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Some Sandwich Theorems for Certain Analytic Functions Defined by Convolution

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results for some analytic functions defined by convolution.

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1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z : z \in C\ , \ |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \ \ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let H(U) denote the class of analytic functions in U and let $H[a,1]$ denote the subclass of the functions $f \in H$(U) of the form:

$$f(z) = a + a_1 z + a_2 z^2 + \dots \quad (a \in C\ ).$$

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Supposing that h and g are two analytic functions in U, let

$$\varphi(r,s,t;z) : C^3 \times \mathrm{U} \to C.$$

If h and $\varphi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$g(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z), \tag{2}$$

then g is a solution of the differential superordination (2). A function $g \in H(\mathrm{U})$ is called a subordinant of (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant.

Recently, Miller and Mocanu [15] obtained sufficient conditions on the functions g, q and φ for which the following implication holds:

$$g(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z) \Rightarrow q(z) \prec h(z).$$

Using the results of Miller and Mocanu [15], Bulboaca [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. [23] obtained sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \text{ and } q_1(z) \prec \frac{z^2f'(z)}{[f(z)]^2} \prec q_2(z)\ ,$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

For functions f given by (1) and $g \in S$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

We observe that for different choices of the function g, the function $(f * g)(z)$ reduces to several interesting operators. For example, if

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \ (c \neq 0, -1, -2, ...; z \in U), \tag{4}$$

where

$$(d)_k = \begin{cases} 1 & (k = 0; d \in C^* = C\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in N; d \in C), \end{cases}$$

we see that, $(f * g)(z) = L(a,c)f(z)$ and $L(a,c)$ is the Carlson-Shaffer operator [6]. If

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}...(\alpha_l)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(1)_{k-1}} z^k, \tag{5}$$

where, $\alpha_i > 0$ $(i = 1,2,...l); \beta_j > 0$ $(j = 1,2,...s), l \leq s+1, l,s \in N_0 = N \cup \{0\}$, where $N = \{1,2,...\}$, we see that, $(f * g)(z) = H_{l,s}(\alpha_1)f(z)$, where $H_{l,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [9] (see also [10] and [11]). The operator $H_{l,s}(\alpha_1)$, contains in tern many interesting operators such as, Hohlov linear operator (see [12]), the Carlson-Shaffer linear operator (see [6] and [21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [13]) and Owa-Srivastava fractional derivative operator (see [18]).

Also, if

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + l + \lambda(k-1)}{1+l} \right]^m z^k \ (\lambda \geqslant 0, l \geqslant 0, m \in N_0), \tag{6}$$

we see that $(f * g)(z) = I(m,\lambda,l)f(z)$, where $I(m,\lambda,l)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [7]. The operator $I(m,\lambda,l)$, contains as special cases, the multiplier transformation (see [8]), the generalized Salăgeăn operator introduced and studied by Al-Oboudi [2] which in tern contains as special case the Salăgeăn operator (see [22]).

In [16], Mostafa et al. obtained some interesting subordination results for the function $\left(\frac{(f * g)(z)}{z}\right)^{\alpha}$ $(\alpha \in C^*)$.

In this paper, we get some interesting subordination results for the function $\left(\frac{z}{(f * g)(z)}\right)^{\delta}$ $(\delta \in C^*)$.

2. Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

Definition 1 ([15]). *Let Q be the set of all functions f that are analytic and injective on $\overline{\mathrm{U}}\backslash E(f)$, where*

$$E(f) = \{\zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathrm{U} \setminus E(f)$.

Lemma 1 ([14]). *Let q be univalent in the unit disc* U*, and let θ and φ be analytic in a domain D containing $q(\mathrm{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathrm{U})$. Set $\psi(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + \psi(z)$ and suppose that* *(i)* ψ *is a starlike function in* U*,*

(ii) $Re\dfrac{zh'(z)}{\psi(z)} > 0,\ z \in \mathrm{U}.$

If p *is analytic in* U *with* $p(0) = q(0)$*,* $p(\mathrm{U}) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{7}$$

then $p(z) \prec q(z)$*, and* q *is the best dominant of (7).*

Lemma 2 ([23])**.** *Let* $\mu, \gamma \in C^*$*, and let* q *be a convex function in* U *with*

$$Re\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\mu}{\gamma}\right) > 0\ \ , z \in \mathrm{U}.$$

