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Almost Hermitian Manifold with Flat Bochner Tensor

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Abstract. Many researchers investigated the flat Bochner tensor on some kinds of almost Hermitian manifold. In the present paper the author studies this tensor on general class almost Hermitian manifold by using a new methodology which is called an adjoint G-structure space. Thus this study generalize the results which are found out by those researchers. It is proved that if M is an almost Hermitian manifold of class R_1 with flat Bochner tensor, then either M is 2-dimensional flat Ricci manifold or n-dimensional $(n > 2)$ flat scalar curvature tensor manifold. As well, it is proved that if M is an almost Hermitian manifold with flat Bochner tensor, then M is a manifold of class R_3 if and only if M is a linear complex manifold. Later on, equivalently of classes R_2 and R_3 is investigated. Finally we prove that if M is flat manifold with flat Bochner tensor, then M is an Einstein manifold with a cosmological constant.
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1. Introduction

The Bochner tensor was introduced by S. Bochner [3] . He defined this tensor on a Kahler manifold as a formal analogy of the Weyle conformal curvature tensor. S. Tachibana [14] gave It the real form and he proved that the Bochner tensor had a meaning on any almost Hermitian manifold. The Kahler manifold with flat Bochner curvature tensor has been studied by many researchers. M. Mastumoto [8] Proved that a Kahler manifold of constant scalar curvature tensor with flat Bochner tensor is local symmetric. S. Tachibana [15] proved that Kahler manifold of a constant scalar curvature tensor with flat Bochner tensor is local-isometric to the product of complex spaces. L. Vanhecke [18] studied the Bochner curvature tensor on almost Hermitian manifold and he obtained some properties which are proved for Kahler manifold. Z. Olsgak [10] gave the classification of 4-dimensional compact flat Bochner of Kahler manifold with non positive scalar curvature tensor. M. Petrovic and L. Vestraclen [11] studied the flat Bochner of Kahler manifold where the Weyles tensor satisfies some conditions. K. Nam [9] proved that if M is a Kahler manifold with flat Bochner curvature tensor whose length of the Ricci tensor is constant, then M is a space of constant holomorphic sectional curvature

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or a locally product space of two spaces of constant holomorphic sectional curvatures. A. Al-Otman [1] studied the Bochner tensor of the class nearly Kahler manifold. He found the classification of the flat Bochner tensor of this class.

In fact you may notice that the most of the mentioned researchers above are studied the Bochner tensor on some kinds of the sixteen classes of almost Hermitian manifold. In the present paper we study the flat Bochner tensor on general class almost Hermitian manifold. This study uses the method of adjoint G-structure space which was introduced by V.F. Krichenko who found two tensors which are the structure and virtual tensors [5]. This method helped the researchers to study the different geometrical properties of almost Hermitian manifold, therefore we use this method to generalize the results which are given by Vanhecke and by the referred researchers.

2. Preliminaries

Let M be $2n$-dimensional smooth manifold, $X(M)$ be a module of smooth vector fields on M and $C^{\infty}(M)$ be an algebra of smooth functions on M.

An almost Hermitian structure (*AH*-structure) on M is a pair $\{J,\ g=<.,.>\}$, where J is an endomorphism of a tangent space $T_p(M)$ with $\left(J_p\right)^2=-id$ and g is a Reimannian metric on M such that $<JX,JY>=<X,Y>,\ X,Y\in X(M)$. A smooth manifold provided *AH*-structure is called an almost Hermitian manifold(*AH*-manifold). We recall that the fundamental (Kahlerian [7]) form is given by $\Omega(X,Y)=<X,JY>$.

As it is well known from [6] that the given *AH*-structure on a manifold M is equivalent to the given an G-structure in principle fiber bundle of all complex frames of M with structure group $U(n)$. This group is called an adjoint G-structure. The frame adapted to the *AH*-structure is called A-frame look as $\{p,\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{\hat{1}},\ldots,\varepsilon_{\hat{n}}\}$ [5], where ε_a are the eigenvectors corresponded to the eigenvalue $i=\sqrt{-1}$ and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $i=-\sqrt{-1}$. Here the index a ranges from 1 to n and $\hat{a}=a+n$.

