$.*

Theorem 8. *Let $\psi : V \to W$ be an open fuzzy and surjective mapping. Then $\psi(M)$ is a closed fuzzy set in W for every saturated closed fuzzy subset M of V. Particularly, for any F-set N, if $N^{\#}$ is a closed fuzzy set in V, then $\psi(N^{\#})$ is a closed fuzzy set in W.*

Proof. Let $\psi : V \to W$ be an open fuzzy and surjective mapping and M be a saturated closed fuzzy subset of V. Then, by Theorem 7, $\psi(M^{\#})$ is a closed fuzzy in W. Now, since M is a saturated fuzzy set, we have, $\mu_{M^{\#}}(u) = \mu_{\psi^{-1}(\psi^{\#}(M))}(u) = \mu_{\psi^{\#}(M)}(\psi(u)) = \inf_{z\in\psi^{-1}(\psi(u))} \mu_M(z) = \mu_M(u)$ which gives $M^{\#} = M$, and which further implies $\psi(M^{\#}) = \psi(M)$ is a closed fuzzy in W. Hence, we prove the desired result.

From the above theorem, we get the following corollary.

Corollary 1. *Let $\psi : V \to W$ be an open fuzzy and surjective mapping. Then, for any F-set N, if $N^{\#}$ is a closed fuzzy set in V, then $\psi(N^{\#})$ is a closed fuzzy set in W.*

4. Characterizations of Closed Fuzzy Mappings

In this section, we discuss some new characterizations of closed fuzzy mappings. The following theorem shows how induced mapping $\psi^{\#}$ enables us to find a characterization of the closed fuzzy mapping in terms of the interior operator rather than the closure operator.

Theorem 9. *Let ψ be a mapping from V to W. Then, ψ is a closed fuzzy mapping if and only if for each subset B of V,*

$$\psi^{\#}(B^{\circ}) \subseteq (\psi^{\#}(B))^{\circ}.$$

Proof. Let ψ be a closed fuzzy mapping and B be a subset of V. By Lemma 2 (b) and Theorem 1, we get

$$\begin{aligned} \psi^{\#}(B^{\circ}) &= (\psi((B^{\circ})^{c}))^{c}, \\ \text{and} \quad (\psi((B^{\circ})^{c}))^{c} &= \psi((\overline{B^{c}}))^{c} \end{aligned}$$

Since ψ is a closed fuzzy mapping, then by Theorem 4, $\psi(\overline{B^{c}})^{c} \subseteq \overline{\psi(B^{c})}^{c}$. But $\overline{\psi(B^{c})}^{c} = \overline{\psi^{\#}(B)^{c}}^{c}$, since Lemma 2 (b) holds. This implies that

$$\begin{aligned} \psi^{\#}(B^{\circ}) &\subseteq \overline{\psi^{\#}(B)^{c}}^{c} \\ &= (\psi^{\#}(B))^{\circ}. \end{aligned}$$

Hence, $\psi^{\#}(B^{\circ}) \subseteq (\psi^{\#}(B))^{\circ}$.
Conversely, let G be closed fuzzy subset of V. Then, by assumption, $\psi^{\#}((G^{c})^{\circ}) \subseteq (\psi^{\#}(G^{c}))^{\circ}$ which implies $\psi^{\#}(G^{c}) \subseteq (\psi^{\#}(G^{c}))^{\circ}$. Since G^{c} is an open fuzzy subset of V, then $\psi^{\#}(G^{c})$ is an open fuzzy subset of W, and hence, $(\psi^{\#}(G^{c}))^{c}$ is a closed fuzzy mapping. Therefore, by Lemma 2 (b), $(\psi^{\#}(G^{c}))^{c} = \psi(G)$ is a closed fuzzy subset of W. Hence, ψ is a closed fuzzy mapping.

The following theorem gives another representation of a closed fuzzy mapping in terms of open fuzzy subsets.

Theorem 10. *Let ψ be any mapping from V to W. Then ψ is a closed fuzzy mapping if and only if for every open fuzzy subset O of V, $\psi^{\#}(O)$ is an open fuzzy set in W.*

Proof. The necessity of the condition follows from Theorem 9.
Conversely, let G be a closed fuzzy subset of V. Then, by assumption, $\psi^{\#}(G^{c})$ is an open fuzzy set in W. But we get by Lemma 2 (b) that $\psi^{\#}(G^{c}) = \psi(G)^{c}$, which implies $\psi(G)^{c}$ is an open fuzzy set in W. Therefore, $\psi(G)$ is a closed fuzzy set in W. Hence, ψ is a closed fuzzy mapping.

