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Novel Categories of Spaces in the Frame of Generalized Fuzzy Topologies via Fuzzy $g\mu$-Closed Sets

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Abstract. One of the known approaches to studying topological concepts is to utilize subclasses of topology, such as clopen sets and generalized closed sets. In this study, we apply the notion of fuzzy generalized μ-closed sets ($Fg\mu$-closed sets) to establish and analyze novel categories of spaces, namely $Fg\mu$-regular, $Fg\mu$-normal, and $F\mu$-symmetric spaces in the frame of generalized fuzzy topology (GFT). We investigate the fundamental properties of these classes, exploring their unique characteristics and preservation theorems under $Fg\mu$-continuous maps. We establish the interrelationships between these classes and the other separation axioms in this setting, and we demonstrate that $F\mu$-regular, $F\mu$-normal, and $F\mu$-symmetric spaces are special cases of $Fg\mu$-regular, $Fg\mu$-normal, and $F\mu$-T_1 spaces, respectively. Additionally, we show that the equivalence for these cases hold when the GFT is $F\mu$-$T_{\frac{1}{2}}$. The connections between these classes and their counterparts in the crisp GT are studied. Finally, we discuss these classes' hereditary and topological properties, further enhancing our comprehension of their behavior and implications.

2020 Mathematics Subject Classifications: 54A40, 54C08, 54D10, 54D15

Key Words and Phrases: Fuzzy μ-closed set; fuzzy $g\mu$-closed set; generalized fuzzy topology; fuzzy $g\mu$-continuous map; fuzzy $g\mu$-regular; fuzzy $g\mu$-normal space

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1. Introduction

Fuzzy sets (F-sets) were proposed by Zadeh [49] in 1965 as a suitable approach to address with uncertainty cases that we cannot be efficiently managed using classical techniques. Over the last decades, the researches of F-sets have a vital role in mathematics and applied sciences and garnered significant attention due to its ability to handle uncertain and vague information in various real-life applications such as artificial intelligence [47, 50], control systems [8, 28], decision-making [17, 23], image processing [1, 46], classifications [22, 24], etc. Chang [16], in 1968, defined the fuzzy topology (FT), allowing the study of topological properties within the frame of F-sets. This development has led to the expansion and investigation of many classical topological notions in the context of FT [2, 7, 9, 10, 42], providing more accurate and flexible models to address problems of uncertainty in various real life ears. Moreover, the hybridization of fuzzy topology with soft topology was introduced and studied by several authors [41, 43, 44].

Generalized closed sets, abbreviated as g-closed sets, is a fundamental notion in both topology and FT. It was proposed in general topology by Levine [29] in 1970. This notion has undergone extensive study in the fields of topology and FT by numerous authors, as in [15, 19, 30, 33, 36, 40, 48]. Since then, it has been widely used as a powerful tool to explore various concepts, including g-regular and g-normal spaces, which have been further generalized and investigated as in [11, 21, 25, 34, 35, 38], and others. These studies have also led to the introduction of new separation axioms that are weaker than T_1. In the fuzzy context, Balasubramanian et al [12] proposed the notion of generalized fuzzy closed sets in 1997, sparking further research by authors like Saraf et al. [45] and Park et al. [37] who extensively studied different forms of generalized fuzzy closed sets.

On the other hand, Császár [20] introduced the concept of generalized topology (or GT), expanding the scope of general topology. Over time, many researchers have endeavored to extend the notion of g-closed sets to the broader framework of GT. Maragathavalli et al [16] notably explored g-closed sets and their fundamental properties within GTS. Prior to that, Chetty [18] extended the concept of GT into a fuzzy environment, leading to the development of GFT. Mandal et al [31] defined the notion of $Fg\mu$-closed sets in $GFTS$ and study the concepts of $F\mu$-regular and $F\mu$-normal in $GFTS$. Furthermore, Chakraborty et al [15] study some properties of $Fg\mu$-closed sets in $GFTS$. They also, investigated various concepts within $GFTS$ as in [13, 14]. However, there are many research gaps and further developments that have not yet been achieved in the context of GFT.

This article aims to contribute to developing the theoretical foundation for GFT by introducing and analyzing novel categories of spaces within the framework of GFT via $Fg\mu$-closed sets. After introductory section, the rest of the article is systematized as follows:

- In section 2. We have review some fundamental definitions and findings that will be utilized throughout this article.
- In section 3. We apply the notion of $Fg\mu$-closed sets to introduce and discuss novel categories of spaces such as $Fg\mu$-regular, $F\mu$-G_3, $F\mu$-$T_{\frac{1}{2}}$, $F\mu$-$T_{2\frac{1}{2}}$, and $F\mu$-symmetric

spaces in the context of GFT. We analyze their basic characteristics and properties. Some related theorems, relations, and implications are discussed.

- In section 4. We introduce new classes of spaces named, $Fg\mu$-normal and $F\mu$-G_4 spaces in GFT via $Fg\mu$-closed sets. We investigate some properties, related theorems, implications, and results in this sequel. We explore the interrelationships between these classes and the other separation axioms with some supporting examples.
- In section 5. The connections of $Fg\mu$-regular ($Fg\mu$-normal) spaces and that in the crisp GT are presented. Moreover, we have explore the basic preservation theorems and discuss the hereditary and topological property of these classes.
- In section 6. Conclusion and future works, we outline the article's contributions and suggest some points to open up new avenues for further research in this area.

2. Basic definitions and results

In this document, $\mathcal{U}$ refers to a universe set, $I^{\mathcal{U}}(I = [0, 1])$ is the class of all F-sets on $\mathcal{U}$, $(\mathcal{U}, \tau)$ means FTS, and $(\mathcal{U}, \mu)$ means $GFTS$. In the following, let's review a few fundamental definitions and findings that will be utilized throughout the rest of this study.

Definition 1. *[49] A fuzzy set (or F-set) H in $\mathcal{U}$ is a map $H : \mathcal{U} \longrightarrow I$. It can be written as $H = \{(u, H(u)) : u \in \mathcal{U}, H(u) \in I\}$. The fuzzy point (or F-point) u_α is an F-set such that $u_\alpha(v) = \alpha > 0$ if $u = v$ and $u_\alpha(v) = 0$ if $u \neq v$ for all $v \in \mathcal{U}$. $u_\alpha \in H$ if $\alpha \leq H(u)$. $FP(\mathcal{U})$ refers to the family of all F-points in $\mathcal{U}$. The constant F-sets $\underline{0}$ and $\underline{1}$ are given by $\underline{0}(u) = 0$ and $\underline{1}(u) = 1$ for any $u \in \mathcal{U}$.*

For $H,\ G \in I^{\mathcal{U}}$, we have the following properties of F-sets (see [16, 39, 49]):

(i) $H \cup G \in I^{\mathcal{U}}$ given by $(H \vee G)(u) = max\{H(u), G(u)\}$ *for every* $u \in \mathcal{U}$.