The matrices of the J, G and Ω in A-frame are given as:

$$\left(g_{ij}\right)=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix},\left(J^i_j\right)=\begin{pmatrix}\sqrt{-1}J_n & 0\\ 0 & -\sqrt{-1}J_n\end{pmatrix},(\Omega_{ij})=\begin{pmatrix}0 & \sqrt{-1}I_n\\ -\sqrt{-1}I_n & 0\end{pmatrix} \quad (1)$$

Where I_n is the identity matrix of order n.

A Bochner tensor on *AH*-manifold M is a tensor of type $(4,0)$ which is defined as the form:

$$\begin{aligned}B(X,Y,Z,W) &= R(X,Y,Z,W)+L(X,W)g(Y,Z)-L(X,Z)g(Y,W)\\ &+ L(Y,Z)g(X,W)-L(Y,W)g(X,Z)+L(JX,W)g(JX,Z)\\ &- L(JX,Z)g(JY,W)+L(JY,Z)g(JX,W)-L(JY,W)g(JX,Z)\\ &- 2L(JX,Y)g(JZ,W)-2L(JZ,W)g(JX,Y),\end{aligned}$$

Where

$$L(X,Y)=-\frac{1}{2n+4}g(rX,Y)+\frac{K}{2(2n+2)(2n+4)}g(X,Y),$$

r is the Ricci tensor and K is the scalar curvature tensor, $X,Y,Z,W \in X(M)$.

Denote $C(X,Y) = L(JX,Y)$. We have $g(JX,Y) = -\Omega(X,Y)$, where Ω is the fundamental(Kahlerian) form.

The components of Bochner tensor at any frame will be as the form:

$$\begin{aligned} B_{ijkl} &= R_{ijkl} + L_{il}g_{jk} - L_{ik}g_{jl} + L_{jk}g_{il} - L_{jl}g_{ik} - C_{il}\Omega_{jk} + C_{ik}\Omega_{jl} - C_{jk}\Omega_{il} \\ &+ C_{jl}\Omega_{ik} + 2C_{ij}\Omega_{kl} + 2C_{kl}\Omega_{ij} \end{aligned} \tag{2}$$

$$L_{ij} = -\frac{1}{2n+4} r_{ij} + \tilde{K} g_{ij} \tag{3}$$

$$C_{ij} = -\frac{1}{2n+4} J_i^k r_{kj} + \tilde{K} J_i^k g_{kj} \tag{4}$$

Where

$$\tilde{K} = \frac{K}{2(2n+2)(2n+4)}$$

Suppose that the indices a,b,c and d in the range $1,2,\ldots,n$. Denote $\hat{a} = a+n$.

In the following proposition we find the components of Bochner tensor on any AH-manifold in the ajoint G-structure space, i.e. in the A-frame:

Proposition 1. *The components of Bochner tensor of AH-manifold are given as the following forms:*

1. $B_{abcd} = R_{abcd}$
2. $B_{\hat{a}bcd} = R_{\hat{a}bcd} + \frac{1}{n+2}(r_{bd}\delta_c^a - r_{bc}\delta_d^a + r_{cd}\delta_b^a)$
3. $B_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} + \frac{1}{2n+4}\left(r_c^a\delta_d^b - r_d^a\delta_c^b + r_d^b\delta_c^a - r_c^b\delta_d^a\right)$
4. $B_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} + \frac{1}{n+2}(r_b^a\delta_c^d - r_c^d\delta_b^a)$
5. $B_{a\hat{b}cd} = R_{a\hat{b}cd} - \frac{1}{n+2} r_{cd}\delta_d^c$
6. $B_{ab\hat{c}d} = R_{ab\hat{c}d} - \frac{1}{n+2} r_{ab}\delta_d^c$
7. $B_{abc\hat{d}} = R_{abc\hat{d}} - \frac{1}{n+2} r_{ab}\delta_c^d$
8. $B_{\hat{a}b\hat{c}d} = R_{\hat{a}b\hat{c}d} + \frac{1}{n+2}\left(r_b^a\delta_d^c + r_d^c\delta_b^a\right) - 4K\delta_b^a\delta_d^c$

And the other components of the Bochner tensor are conjugate of the above components.