If p *is analytic in* U *and*

$$\mu p(z) + \gamma zp'(z) \prec \mu q(z) + \gamma zq'(z), \tag{8}$$

then $p(z) \prec q(z)$*, and* q *is the best dominant of (8).*

Lemma 3 ([5])**.** *Let* q *be convex univalent function in* U *and let* θ *and* φ *be analytic in a domain* D *containing* $q(\mathrm{U})$*. Suppose that:*

(i) $Re\frac{\theta'(q(z))}{\varphi(q(z))} > 0,\ z \in U,$

(ii) $h(z) = zq'(z)\varphi(q(z))$ *is starlike in* U*.*

If $p \in H[q(0), 1] \cap Q$ *with* $p(U) \subset D$*, the function* $\theta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* U *and*

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{9}$$

then $q(z) \prec p(z)$*, and* q *is the best subordinant of (9).*

Lemma 4 ([19])**.** *The function* $q(z) = (1 - z)^{-2ab}$ *is univalent in* U *if and only if* $|2ab - 1| \leq 1$ *or* $|2ab + 1| \leq 1$*.*

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that, $\delta, \eta \in C^*, z \in U$ and the power is the principal one.

Theorem 1. *Let* q *be univalent in* U *and satisfies*

$$Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\delta}{\eta}\} > 0. \tag{10}$$

If $f, g \in S$ *with* $(f * g)(z) \neq 0, z \in U^* = U\backslash\{0\}$ *satisfy the subordination:*

$$\chi_g(\eta, \delta, f) \prec q(z) + \frac{\eta}{\delta} zq'(z), \tag{11}$$

where $\chi_g(\eta,\delta,f)$ is given by

$$\chi_g(\eta,\delta,f) = (1+\eta)\left(\frac{z}{(f*g)(z)}\right)^{\delta} - \eta\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\delta}, \tag{12}$$

then

$$(\frac{z}{(f*g)(z)})^{\delta} \prec q(z) \tag{13}$$

and q is the best dominant.

Proof. Define a function p by

$$p(z) = (\frac{z}{(f*g)(z)})^{\delta}. \tag{14}$$

Then the function p is analytic in U and $p(0)=1$. Therefore, by differentiating (14) logarithmically with respect to z, we have

$$p(z) + \frac{\eta}{\delta}zp'(z) = (1+\eta)\left(\frac{z}{(f*g)(z)}\right)^{\delta} - \eta\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\delta}. \tag{15}$$

Using (11) and (15), we have

$$p(z) + \frac{\eta}{\delta}zp'(z) \prec q(z) + \frac{\eta}{\delta}zq'(z). \tag{16}$$

Hence, the assertion (13) now follows by using Lemma 2 with $\gamma = \frac{\eta}{\delta}$ and $\mu = 1$.

Putting $q(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 1, the condition (10) becomes

$$\mathrm{Re}\left\{\frac{1-Bz}{1+Bz} + \frac{\delta}{\eta}\right\} > 0, z \in U. \tag{17}$$

It is easy to check that the function $\phi(z) = \frac{1-\zeta}{1+\zeta}, |\zeta| < |B| \leq 1$, is convex in U, and since $\phi(\overline{\zeta}) = \overline{\phi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\phi(U)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\mathrm{Re}\frac{1-Bz}{1+Bz}\right\} = \frac{1-|B|}{1+|B|} \geqslant 0.$$

Then, the inequality (17) is equivalent to

$$\mathrm{Re}\frac{\eta}{\delta} \geqslant \frac{|B|-1}{1+|B|}, \tag{18}$$

hence, we have the following corollary.

Corollary 1. *Let $-1 \leq B < A \leq 1$ and (18) holds. If $f(z) \in S$ with $(f * g)(z) \neq 0, z \in U^*$ and*

$$\chi_g(\eta, \delta, f) \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

where $\chi_g(\eta, \delta, f)$ is given by (12), then

$$\left(\frac{z}{(f * g)(z)}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $g(z) = z(1-z)^{-1}$ and $g(z) = z(1-z)^{-2}$, respectively, in Theorem 1, we have the result obtained by Shanmugam et al. [24, Corollaries 3.2 and 3.3, respectively].

Taking $g(z)$ of the form (5), and using the identity (see [9])

$$z\left(H_{l,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{l,s}(\alpha_1+1)f(z) - (\alpha_1 - 1)H_{l,s}(\alpha_1)f(z), \tag{19}$$

then we have the following corollary.