(ii) $H \cap G \in I^{\mathcal{U}}$ given by $(H \wedge G)(u) = min\{H(u), G(u)\}$ *for every* $u \in \mathcal{U}$.

(iii) $H^c \in I^{\mathcal{U}}$ given by $H^c(u) = 1 - H(u)$ for all $u \in \mathcal{U}$.

(iv) For $A \subset \mathcal{U}$, the characteristic function χ_A is an F-set on $\mathcal{U}$.

(v) The support of $H \in I^{\mathcal{U}}$ is denoted by $S(H)$ and given by $S(H) = \{u \in \mathcal{U} : H(u) > 0\}$.

(vi) For a map $f : \mathcal{U} \longrightarrow \mathcal{W}$ and $H \in I^{\mathcal{U}},\ G \in I^{\mathcal{W}}$, we have:

(a) $f(H)$ is an F-set on $\mathcal{W}$ given as $f(H)(w) = \sup\{H(u) : u \in f^{-1}(w)\}$ if $f^{-1}(w) \neq \emptyset$ and $f(H)(w) = \underline{0}$ if $f^{-1}(w) = \emptyset$.

(b) $f^{-1}(G)$ is an F-set on $\mathcal{U}$ given as $f^{-1}(G)(u) = G(f(u))$ for every $u \in \mathcal{U}$.

Definition 2. *[16] An FTS is the pair $(\mathcal{U}, \tau)$, where $\tau \subseteq I^{\mathcal{U}}$ which is closed under finite intersections, arbitrary union, and $\underline{0}$, $\underline{1}$ in τ. An F-set H is called F-open set if $H \in \tau$ and the complement of H is called F-closed set. For an F-set H in $(\mathcal{U}, \tau)$, the F-complement, F-interior, and F-closure of H are written as H^c, $int(H)$, and $cl(H)$ respectively.*

Definition 3. *[39] An F-point u_α is called quasi-coincident with a F-set H in $\mathcal{U}$, symbolized by $u_\alpha q H$, if there is $u \in U$ such that $\alpha + H(u) > 1$. In general, HqG if $H(u)+G(u) > 1$ for some $u \in U$. If H is not quasi-coincident with G, then we write $H\tilde{q}G$.*

Definition 4. *[16, 27, 32, 39] For any two F-sets H, G in $(\mathcal{U}, \tau)$ and $u_\alpha \in FP(\mathcal{U})$, we have:*
(1) $u_\alpha \tilde{q} H \Longleftrightarrow u_\alpha \in H^c$,in general $H\tilde{q}G \Longleftrightarrow H \subseteq G^c$
(2) $H \cap G = \underline{0} \Longrightarrow H\tilde{q}G$
(3) $H\tilde{q}G,\ F \subseteq G \Longrightarrow H\tilde{q}F$
(4) $H \subseteq G \Longleftrightarrow (u_\alpha q H \Longrightarrow u_\alpha q G)$ for all $u_\alpha \in FP(U)$
(5) $u_\alpha \tilde{q} v_\beta \Longleftrightarrow u \neq v$ or ($u = v$ and $\alpha + \beta > 1$).
For a map $f : \mathcal{U} \longrightarrow \mathcal{V}$, $H \in I^{\mathcal{U}}, G \in I^{\mathcal{V}}$, and $u_\alpha \in FP(\mathcal{U})$, we have:
(i) $f(u_\alpha) q G \Longrightarrow u_\alpha q f^{-1}(G)$, and $u_\alpha q H \Longrightarrow f(u_\alpha) q f(H)$.
(ii) $u_\alpha q f^{-1}(G)$ if $f(u_\alpha) \in G$, and $f(u_\alpha) \in f(H)$ if $u_\alpha \in H$.

Definition 5. *[18] A collection $\mu \subseteq I^{\mathcal{U}}$ is called GFT on $\mathcal{U}$ iff $\underline{0} \in \mu$ and $\bigvee_{i \in J} H_i \in \mu$ for any class $\{H_i : i \in J\} \subset \mu$. The structure $(\mathcal{U}, \mu)$ is called an $GFTS$. Every member of μ is called a fuzzy μ-open set (in short, $F\mu$-open set) and the complement of a $F\mu$-open set is called $F\mu$-closed set. The family $F\mu O(\mathcal{U})$ (resp. $F\mu C(\mathcal{U})$ denotes to the class of all $F\mu$-open (resp. $F\mu$-closed) sets on $\mathcal{U}$.*

For an $GFTS$ $(\mathcal{U}, \mu)$ and $H \in I^{\mathcal{U}}$. The $F\mu$-closure of H is the smallest $F\mu$-closed set containing H, it is symbolized by $cl_\mu(H)$ and the $F\mu$-interior of H, symbolized by $int_\mu(H)$ is the largest $F\mu$-open set contained in H.

Evidently, $H \in I^{\mathcal{U}}$ is $F\mu$-open (resp. $F\mu$-closed) if and only if $H = int_\mu(H)$ (resp. $H = cl_\mu(H)$) It is clear that int_μ and cl_μboth are monotonic and idempotent operators.

Notation. For an $GFTS$ $(\mathcal{U}, \mu)$ and $u_\alpha \in FP(\mathcal{U})$. O_{u_α} refers to an $F\mu$-open set containing u_α and it is called an $F\mu$-open neighborhood (or $F\mu$-open nbd) of u_α. In general, O_H refers to an $F\mu$-open set containing H.

Definition 6. *Let $(\mathcal{U}, \mu)$ be an $GFTS$ and $V \subseteq \mathcal{U}$. The family $\mu_V = \{\chi_V \cap H : H \in \mu\}$ is an GFT on V. The pair (V, μ_V) is called an GFT-subspace (or $GFTSS$) of $(\mathcal{U}, \mu)$.*

Lemma 1. *[27] Let $(\mathcal{U}, \mu)$ be an $GFTS$, $H \in I^{\mathcal{U}}$ and $u_\alpha \in FP(\mathcal{U})$, we have:*

(i) for any two $F\mu$-open sets F and G, if $F\tilde{q}G$, then $cl_\mu(F)\tilde{q}G$ and $F\tilde{q}cl_\mu(G)$.

(ii) $u_\alpha q cl_\mu(H)$ if and only if $O_{u_\alpha} q H$ for all $O_{u_\alpha} \in \mu$.

Definition 7. *[14] An F-set H in $GFTS$ $(\mathcal{U}, \mu)$ is called:*

(i) $F\mu$-regular closed (resp., $F\mu$-regular open) if $H = cl_\mu(int_\mu(H))$(resp., $H = int_\mu(cl_\mu(H))$).

(ii) $F\mu$-locally closed if there is $F \in \mu$ and $G \in F\mu C(\mathcal{U})$ such that $H = F \cap G$.