The following example illustrates the above theorem.

Example 2. *Assume* $V = \{v_1, v_2\}$ *and* $W = \{w_1, w_2\}$*, consider* F*-sets* M *and* N *as* $M = \{< v_1, 0.6 >, < v_2, 0.7 >\}$ *and* $N = \{< w_1, 0.6 >, < w_2, 0.5 >\}$*. Then* $T_1 = \{\emptyset, M, V\}$ *and* $T_2 = \{\emptyset, N, W\}$ *be fuzzy topologies on* V *and* W*, respectively. We define a fuzzy mapping* $\psi : (V, T_1) \to (W, T_2)$ *as follows* $\psi(v_1) = w_1, \psi(v_2) = w_2$*. One can check that* ψ *is not a closed fuzzy mapping. On the other hand, let* M *be an open fuzzy set in* V*. Then,* $\psi^{\#}(M) = \{< w_1, 0.6 >, < w_2, 0.7 >\}$ *is not an open fuzzy set in* W*.*

In the next theorem, we discuss another representation of a closed fuzzy mapping when a mapping is surjective.

Theorem 11. *Let* $\psi : V \to W$ *be a surjective mapping. Then* ψ *is a closed fuzzy mapping if and only if for each open fuzzy subset* O *of* W*,* $\psi(O^{\#})$ *is an open fuzzy set in* W*.*

Proof. The proof is similar to that of Theorem 7.

In the final result about closed fuzzy mappings, we see that image of saturated open fuzzy sets under the surjective closed fuzzy mapping is an open fuzzy set.

Theorem 12. *Let* $\psi : V \to W$ *be a surjective closed fuzzy mapping. Then* $\psi(O)$ *is an open fuzzy set in* W *for every saturated open fuzzy subset* O *in* V*.*

Proof. The proof is similar to the above Theorem 8.

Corollary 2. *Let* $\psi : V \to W$ *be an closed fuzzy mapping and surjective. Then, for any open fuzzy subset* $B^{\#}$ *of* V*, we have* $\psi(B^{\#})$ *is an open fuzzy set in* W*.*

5. Conclusion

Applications of operators which are either obtained from a prior structure or an operator acting on a given mapping, induce useful mappings which enable us to study the important topological concepts or enable us to give a new characterization of such important concepts as continuity, open and closed mappings. In view of the research done on F-set theory, much needs to be investigated in the application of mappings induced by operators defined by F-sets. In this paper, we have established new characterizations of open and closed fuzzy mappings in connection with induced fuzzy mapping rather than the well-known results of open fuzzy mappings in terms of interiors and closed fuzzy mapping in terms of closures. The significat of these characterizations is that it leads to the growth of the theoretical study of FTSs and offers various descriptions for fuzzy mappings.

Our next target will be to study the concepts of invertedly open fuzzy mappings and invertedly closed fuzzy mappings by using this induced mapping. Also, we look at the validity of the results introduced herein via some generalizations of fuzzy topology, such as infra fuzzy (soft) topology [3] and supra fuzzy (soft) topology [4, 5].

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FFR-2025-1166-01"

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] W.F. Al-Omeri. On mixed b-fuzzy topological spaces. *International Journal of Fuzzy Logic and Intelligent Systems*, 20(3):242–246, 2020.

[2] W.F. Al-Omeri, O.H. Khalil, and A. Ghareeb. Degree of (l, m)-fuzzy semi-precontinuous and (l, m)-fuzzy semi-preirresolute functions. *Demonstratio Mathematica*, 51(1):182–197, 2018.

[3] T. M. Al-shami. New soft structure: Infra soft topological spaces. *Mathematical Problems in Engineering*, 2021:Article ID 3361604, 12 pages, 2021.

[4] T. M. Al-shami and M. E. El-Shafei. On supra soft topological ordered spaces. *Arab Journal of Basic and Applied Sciences*, 26(1):433–445, 2019.

[5] T. M. Al-shami and M. E. El-Shafei. Two types of separation axioms on supra soft separation spaces. *Demonstratio Mathematica*, 52(1):147–165, 2019.

[6] T.M. Al-shami. Compactness on soft topological ordered spaces and its application on the information system. *Journal of Mathematics*, 2021:12, 2021.

[7] T.M. Al-shami. Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets. *Soft Computing*, 25(23):14449–14460, 2021.