Proof. 1 - Set $i=a$, $j=b$, $k=c$, $l=d$, thus equation (2) becomes:

$$\begin{aligned} B_{abcd} &= R_{abcd} + L_{ad}g_{bc} - L_{ac}g_{bd} + L_{bc}g_{ad} - L_{bd}g_{ac} - C_{ad}\Omega_{\mathrm{bc}} + \mathrm{C}_{\mathrm{ac}}\Omega_{\mathrm{bd}} - \mathrm{C}_{\mathrm{bc}}\Omega_{\mathrm{ad}} \\ &+ C_{BD}\Omega_{\mathrm{ac}} + 2\mathrm{C}_{\mathrm{ab}}\Omega_{\mathrm{cd}} + 2C_{cd}\Omega_{ab} \end{aligned}$$

According to equations (1) we get $B_{abcd} = R_{abcd}$.

2 - Set $i=\hat{a},\ j=b,\ k=c,\ l=d$, the equation (2) becomes:

$$\begin{aligned} B_{\hat{a}bcd} &= R_{\hat{a}bcd}+L_{\hat{a}d}g_{bc}-L_{\hat{a}c}g_{bd}+L_{bc}g_{\hat{a}d}-L_{bd}g_{\hat{a}c}-C_{\hat{a}d}\Omega_{bc}+C_{\hat{a}c}\Omega_{bd}-C_{bc}\Omega_{\hat{a}d} \\ &+ C_{bd}\Omega_{\hat{a}c}+2C_{ab}\Omega_{cd}+2C_{cd}\Omega_{\hat{a}b} \end{aligned}$$

Using (1), (3) and (4), we obtained:

$$B_{\hat{a}bcd}=R_{\hat{a}bcd}+\frac{1}{n+2}(r_{bd}\delta_c^a-r_{bc}\delta_d^a+r_{cd}\delta_b^a)$$

In the same manner we can get the other components.

3. Main Results

A. Gray [4] defined three special classes of *AH*-manifold, which are given as the following form:

1. Class R_1 if $<R(X,Y)Z,W>=<R(JX,JY)Z,W>$.
2. Class R_2 if $<R(X,Y)Z,W>=<R(JX,JY)Z,W>+<R(JX,Y)JZ,W>+R<JX,Y)Z,JW>$.
3. Class R_3 if $<R(X,Y)Z,W>=<R(JX,JY)JZ,JW>$.

Gray proved that for a random *AH*-manifold , the relation among them is, $R_1\subset R_2\subset R_3$.

The manifold of class R_1 is called a parakahler manifold [12]. The manifold of class R_3 has been studied by the name *RK*-manifold [17]. The following lemma gives the necessary and sufficient conditions that a random *AH*-manifold is one of the above classes in the adjoint *G*-space.

Lemma 1 ([16])**.** *In the adjoint G-structure space, an AH-manifold is a manifold of:*

1. *Class R_1 if, and only if,* $R_{\hat{a}bcd}=0$, $R_{abcd}=0$, $R_{\hat{a}\hat{b}cd}=0$
2. *Class R_2 if, and only if,* $R_{\hat{a}bcd}=0$, $R_{abcd}=0$
3. *Class R_3 if, and only if,* $R_{\hat{a}bcd}=0$

Recall that an *AH*-manifold has J-invariant Ricci tensor if, $r\circ J=J\circ r$ [16].

Lemma 2 ([16])**.** *An AH-manifold has J-invariant Ricci tensor if, and only if, in the adjoint G-structure space,* $r_{ab}=0$.

Theorem 1. *Suppose that M is AH-manifold with flat Bochner tensor, then M is a manifold of class R_3 if, and only if, M is linear complex manifold.*

Proof. By proposition 1 we have:

$$B_{\hat{a}bcd} = R_{\hat{a}bcd} + \frac{1}{n+2}(r_{bd}\delta^a_c - r_{bc}\delta^a_d + r_{cd}\delta^a_b)$$

Suppose that M is *AH*-manifold of class R_3 with flat Bochner tensor. That means $B_{\hat{a}bcd} = 0$ and $R_{\hat{a}bcd} = 0$. Thus we get:

$$r_{bd}\delta^a_c - r_{bc}\delta^a_d + r_{cd}\delta^a_b = 0 \tag{5}$$

Contracting (5) by the indexes c and a, we obtained:

$$nr_{bd} - r_{bd} + r_{bd} = 0$$

Which means that $r_{bd} = 0$. By Lemma 2 we have $r_{bd} = 0$ if, and only if, $r \circ J = J \circ r$. Hence, from [2] M is linear complex manifold.