Definition 8. *[31] An F-set H in $GFTS$ $(\mathcal{U}, \mu)$ is called fuzzy generalized μ-closed (or $Fg\mu$-closed) if $cl_\mu(H) \subseteq G$ whenever $H \subseteq G$ and $G \in F\mu O(\mathcal{U})$. The class of all $F\mu g$-closed sets in $(\mathcal{U}, \mu)$ is symbolized by $Fg\mu C(\mathcal{U})$. The complement of $Fg\mu$-closed set is called an $Fg\mu$-open set.*

Remark 1. *[15] In $GFTS$ $(\mathcal{U}, \mu)$, we have:*

(i) Every $F\mu$-closed (resp., $F\mu$-open) set is an $Fg\mu$-closed (resp., $Fg\mu$-open) set,

(ii) Every $F\mu$-closed ($F\mu$-open) set is an $F\mu$-locally closed set, but not conversely.

Remark 2. *The concepts of $Fg\mu$-closed sets and $F\mu$-locally closed sets are generalizations of $F\mu$-closed sets but both are independent to each other. For examples see [15] .*

Proposition 1. *[15] An $Fg\mu$-closed set in an $GFTS$ $(\mathcal{U}, \mu)$ is an $F\mu$-closed set if and only if it is $F\mu$-locally closed.*

According to definition provided by Kandil et al. [26], the next definition is obtained by taking $\mu = \delta$ and replacing F-open sets with $F\mu$-open sets.

Definition 9. *An $GFTS$ $(\mathcal{U}, \mu)$ is said to be:*

(i) $F\mu$-T_0 iff for any $u_\alpha, v_\beta \in FP(\mathcal{U})$ with $u_\alpha \tilde{q} v_\beta$ implies *$u_\alpha \tilde{q} cl_\mu(v_\beta)$ or $cl_\mu(u_\alpha) \tilde{q} v_\beta$.*

(ii) $F\mu$-T_1 iff for any $u_\alpha, v_\beta \in FP(\mathcal{U})$ with $u_\alpha \tilde{q} v_\beta$ implies *$u_\alpha \tilde{q} cl_\mu(v_\beta)$ and $cl_\mu(u_\alpha) \tilde{q} v_\beta$.*

(iii) $F\mu$-T_2 iff for any $u_\alpha, v_\beta \in FP(\mathcal{U})$ with $u_\alpha \tilde{q} v_\beta$, there are $G, H \in \mu$ such that $u_\alpha \in G, v_\beta \in H$ and $G \tilde{q} H$.

(iv) $F\mu$-regular (or $F\mu$-R_2) iff for any $u_\alpha \in FP(\mathcal{U})$ and any $H \in F\mu C(\mathcal{U})$ with $u_\alpha \tilde{q} H$, there are $F, G \in \mu$ such that $u_\alpha \in F, H \subseteq G$ and $F \tilde{q} G$.

*(v) $F\mu$-*normal *(or $F\mu$-R_3) iff for any $F\mu$-closed sets F_1, F_2 with $F_1 \tilde{q} F_2$, there are $H, G \in \mu$ such that $F_1 \subseteq H, F_2 \subseteq G$ and $H \tilde{q} G$.*

(vi) $F\mu$-T_3 (resp. $F\mu$-T_4) iff it is both $F\mu$-R_2 (resp. $F\mu$-R_3) and $F\mu$-T_1.

Note. Evidently, $F\mu\text{-}T_4 \Longrightarrow F\mu\text{-}T_3 \Longrightarrow F\mu\text{-}T_2 \Longrightarrow F\mu\text{-}T_1$.

Definition 10. *A map $f : (\mathcal{U}, \mu_1\ ) \longrightarrow (\mathcal{V}, \mu_2)$ is called:*

(i) $F\mu$-continuous iff $f^{-1}(H) \in F\mu C(\mathcal{U})$ for each $H \in F\mu C(\mathcal{V})$[31].

(ii) $Fg\mu$-continuous iff $f^{-1}(H) \in Fg\mu C(\mathcal{U})$ for each $H \in F\mu C(\mathcal{V})$[15].

(iii) $Fg\mu$-closed($Fg\mu$-open) iff $f(H)$ is $Fg\mu$-closed($Fg\mu$-open) set in $(\mathcal{V}, \mu_2)$ for every $F\mu$-closed($F\mu$-open) set H in $(\mathcal{U}, \mu_1)$ [14].

Note. Evidently, every $F\mu$-continuous is $Fg\mu$-continuous.

3. Fuzzy $g\mu$-regular spaces

In this part, we introduce and discuss some characteristics and properties of a new class of spaces named, $Fg\mu$-regular spaces in $GFTS$. First let's give the next definition.

Definition 11. *An $GFTS$ $(\mathcal{U}, \mu)$ is named:*

(i) $F\mu$-$T_{\frac{1}{2}}$ iff every $Fg\mu$-closed set in $(\mathcal{U}, \mu)$ is an $F\mu$-closed set.

(ii) $F\mu$-$T_{2\frac{1}{2}}$ *iff for each* u_α,$v_\beta \in FP(\mathcal{U})$ *with* $u_\alpha \tilde{q} v_\beta$, *there are* $G, H \in \mu$ *such* $u_\alpha \in G, v_\beta \in H$ *and* $cl_\mu(G)\tilde{q}cl_\mu(H)$.

Proposition 2. *For an GFTS* $(\mathcal{U}, \mu)$, *the next items are equivalent:*

(1) $(\mathcal{U}, \mu)$ *is* $F\mu$-$T_{\frac{1}{2}}$.

(2) Every $Fg\mu$*-closed set is an* $F\mu$*-locally closed set.*

Proof. (1) $\Longrightarrow$ (2). Let $(\mathcal{U}, \mu)$ be an $F\mu$-$T_{\frac{1}{2}}$ space, then every $Fg\mu$-closed set is $F\mu$-closed set. By Remark 1, every $F\mu$-closed set is $F\mu$-locally closed set. The result holds.
(2) $\Longrightarrow$ (1). It follows directly from Proposition 1.

Definition 12. *An GFTS* $(\mathcal{U}, \mu)$ *is called* $Fg\mu$*-regular (or* FG-μR_2*) iff for every* $Fg\mu$*-closed set* H *with* $u_\alpha \tilde{q} H$ *for each* F*-point* u_α, *there are* $F\mu$*-open sets* F, G *containing* u_α, H *respectively, such that* $F\tilde{q}G$.