Corollary 2. *Let q be univalent in U and satisfies (10). If $f \in S$ with $H_{l,s}(\alpha_1)f(z) \neq 0, z \in U^*$, and satisfies the subordination*

$$\chi_1(\alpha_1, \eta, \delta, f) \prec q(z) + \frac{\eta}{\delta} z q'(z),$$

where $\chi_1(\alpha_1, \eta, \delta, f)$ is given by

$$\chi_1(\alpha_1, \eta, \delta, f) = (1+\alpha_1\eta)\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} - \alpha_1\eta\frac{H_{l,s}(\alpha_1+1)f(z)}{H_{l,s}(\alpha_1)f(z)}\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta}, \tag{20}$$

then

$$\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} \prec q(z)$$

and q is the best dominant.

Letting g be of the form (6), and using the identity (see [7])

$$\lambda z\left(I^m(\lambda, l)f(z)\right)' = (l+1)I^{m+1}(\lambda, l)f(z) - (1+l-\lambda)I^m(\lambda, l)f(z)\ (\lambda > 0; l \geqslant 0; m \in N_0), \tag{21}$$

then we have the following corollary.

Corollary 3. *Let q be univalent in U and satisfies (10), $\lambda > 0, l \geqslant 0$ and $m \in N_0$. If $f \in S$ with $I^m(\lambda, l)f(z) \neq 0, z \in U^*$, and satisfies the subordination*

$$\chi_2(l, m, \lambda, \eta, \delta, f) \prec q(z) + \frac{\eta}{\delta} z q'(z),$$

where $\chi_2(l,m,\lambda,\eta,\delta,f)$ is given by

$$\chi_2(l,m,\lambda,\eta,\delta,f)=(1+\frac{\eta(l+1)}{\lambda})\left(\frac{z}{I^m(\lambda,l)f(z)}\right)^\delta-\frac{\eta(l+1)}{\lambda}\frac{I^{m+1}(\lambda,l)f(z)}{I^m(\lambda,l)f(z)}\left(\frac{z}{I^m(\lambda,l)f(z)}\right)^\delta, \quad (22)$$

then

$$\left(\frac{z}{I^m(\lambda,l)f(z)}\right)^\delta \prec q(z)$$

and q is the best dominant.

Theorem 2. *Let $\gamma \in C^*$ and let q be univalent in U with $q(0)=1, q(z)\neq 0, z\in U$ and satisfies the condition:*

$$Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\ z\in U. \quad (23)$$

If $f,g\in S$ with $(f*g)(z)\neq 0, z\in U^*$ and satisfies the subordination:

$$1+\gamma\delta\left(1-\frac{z(f*g)'(z)}{(f*g)(z)}\right)\prec 1+\gamma\frac{zq'(z)}{q(z)}. \quad (24)$$

then,

$$\left(\frac{z}{(f*g)(z)}\right)^\delta \prec q(z),$$

and q is the best dominant of (24).

Proof. Let a function p defined by (14), then the function p is analytic in U and $p(0)=1$. Therefore, by differentiating (14) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)}=\delta\left(1-\frac{z(f*g)'(z)}{(f*g)(z)}\right).$$

Using the above relation in (24), we have

$$1+\gamma\frac{zp'(z)}{p(z)}\prec 1+\gamma\frac{zq'(z)}{q(z)}.$$

Taking $\theta(w)=1$ and $\varphi(w)=\gamma/w$, then φ and θ are analytic in C^*. Simple computations show that

$$\begin{aligned}\psi(z) &= zq'(z)\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)},\\ h(z) &= \theta(q(z))+\psi(z)=1+\gamma\frac{zq'(z)}{q(z)},\end{aligned}$$

and it is easily to see that the conditions of Lemma 1 are satisfied whenever (23) holds. Then, applying Lemma 1, the proof of Theorem 2 is completed.

Putting $q(z)=(1+Az)/(1+Bz)$ $(-1\le B<A\le 1)$ in Theorem 2, it is easy to check that the condition (23) holds whenever $-1\le B<A\le 1$, hence we obtain:

Corollary 4. *Let $-1 \le B < A \le 1$ Let $f, g \in S$ with $(f * g)(z) \ne 0$, $z \in U^*$, suppose that*

$$1+\gamma\delta\left(1-\frac{z(f*g)'(z)}{(f*g)(z)}\right) \prec 1+\frac{\gamma(A-B)z}{(1+Az)(1+Bz)}.$$

Then,

$$\left(\frac{z}{(f*g)(z)}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

and $(1+Az)/(1+Bz)$ is the best dominant.