[8] T.M. Al-shami. On soft separation axioms and their applications on decision-making problem. *Mathematical Problems in Engineering*, 2021:12, 2021.

[9] T.M. Al-shami. Soft somewhat open sets: Soft separation axioms and medical application to nutrition. *Computational and Applied Mathematics*, 41, 2022.

[10] T.M. Al-shami. Topological approach to generate new rough set models. *Complex & Intelligent Systems*, 8:4101–4113, 2022.

[11] T.M. Al-shami, H.Z. Ibrahim, A. Mhemdi, and Radwan Abu-Gdairi. n^{th} power root fuzzy sets and its topology. *International Journal of Fuzzy Logic and Intelligent Systems*, 22(4):350–365, 2022.

[12] T.M. Al-shami, S. Kaur, A. Özkan, M. Hosny, and A. Mhemdi. Some characterizations of soft continuous mappings using soft graphs. *Filomat*, 38(22):7823–7830, 2024.

[13] T.M. Al-shami, A. Rawshdeh, H. Al-jarrah, and A. Mhemdi. Connectedness and covering properties via infra topologies with application to fixed point theorem. *AIMS Mathematics*, 8(4):8928–8948, 2023.

[14] M. Alimohammady, E. Ekici, S. Jafari, and M. Roohi. On fuzzy upper and lower contra-continuous multifunctions. *Iranian Journal of Fuzzy Systems*, 8(3):149–158, 2011.

[15] Z.A. Ameen, R.A. Mohammed, T.M. Al-shami, and B.A. Asaad. Novel fuzzy topologies formed by fuzzy primal frameworks. *Journal of Intelligent & Fuzzy Systems*, 2024.

[16] A.V. Arkhangel'skii and V.I. Ponomarev. *Fundamentals of General Topology: Problems and Exercises*. Mathematics and its Applications. Hindustan Publishing Corporation, New Delhi, India, 1984.

[17] C.L. Chang. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 24:182–190, 1968.

[18] S. Demiralp, T.M. Al-shami, A.M. Abd El-latif, and F.A. Abu Shaheen. Topologically indistinguishable relations and separation axioms. *AIMS Mathematics*, 9(6):15701–15723, 2024.

[19] M. Hosny and T.M. Al-shami. Employing a generalization of open sets defined by ideals to initiate novel rough approximation spaces with a chemical application. *European Journal of Pure and Applied Mathematics*, 17(4):3436–3463, 2024.

[20] S. Kaur, T.M. Al-shami, A. Ozkan, and M. Hosny. A new approach to soft continuity. *Mathematics*, 11:3164, 2023.

[21] S. Kaur and N. Goyal. On induced map $\psi^{\#}$ in fuzzy set theory and its applications, n.d.

[22] S.R. Malghan and S.S. Benchalli. On fuzzy topological spaces. *Glassnik Mathematicki*, 16(36):313–325, 1981.

[23] S.R. Malghan and S.S. Benchalli. Open maps, closed maps and local compactness in fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 99:338–349, 1984.

[24] N. Malik, M. Shabir, T. M. Al-shami, R. Gul, and M. Arar. A novel decision-making technique based on t-rough bipolar fuzzy sets. *Journal of Mathematics and Computer Science*, 33(3):275–289, 2024.

[25] N. Malik, M. Shabir, T. M. Al-shami, R. Gul, M. Arar, and M. Hosny. Rough bipolar fuzzy ideals in semigroups. *Complex & Intelligent Systems*, 9(6):7197–7212, 2023.

[26] P. PAO-Ming and Ying-Ming LIU. Fuzzy topology i. neighborhood structure of a fuzzy point and moore-smith convergence. *Journal of Mathematical Analysis and Applications*, 76:571–599, 1980.

[27] S. Saleh, R. Abu-Gdairi, T.M. Al-shami, and Mohammed S. Abdo. On categorical property of fuzzy soft topological spaces. *Applied Mathematics & Information Sciences*, 16(4):635–641, 2022.

[28] S. Saleh, T.M. Al-shami, and A. Mhemdi. On some new types of fuzzy soft compact spaces. *Journal of Mathematics*, 2023:8, 2023.

[29] C.K. Wong. Fuzzy points and local properties of fuzzy topology. *Journal of Mathematical Analysis and Applications*, 46:316–328, 1974.

[30] L.A. Zadeh. Fuzzy sets. *Information and Computation*, 8:338–353, 1965.