Remark 3. *Evidently, any* FG-μR_2 *space is* $F\mu$-R_2 *but not conversely.*

Example 1. *Let* $\mathcal{U} = \{u, v\}$ *and* $\mu = \{\underline{0}, \underline{1}, H, G\}$, *where* $H = (u_{0.3}, v_{0.5}), G = (u_{0.7}, v_{0.5})$, *then* μ *is an GFT on* $\mathcal{U}$. *One can check that* $(\mathcal{U}, \mu)$ *is* $F\mu$-R_2 *but not* FG-μR_2. *Indeed, for* $u_{0.5} \in FP(\mathcal{U})$ *and* $Fg\mu$*-closed set* $F = (u_{0.4}, v_{0.7})$ *with* $u_{0.5}\tilde{q}F$, *there are* $O_{u_{0.5}} = G \in \mu$ *and* $O_F = \underline{1} \in \mu$ *but* $O_{u_{0.5}} q O_F$. *Hence* $(\mathcal{U}, \mu)$ *is not* FG-μR_2.

Theorem 1. *An GFTS* $(\mathcal{U}, \mu)$ *is* FG-μR_2 *if and only if is both* $F\mu$-R_2 *and* $F\mu$-$T_{\frac{1}{2}}$.

Proof. Assume that $(\mathcal{U}, \mu)$ is FG-μR_2. By Remark 3, it is $F\mu$-R_2. Let H be any $Fg\mu$-closed set with $u_\alpha \tilde{q} H$ for each $u_\alpha \in FP(\mathcal{U})$ that is, $u_\alpha \in H^c$, there are $F, G \in \mu$ such that $O_{u_\alpha} \in F$, $H \subseteq G$ and $F\tilde{q}G$ implies that $F\tilde{q}H$. From Lemma 1, we have $u_\alpha \tilde{q} cl_\mu(H)$ that is, $u_\alpha \in (cl_\mu(H))^c$. Therefore, $H^c \subseteq (cl_\mu(H))^c$ implies $cl_\mu(H) \subseteq H$ and so, $H = cl_\mu(H)$ this means that, any $Fg\mu$-closed set in $(\mathcal{U}, \mu)$ is an $F\mu$-closed set. Hence $(\mathcal{U}, \mu)$ is $F\mu$-$T_{\frac{1}{2}}$. Conversely, it is obvious.

Theorem 2. *Let* $(\mathcal{U}, \mu)$ *be GFTS and* $u_\alpha \in FP(\mathcal{U})$. *The next items are equivalent:*

(1) $(\mathcal{U}, \mu)$ *is* FG-μR_2,

(2) For any $Fg\mu$*-open set* O_{u_α} *containing* u_α, *there is* $O^*_{u_\alpha} \in \mu$ *such that* $cl_\mu(O^*_{u_\alpha}) \subseteq O_{u_\alpha}$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $(\mathcal{U}, \mu)$ is FG-μR_2 and O_{u_α} is an $Fg\mu$-open set containing u_α, we have $O^c_{u_\alpha} = H \in Fg\mu C(\mathcal{U})$. Clearly, $O_{u_\alpha} \tilde{q} H$ that is, $u_\alpha \tilde{q} H$. Since $(\mathcal{U}, \mu)$ is FG-μR_2, there are $O^*_{u_\alpha}, O_H \in \mu$ such that $O^*_{u_\alpha} \tilde{q} O_H$ implies that $O^*_{u_\alpha} \subseteq O^c_H$ so that, $cl_\mu(O^*_{u_\alpha}) \subseteq O^c_H$. Since $H \subseteq O_H$, we have $O^c_H \subseteq H^c = O_{u_\alpha}$. Therefore $cl_\mu(O^*_{u_\alpha}) \subseteq O_{u_\alpha}$.
(2) $\Longrightarrow$ (1). Let $G \in Fg\mu C(\mathcal{U})$ with $u_\alpha \tilde{q} G$, then $u_\alpha \in G^c = O_{u_\alpha}$ which is $Fg\mu$-open set containing u_α. By given, there is $F\mu$-open set $O^*_{u_\alpha}$ such that $cl_\mu(O^*_{u_\alpha}) \subseteq O_{u_\alpha} = G^c$ that is, $G \subseteq (cl_\mu(O^*_{u_\alpha}))^c = O_G$ and $cl_\mu(O^*_{u_\alpha}) \tilde{q}\ (cl_\mu(O^*_{u_\alpha}))^c = O_G$. Therefore $O^*_{u_\alpha} \tilde{q} O_G$. This completes the proof.

Theorem 3. *For an GFTS* $(\mathcal{U}, \mu)$ *and* $u_\alpha \in FP(\mathcal{U})$*. The next items are equivalent:*

(1) $(\mathcal{U}, \mu)$ *is FS-*GR_2*,*

(2) For any $Fg\mu$*-closed set* G *with* $u_\alpha \tilde{q} G$*, there are* $O_{u_\alpha}, O_G \in \mu$ *such that* $cl_\mu(O_{u_\alpha})\tilde{q}cl_\mu(O_G)$*.*

Proof. Necessity. Let $(\mathcal{U}, \mu)$ be FG-μR_2 and $G \in Fg\mu C(\mathcal{U})$ with $u_\alpha \tilde{q} G$, there are $O^*_{u_\alpha}$ and $O_G \in \mu$ such that $O_G \tilde{q} O^*_{u_\alpha}$. By Lemma 1, we have $cl_\mu(O_G)\tilde{q}O^*_{u_\alpha}$ implies $cl_\mu(O_G)\tilde{q}u_\alpha$. In similar, since $(\mathcal{U}, \mu)$ is FG-μR_2, there are $O^{**}_{u_\alpha}$ and $O_{cl_\mu(O_G)} \in \mu$ such that $O^{**}_{u_\alpha}\tilde{q}O_{cl_\mu(O_G)}$. By Lemma 1, we get $cl_\mu(O^{**}_{u_\alpha})\tilde{q}O_{cl_\mu(O_G)}$. Take $O_{u_\alpha} = O^*_{u_\alpha} \cup O^{**}_{u_\alpha} \in \mu$. By the above theorem, there is $O_{u_\alpha} \in \mu$ such that $cl_\mu(O_{u_\alpha}) \subseteq O^*_{u_\alpha}$. Since $cl_\mu(O_G)\tilde{q}O^*_{u_\alpha}$, we have $cl_\mu(O_G)\tilde{q}cl_\mu(O_{u_\alpha})$.
Conversely, It follows by the hypothesis.

Corollary 1. *An GFTS* $(\mathcal{U}, \mu)$ *is FS-*GR_2 *if and only if for any* $H \in I^{\mathcal{U}}$ *and any* $Fg\mu$*-closed set* G *with* $H\tilde{q}G$*, there are* $O_H, O_G \in Fg\mu O(\mathcal{U})$ *such that* $O_H \tilde{q} O_G$*.*

Proof. It can be obtained from Definition 12 and Remark 1.