Corollary 1. *Suppose that M is an AH-manifold with flat Bochner tensor, then M is a manifold of class R_3 if, and only if, M is a manifold of class R_2.*

Proof. This is directly from the condition of the class R_2 .

Theorem 2. *Suppose that M is an AH-manifold with flat Bochner tensor. If M is a manifold of class R_1, then M is either n-dimensional Ricci flat manifold for $n > 2$ or 2-dimensional flat scalar curvature manifold.*

Proof. Suppose that M is *AH*-manifold of class R_1 with flat Bochner tensor. According to Lemma 1 we have $R_{\hat{a}bcd} = 0$, $R_{abcd} = 0$, $R_{\hat{a}\hat{b}cd} = 0$. Thus

$$r^a_c\delta^b_d - r^a_d\delta^b_c + r^b_d\delta^a_c - r^b_c\delta^a_d = 0 \tag{6}$$

Contracting the equation (6) by the indexes a and c we get:

$$r^a_a\delta^b_d - r^b_d + nr^b_d - r^b_d = 0 \tag{7}$$

Contracting the equation (7) by the indexes b and d we obtained:

$$nr^a_a - r^a_a + nr^a_a - r^a_a = 0$$

Hence

$$r^a_a = 0$$

Thus, the equation (7) will be as the form:

$$(n-2)\,r^b_d = 0$$

If $n \neq 2$ we get:

$$r^b_d = 0$$

Therefore M is Ricci flat manifold.

If $n = 2$, we shall discuss the cases of the values a, b, c, and d in the equation (6):

1. Put $a = 2$, $b = 2$, $c = 1$, $d = 1$ we get:

$$r_1^1 + r_2^2 = 0$$

2. Put $a = 2$, $b = 1$, $c = 1$, $d = 2$ we get:

$$-r_1^1 - r_2^2 = 0$$

Thus in all possible other cases of the values a, b, c, and d we obtained:

$$r_1^1 + r_2^2 = 0 \text{ or } -r_1^1 - r_2^2 = 0$$

This means $r_i^i = 0$.

It is well known, that the scalar curvature tensor is given by the form $K = r_i^i$. Therefore M is a manifold of flat scalar curvature tensor.

Theorem 3. *Suppose that M is AH-manifold with flat Bochner tensor, if M is flat manifold, then M is an Einstein manifold with cosmological constant $\frac{K}{2n}$.*

Proof. By the proposition 1 we have:

$$B_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} + \frac{1}{n+2}(r_b^a \delta_c^d - r_c^d \delta_b^a)$$

Suppose that M is flat manifold with flat Bochner tensor. This means that the Riemannian and Bochner tensors are vanishing. Thus we obtained:

$$r_b^a \delta_c^d - r_c^d \delta_b^a = 0 \qquad (8)$$

Contracting (8) by the indexes c and d , we get:

$$n r_b^a = r_c^c \delta_b^a \qquad (9)$$

We have $K = r_i^i = r_a^a + r_{\hat{a}}^{\hat{a}} = 2r_a^a$. Thus

$$r_c^c = \frac{K}{2}$$

So the equation (9) becomes:

$$r_b^a = \frac{K}{2n}\delta_b^a$$

$$\frac{K}{2n}\delta_{\hat{b}}^{\hat{a}} = \frac{K}{2n}\delta_b^a = \frac{\overline{K}}{2n}\overline{\delta_b^a} = \overline{r_b^a} = r_{\hat{b}}^{\hat{a}} = \frac{K}{2n}\delta_{\hat{b}}^{\hat{a}}$$

Hence

$$r_j^i = \frac{K}{2n}\delta_j^i$$

Therefore, from [13] M is Einstein manifold with cosmological constant $\frac{K}{2n}$.

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