Definition 13. *An GFTS* $(\mathcal{U}, \mu)$ *is said to be* $F\mu$*-symmetric iff* $u_\alpha \tilde{q} cl_\mu(v_\beta)$ *implies* $v_\beta \tilde{q} cl_\mu(u_\alpha)$ *for any* $u_\alpha,\ v_\beta \in FP(\mathcal{U})$*.*

Theorem 4. *For an GFTS* $(\mathcal{U}, \mu)$*. The next items are equivalent:*

(1) $(\mathcal{U}, \mu)$ *is* $F\mu$*-symmetric,*

(2) $cl_\mu(u_\alpha)\tilde{q}G$ *for any* $G \in F\mu C(\mathcal{U})$ *with* $u_\alpha \tilde{q} G$*.*

Proof. Necessity. Let $G \in F\mu C(\mathcal{U})$ such that $u_\alpha \tilde{q} G$, then $cl_\mu(\mathcal{V}_\beta) \subseteq G$ for any $v_\beta \in G$. This implies that $u_\alpha \tilde{q} cl_\mu(v_\beta)$. Since $(\mathcal{U}, \mu)$ is $F\mu$-symmetric, we have $v_\beta \tilde{q} cl_\mu(u_\alpha)$ for any $v_\beta \in G$ and so, there is $O_{v_\beta} \in \mu,\ v_\beta \in O_{v_\beta}$ with $u_\alpha \tilde{q} O_{v_\beta}$. Now take $H = \cup\{O_{v_\beta} : v_\beta \in G$ and $u_\alpha \tilde{q} O_{v_\beta}\}$, then $H = O_G$ and $u_\alpha \tilde{q} H$ implies $u_\alpha \in H^c$ and so, $cl_\mu(u_\alpha) \subseteq H^c$ that is, $cl_\mu(u_\alpha)\tilde{q}H$. Therefore $cl_\mu(u_\alpha)\tilde{q}G$.
Conversely. It is obvious.

Corollary 2. *An GFTS* $(\mathcal{U}, \mu)$ *is* $F\mu$*-symmetric iff* u_α *is an* $Fg\mu$*-closed set for any* $u_\alpha \in FP(\mathcal{U})$*.*

Remark 4. *Evidently, every* $F\mu$*-*T_1 *space is* $F\mu$*-symmetric but not conversely.*

Example 2. *Consider* $\mathcal{U} = \{u\}$ *and* $\mu = \{\underline{0}, \underline{1}, u_{0.5}\}$*, then* μ *is an GFT on* $\mathcal{U}$*. One can check that* μ *is* $F\mu$*-symmetric but not* $F\mu$*-*T_1*. Further,* μ *is not* $F\mu$*-*$T_{\frac{1}{2}}$ *.*

Proposition 3. *An GFTS* $(\mathcal{U}, \mu)$ *is* $F\mu$*-*T_1 *iff is both* $F\mu$*-symmetric and* $F\mu$*-*T_0*.*

Proof. Evidently, if $(\mathcal{U}, \mu)$ is $F\mu$-T_1, then it is $F\mu$-symmetric and $F\mu$-T_0.
Conversely, let $(\mathcal{U}, \mu)$ be $F\mu$-symmetric and $F\mu$-T_0. Suppose $u_\alpha \tilde{q} v_\beta$, we have either $u_\alpha \tilde{q} cl_\mu(v_\beta)$ or $v_\beta \tilde{q} cl_\mu(u_\alpha)$. By $F\mu$-symmetric, we get $u_\alpha \tilde{q} cl_\mu(v_\beta)$ and $v_\beta \tilde{q} cl_\mu(u_\alpha)$ for any $u_\alpha, v_\beta \in FP(\mathcal{U})$. This completes the proof.

From the pervious results, one can verify the following proposition.

Proposition 4. *For an $F\mu$-symmetric space $(\mathcal{U},\mu)$. The next items are equivalent:*

(1) $(\mathcal{U},\mu)$ *is* $F\mu$-T_0,

(2) $(\mathcal{U},\mu)$ *is* $F\mu$-$T_{\frac{1}{2}}$,

(3) $(\mathcal{U},\mu)$ *is* $F\mu$-T_1.

Definition 14. *An GFTS $(\mathcal{U},\mu)$ is called $F\mu$-G_3 iff it is FG-μR_2 and $F\mu$-symmetric.*

Theorem 5. *Every $F\mu$-G_3 space is $F\mu$-$T_{2\frac{1}{2}}$.*

Proof. Let $(\mathcal{U},\mu)$ be $F\mu$-G_3 and $u_\alpha,\ v_\beta \in FP(\mathcal{U})$ with $u_\alpha \tilde{q} v_\beta$. Since $(\mathcal{U},\mu)$ is $F\mu$-symmetric and so, u_α is a $Fg\mu$-closed set for any $u_\alpha \in FP(\mathcal{U})$. By Theorem 3, there are $O_{u_\alpha},\ O_{v_\beta} \in \mu$ such that $cl(O_{u_\alpha})\tilde{q}cl(O_{v_\beta})$. Therefore $(\mathcal{U},\mu)$ is $F\mu$-$T_{2\frac{1}{2}}$.

Corollary 3. *Clearly, every $F\mu$-G_3 space is $F\mu$-T_2 but not conversely.*

Example 3. *Let $\mathcal{U}$ be an infinite set. For $u, v \in \mathcal{U},\ u \neq v$, let $H_{u,v} \in I^{\mathcal{U}}$ defined as:*

$$H_{u,v}(w) = \begin{cases} 1, & \textit{if } w = u \\ 0, & \textit{if } w = v \\ 0.5, & \textit{if } w \neq u \textit{ and } w \neq v \textit{ for all } w \in \mathcal{U}. \end{cases}$$

Consider the GFT μ on $\mathcal{U}$ which is induced by the class $\{H_{u,v} : u, v \in \mathcal{U},\ u \neq v\}$. One can verify that μ is $F\mu$-T_2 but not FG-μR_2 and so, is not $F\mu$-G_3.

Theorem 6. *For an GFTS $(\mathcal{U},\mu)$. The next items are equivalent:*

(1) $(\mathcal{U},\mu)$ *is* $F\mu$-G_3,

(2) $(\mathcal{U},\mu)$ *is* $F\mu$-T_3.

Proof. (1) $\Longrightarrow$ (2). Assume that $(\mathcal{U},\mu)$ is $F\mu$-G_3, we have it is both FG-μR_2 and $F\mu$-symmetric. Clearly, every FG-μR_2 is $F\mu$-R_2 also, every $F\mu$-G_3 is $F\mu$-T_2. Hence $(\mathcal{U},\mu)$ is $F\mu$-R_2 and $F\mu$-T_1 that is, $(\mathcal{U},\mu)$ is $F\mu$-T_3.
(2) $\Longrightarrow$(1). Let $(\mathcal{U},\mu)$ be $F\mu$-T_3, then it is both $F\mu$-R_2 and $F\mu$-T_1. This implies that $(\mathcal{U},\mu)$ is $F\mu$-$T_{\frac{1}{2}}$ and $F\mu$-symmetric. Thus, $(\mathcal{U},\mu)$ is $F\mu$-R_2 and $F\mu$-$T_{\frac{1}{2}}$ implies that $(\mathcal{U},\mu)$ is FG-μR_2 as well as, it is $F\mu$-symmetric. Therefore $(\mathcal{U},\mu)$ is $F\mu$-G_3.

4. Fuzzy $g\mu$-normal spaces

In this part, we introduce a new class of spaces called, $Fg\mu$-normal spaces in the frame of $GFTS$ via $Fg\mu$-closed sets and discuss some properties and related theorems in this sequel.

Definition 15. *An GFTS $(\mathcal{U},\mu)$ is called $Fg\mu$-normal (or FG-μR_3) iff for any $G, H \in Fg\mu C(\mathcal{U})$ with $G\tilde{q}H$, there are $O_G, O_H \in \mu$ containing G, H respectively, such that $O_G\tilde{q}O_H$.*

Remark 5. *Evidently, every FG-μR_3 space is $F\mu$-R_3.*

Corollary 4. *An GFTS $(\mathcal{U}, \mu)$ is FG-μR_3 if and only if for any two $Fg\mu$-closed sets G, H with $G\tilde{q}H$, there are $O_G, O_H \in Fg\mu O(\mathcal{U})$ such that $O_G\tilde{q}\ O_H$.*

Proof. It can be obtained from Definition 15 and Remark 1.

Theorem 7. *For an GFTS $(\mathcal{U}, \mu)$. The next items are equivalent:*

(1) $(\mathcal{U}, \mu)$ is FG-μR_3,

(2) For any $H \in Fg\mu C(\mathcal{U})$ and any $O_H \in \mu$ containing H, there is $O^*_H \in \mu$ such that $cl_\mu(\ O^*_H) \subseteq O_H$.

Proof. Necessity. Assume that $(\mathcal{U}, \mu)$ be FG-μR_3, $H \in Fg\mu C(\mathcal{U})$, and $O_H \in F\mu O(\mathcal{U})$ containing H, we have $O^c_H \in F\mu C(\mathcal{U})$. Clearly, $O_H\tilde{q}O^c_H$ that implies $H\tilde{q}O^c_H$. Since $(\mathcal{U}, \mu)$ is FG-μR_3, there are $O^*_H, O_{O^c_H} \in \mu$ such that $O^*_H\tilde{q}O_{O^c_H}$ implies that $O^*_H\subseteq(O_{O^c_H})^c$ and so, $cl_\mu(O^*_H)\subseteq(O_{O^c_H})^c$. Since $O^c_H\subseteq O_{O^c_H}$, we have $(O_{O^c_H})^c\subseteq O_H$ and $cl_\mu(\mathrm{O}^*_H)\subseteq(O_{O^c_H})^c\subseteq O_H$. The result holds.
Conversely, it follows directly by the hypothesis.

Theorem 8. *For an GFTS $(\mathcal{U}, \mu)$. The following items are equivalent:*

(1) $(\mathcal{U}, \mu)$ is FG-μR_3,

(2) For any $F, G \in Fg\mu C(\mathcal{U})$ with $F\tilde{q}G$, there are $O_F, O_G \in \mu$ containing F, G respectively, such that $cl_\mu(O_F)\tilde{q}cl_\mu(O_G)$.

Proof. Necessity. Assume that $(\mathcal{U}, \mu)$ is FG-μR_3 and $F, G \in Fg\mu C(\mathcal{U})$ with $F\tilde{q}G$, there are $O^*_F, O_G \in \mu$ such that $O^*_F\tilde{q}O_G$ implies that $O^*_F\tilde{q}cl_\mu(O_G)$ (by Lemma 1). Again, $(\mathcal{U}, \mu)$ is FG-μR_3, there are $O^{**}_F, O_{cl_\mu(O_G)} \in \mu$ such that $O^{**}_F\tilde{q}O_{cl_\mu(O_G)}$. This implies that $cl_\mu(O^{**}_F)\tilde{q}O_{cl_\mu(O_G)}$ (by Lemma 1). Take $O_F = O^*_F \cup O^{**}_F \in \mu$. Since $(\mathcal{U}, \mu)$ is FG-μR_3 and $O^*_F \in \mu$. So by the above theorem, there is $O_F \in \mu$ such that $cl_\mu(O_F) \subseteq O^*_F$. Since $O^*_F\tilde{q}cl_\mu(O_G)$, we have $cl_\mu(O_F)\tilde{q}cl_\mu(O_G)$.
Conversely, it follows directly from the hypothesis.

Definition 16. *An GFTS $(\mathcal{U}, \mu)$ is called $F\mu$-G_4 iff it is FG-μR_3 and $F\mu$-symmetric.*

Theorem 9. *Every $F\mu$-G_4 space is $F\mu$-G_3.*

Proof. Suppose that $(\mathcal{U}, \mu)$ is $F\mu$-G_4, then it is FG-μR_3 and $F\mu$-symmetric. Consider H is $Fg\mu$-closed set with $u_\alpha\tilde{q}H$, then u_α is an $Fg\mu$-closed set (as $(\mathcal{U}, \mu)$ is $F\mu$-symmetric). Since $(\mathcal{U}, \mu)$ is FG-μR_3, there are $O_{u_\alpha}, O_H \in \mu$ such that $O_{u_\alpha}\tilde{q}O_H$. Hence $(\mathcal{U}, \mu)$ is FG-μR_2. Therefore $(\mathcal{U}, \mu)$ is $F\mu$-G_3.

Corollary 5. *If $(\mathcal{U}, \mu)$ is FG-μR_3 and $F\mu$-symmetric space, then $(\mathcal{U}, \mu)$ is FG-μR_2 .*

Proposition 5. *An GFTS $(\mathcal{U}, \mu)$ is FG-μR_3 iff it is both $F\mu$-R_3 and $F\mu$-$T_{\frac{1}{2}}$.*

Proof. It is analogues to that of Theorem 1.

Theorem 10. *An GFTS* $(\mathcal{U}, \mu)$ *is* $F\mu$-G_4 *if and only if it is* $F\mu$-T_4.

Proof. It is analogues to that of Theorem 6.

From the definitions and discussions in section 3 and 4. The following implications hold.

Corollary 6. *The following implications hold.*

$$\begin{array}{l} F\mu-T_4 \Longrightarrow F\mu-T_3 \Longrightarrow F\mu-T_{2\frac{1}{2}} \Longrightarrow F\mu-T_2 \Longrightarrow F\mu-T_1 \Longrightarrow F\mu-T_0 \\ \quad\ \Updownarrow \qquad\quad \Updownarrow \\ F\mu-G_4 \Longrightarrow F\mu-G_3 \Longleftrightarrow FG-\mu R_2 \wedge F\mu-symmetric \\ \quad\ \Updownarrow \\ FG-\mu R_3 \wedge F\mu-symmetric \Longrightarrow FG-\mu R_2 \Longrightarrow F\mu-R_2 \end{array}$$

5. Further applications and relations

In the following discussion, we will explore the basic preservation theorems and some relations of FG-μR_2 and FG-μR_3.

Definition 17. *For an GTS* $(\mathcal{U}, \theta)$. *The class* $\mu_\theta = \{\chi_H : H \in \theta\}$ *forms an GFT on* $\mathcal{U}$ *generated by* θ.

Theorem 11. $(\mathcal{U}, \mu_\theta)$ *is* FG-μR_2 $\Longleftrightarrow (\mathcal{U}, \theta)$ *is* μ*-regular.*

Proof. Necessity. Suppose that $(\mathcal{U}, \mu_\theta)$ is FG-μR_2 and G is any μ-closed set in $(\mathcal{U}, \theta)$ such that $u \notin G$, then $\chi_G = H \in F\mu C(\mathcal{U})$ which is also, $Fg\mu$-closed set in $(\mathcal{U}, \mu_\theta)$ with $u_1 \tilde{q} H$. Since $(\mathcal{U}, \mu_\theta)$ is FG-μR_2, there are $O_{u_1}, O_H \in \mu_\theta$ such that $O_{u_1} \tilde{q} O_H$. Thus, there are $O_u, O_G \in \theta$ such that $O_{u_1} = \chi_{O_u}, O_H = \chi_{O_G}$ and $O_u \cap O_G = \emptyset$. Hence $(\mathcal{U}, \theta)$ is μ-regular.
Conversely, let $(\mathcal{U}, \theta)$ be μ-regular and H any $Fg\mu$-closed set in $(\mathcal{U}, \mu_\theta)$ such that $u_\alpha \tilde{q} H$, there is μ-closed set B in $(\mathcal{U}, \theta)$ such that $H = \chi_{O_B}$ and $u \notin B$. Since $(\mathcal{U}, \theta)$ is μ-regular, there are $O_u, O_B \in \theta$ such that $O_u \cap O_B = \emptyset$ and so, there are O_{u_α} and $O_H \in \mu_\theta$ such that $O_{u_\alpha} = \chi_{O_u},\ O_H = \chi_{O_B}$ with $O_{u_\alpha} \tilde{q} O_H$. Therefore $(\mathcal{U}, \mu_\theta)$ is FG-μR_2.

Theorem 12. $(\mathcal{U}, \mu_\theta)$ *is* FG-μR_3 $\Longleftrightarrow (\mathcal{U}, \theta)$ *is* μ*-normal.*

Proof. It can be obtained by a similar way of that in Theorem 11.

Definition 18. *For two GFTSs* $(\mathcal{U}, \mu_1)$, $(\mathcal{V}, \mu_2)$. *A map* $f{:}(\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ *is called* $Fg\mu c$*-irresolute iff* $f^{-1}(G) \in Fg\mu C(\mathcal{U})$ *for any* $G \in Fg\mu C(\mathcal{V})$.

Note. Evidently, every $Fg\mu c$-irresolute map is $Fg\mu$-continuous.

Theorem 13. *Let* $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ *be an* $F\mu$*-open and* $Fg\mu$*-continuous bijection map. If* H *is* $Fg\mu$*-closed set in* $(\mathcal{V}, \mu_2)$, *then* $f^{-1}(H)$ *is* $Fg\mu$*-closed set in* $(\mathcal{U}, \mu_1)$.

Proof. Consider $H \in Fg\mu C(\mathcal{V})$ and $f^{-1}(H) \subseteq G$ where $G \in \mu_1$, then $H \subseteq f(G)$. Since f is $F\mu$-open, we have $f(G) \in \mu_2$. By given, H is a $Fg\mu$-closed set in $(\mathcal{V}, \mu_2)$, then $cl_\mu(H) \subseteq f(G)$. So that $f^{-1}(cl_\mu(H)) \subseteq G$ (as f is injective). Evidently, f is $Fg\mu$-continuous, then $f^{-1}(cl_\mu(H))$ is $Fg\mu$-closed set in $(\mathcal{U}, \mu_1)$. Hence $f^{-1}(cl_\mu(H)) \subseteq cl_\mu(f^{-1}(cl_\mu(H))) = f^{-1}(cl_\mu(H)) \subseteq G$. Therefore, $f^{-1}(H)$ is an $Fg\mu$-closed set in $(\mathcal{U}, \mu_1)$.

Corollary 7. *For two GFTSs $(\mathcal{U}, \mu_1)$, $(\mathcal{V}, \mu_2)$. If $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ is an $F\mu$-open and $Fg\mu$-continuous bijection map, then f is $Fg\mu c$-irresolute.*

Theorem 14. *Let $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ be a $F\mu$-open and $Fg\mu$-continuous bijection map. If $(\mathcal{U}, \mu_1)$ is FG-μR_2, then $(\mathcal{V}, \mu_2)$ also, is FG-μR_2.*

Proof. Let $H \in Fg\mu C(\mathcal{V})$ and $v_\alpha \tilde{q} H$. Since f is $F\mu$-open and $Fg\mu$-continuous bijective, we have by Theorem 13, $f^{-1}(H)$ is $Fg\mu$-closed. Put $f(u_\alpha) = v_\alpha$, then $u_\alpha \tilde{q} f^{-1}(H)$. Since $(\mathcal{U}, \mu_1)$ is FG-μR_2, there are O_{u_α}, $O_{f^{-1}(H)} \in \mu_1$ such that $O_{u_\alpha} \tilde{q} O_{f^{-1}(H)}$. Since f_{up} is $F\mu$-open and bijective, we get $f(O_{u_\alpha}), f(O_{f^{-1}(H)}) \in \mu_2$ such that $v_\alpha \in f(O_{u_\alpha})$, $H \subseteq f(O_{f^{-1}(H)})$ and $f(O_{u_\alpha}) \tilde{q} f(O_{f^{-1}(H)})$. Therefore, $(\mathcal{V}, \mu_2)$ is FG-μR_2.

Theorem 15. *Consider $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ is $F\mu$-open and $Fg\mu$-continuous bijection. If $(\mathcal{U}, \mu_1)$ is FG-μR_3, then $(\mathcal{V}, \mu_2)$ also, is FG-μR_3.*

Proof. It can be obtained by a similar way of that in Theorem 14.

Theorem 16. *For an $F\mu$-continuous and $Fg\mu$-closed injective map $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$. If $(\mathcal{V}, \mu_2)$ is FG-μR_2, then $(\mathcal{U}, \mu_1)$ also, is FG-μR_2.*

Proof. Let $H \in Fg\mu C(\mathcal{U})$ with $u_\alpha \tilde{q} H$. By $F\mu$-continuity and $Fg\mu$-closedness, we have $f(H) \in Fg\mu C(\mathcal{V})$. In fact, if $f(H) \subseteq G$ and $G \in \mu_2$, then $H \subseteq f^{-1}(G)$ and so, $cl_\mu(H) \subseteq f^{-1}(G)$ implies that $f(H) \subseteq f(cl_\mu(H)) \subseteq ff^{-1}(G) \subseteq G$ that is, $f(H) \subseteq G$. Hence $f(H)$ is $Fg\mu$-closed. Since f is injective, we have $f(u_\alpha) \tilde{q} f(H)$. Given that $(\mathcal{V}, \mu_2)$ is FG-μR_2, there are $O_{f(u_\alpha)}, O_{f(H)} \in \mu_2$ such that $O_{f(u_\alpha)} \tilde{q} O_{f(H)}$. Since f is $F\mu$-continuous, we have $f^{-1}(O_{f(u_\alpha)}), f^{-1}(O_{f(H)}) \in \mu_1$ such that $u_\alpha \in f^{-1}(O_{f(u_\alpha)})$, $H \subseteq f^{-1}(O_{f(H)})$, and $f^{-1}(O_{f(u_\alpha)}) \tilde{q} f^{-1}(O_{f(H)})$. Therefore, $(\mathcal{U}, \mu_1)$ is FG-μR_2.

Theorem 17. *Let $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ be $F\mu$-continuous, $Fg\mu$-closed injective. If $(\mathcal{V}, \mu_2)$ is FG-μR_3, then $(\mathcal{U}, \mu_1)$ is FG-μR_3.*

Proof. Suppose that $H, G \in Fg\mu C(\mathcal{U}))$ with $H \tilde{q} G$. As in the above theorem $f(H), f(G) \in Fg\mu C(\mathcal{V})$. Since f_{up} is injective, then $f(H) \tilde{q} f(G)$. By given $(\mathcal{V}, \mu_2)$ is FG-μR_3, there are $O_{f(H)}, O_{f(G)} \in \mu_2$ with $O_{f(H)} \tilde{q} O_{f(G)}$. Since f is $F\mu$-continuous, we have $f^{-1}(O_{f(H)}), f^{-1}(O_{f(G)}) \in \mu_1$ and $H \subseteq f^{-1}(O_{f(H)})$, $G \subseteq f^{-1}(O_{f(G)})$ with $f^{-1}(O_{f(H)}) \tilde{q} f^{-1}(O_{f(G)})$. Therefore, $(\mathcal{U}, \mu_1)$ is FG-μR_2.

Theorem 18. *Let $f : (\mathcal{U}, \mu_1) \longrightarrow (\mathcal{V}, \mu_2)$ be $Fg\mu c$-irresolute and $F\mu$-open surjective. If $(\mathcal{U}, \mu_1)$ is FG-μR_3, then $(\mathcal{V}, \mu_2)$ is FG-μR_3.*

Proof. Let $G, H \in Fg\mu C(\mathcal{V})$ such that $G\tilde{q}H$, then $f^{-1}(G), f^{-1}(H)$ are $Fg\mu$-closed sets in $(\mathcal{U}, \mu_1)$ with $f^{-1}(G)\,\tilde{q}f^{-1}(H)$. Since $(\mathcal{U}, \mu_1)$ is FG-μR_3, there are $O_{f^{-1}(G)}, O_{f^{-1}(H)} \in \mu_1$ containing $f^{-1}(G), f^{-1}(H)$, respectively with $O_{f^{-1}(G)}\tilde{q}O_{f^{-1}(H)}$. Since f is surjective, we have $G \subseteq f(O_{f^{-1}(G)})$, $H \subseteq f(O_{f^{-1}(H)})$ and $f(O_{f^{-1}(G)}), f(O_{f^{-1}(H)}) \in \mu_2$(As f is $F\mu$-open) with $(O_{f^{-1}(G)})\tilde{q}f(O_{f^{-1}(H)})$. Hence $(\mathcal{V}, \mu_2)$ is FG-μR_3.

From Theorems 14 and 15, one can verify the next theorem.

Theorem 19. *The property of being FG-μR_2 (FG-μR_3) is a $Fg\mu$-topological property.*

In the following, we show that the being FG-μR_2 (FG-μR_3) are hereditary property.

Theorem 20. *Every $F\mu$-subspace $(\mathcal{V}, \mu_V)$ of FG-μR_2 is FG-μR_2.*

Proof. Let $(\mathcal{U}, \mu)$ be FG-μR_2, $u_\alpha \in FP(\mathcal{V})$ and $H \in Fg\mu C(\mathcal{V})$ with $u_\alpha\tilde{q}H$, there is an $Fg\mu$-closed set G in $(\mathcal{U}, \mu)$ such that $H = \chi_V \cap G$ and $u_\alpha\tilde{q}G$. Since $(\mathcal{U}, \mu)$ is FG-μR_2, we have $O_{u_\alpha}, O_G \in \mu$ such that $O_{u_\alpha}\tilde{q}O_G$. Put $O^*_{u_\alpha} = \chi_V \cap O_{u_\alpha} \in \mu_V$ and $O^*_G = \chi_V \cap O_G \in \mu_V$ which are containing u_α and G respectively, and $O^*_{u_\alpha}\tilde{q}O^*_G$. This completes the proof.

Theorem 21. *Every $F\mu$-closed subspace $(\mathcal{V}, \mu_V)$ of FG-μR_3 is FG-μR_3.*

Proof. It follows by a similar way of that in Theorem 20.

6. Conclusion and future work

Topology is a branch of mathematics that studies the properties of space preserved under continuous transformations. It allows mathematicians to analyze and classify spaces, leading to applications in various fields, where understanding the fundamental structure of spaces is of great importance.

This study focuses on the applications of $Fg\mu$-closed sets in generalized fuzzy topology. Some classes, namely $Fg\mu$-regular, $Fg\mu$-normal, $F\mu$-symmetric, have been introduced and analyzed. The paper thoroughly have investigated the fundamental properties and unique characteristics of these classes. A comprehensive is framework has been established through the presentation of related theorems and relations, demonstrating their interrelationships with other separation axioms in this context. Additionally, the hereditary and topological properties of these classes have been explored.

The present results in this article are very useful and contribute to the development of the theoretical and practical foundations of the GFT, and will open up the door for future works in this area such as:

- We plan to further study for $Fg\mu$-closed sets with some separation axioms in this settings.
- We intend to discuss some weaker forms of $Fg\mu$-closed sets with some applications of them in FGT-spaces.

- We plan to extend the characterizations of these classes within infra soft topological spaces [3, 4], and explore their potential applications.
- Exploring the proposed concepts through well-known topological structures [5, 6] is an interesting idea for researchers and scholars interested in topological studies, which we have left for future investigations.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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