Taking $\gamma = \frac{-1}{ab}$ $(a, b \in C^*), \delta = a$ and $q(z) = (1-z)^{-2ab}$ in Theorem 2, then combining this together with Lemma 4, we obtain the following corollary.

Corollary 5. *Let $a, b \in C^*$ such that $|2ab-1| \le 1$ or $|2ab+1| \le 1$. Let$f \in S$ and suppose that $\frac{(f*g)(z)}{z} \ne 0$ for all $z \in U^*$. If*

$$1+\frac{1}{b}\left(\frac{z(f*g)'(z)}{(f*g)(z)}-1\right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{z}{(f*g)(z)}\right)^{a} \prec (1-z)^{-2ab},$$

and $(1-z)^{-2ab}$ is the best dominant.

Remark 1. *(i) Taking $g(z) = \frac{z}{1-z}$ in Corollary 5, we obtain the result due to Obradović et al. [17, Theorem 1];*

(ii) Taking $g(z) = \frac{z}{1-z}$ and $a = 1$ in Corollary 5, we obtain the recent result of Srivastava and Lashin [25, Theorem 3];

(iii) Taking $g(z) = \frac{z}{1-z}, \gamma = \frac{e^{i\lambda}}{ab\cos\lambda}$ $(a, b \in C^*; |\lambda| < \frac{\pi}{2})$, $\alpha = a$ and $q(z) = (1-z)^{-2ab\cos\lambda e^{-i\lambda}}$ in Corollary 5, we obtain the result due to Aouf et al. [3, Theorem 1].

Theorem 3. *Let q be convex univalent in U, $\delta, \eta \in C^*$ and satisfies*

$$Re\{\frac{\delta}{\eta}\} > 0. \tag{25}$$

Let $f, g \in S$, $(f*g)(z) \ne 0$, $z \in U^*$, suppose that $\left(\frac{z}{(f*g)(z)}\right)^{\delta} \cap H[q(0), 1] \in Q$ and that $\chi_g(\alpha, \eta; f)$ is univalent in U, *where $\chi_g(\delta, \eta; f)$ is given by (12). Then*

$$q(z)+\frac{\eta}{\delta}zq'(z) \prec \chi_g(\delta, \eta; f)(z), \tag{26}$$

implies

$$q(z) \prec \left(\frac{z}{(f*g)(z)}\right)^{\delta},$$

and q is the best subordinant of (26).

Proof. Define a function p defined by (14). Then simple computations show that

$$p(z)+\frac{\eta}{\delta}zp'(z)=\chi_g(\delta,\eta,f).$$

Putting $\theta(w)=w$ and $\varphi(w)=\eta/\delta$, then θ and φ are analytic in C, and

$$\mathrm{Re}\frac{\theta'(q(z))}{\varphi(q(z))}=\mathrm{Re}\frac{\delta}{\eta}q'(z)>0\ (z\in U).$$

Since q is a convex function, it follows that $h(z)=zq'(z)\varphi(q(z))=\frac{\eta zq'(z)}{\delta}$ is starlike in U. Then by applying Lemma 3, the proof is completed.

Letting g be of the form (5) in Theorem 3 and using the identity (19), we get the following result obtained the following result:

Corollary 6. *Let q be convex in* U*, and suppose that $\delta,\eta\in C^*$ satisfies the condition (25). For all functions $f\in S$ with $H_{l,s}(\alpha_1)f(z)\neq 0$, $z\in U^*$, suppose that* $\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\alpha}\in H[q(0),1]\cap Q$, *and that $\chi_1(\alpha_1;\delta,\eta;f)$ is univalent in* U*, where $\chi_1(\alpha_1;\delta,\eta;f)$ is given by (20).*

Then,

$$q(z)+\frac{\eta}{\delta}zq'(z)\prec\chi_1(\alpha_1;\delta,\eta;f)(z), \tag{27}$$

implies

$$q(z)\prec\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta},$$

and q is the best subordinant of (27).

Letting g be of the form (6) in Theorem 3 and using the identity (21), we have:

Corollary 7. *Let q be convex in* U*, and suppose that $\alpha,\eta\in C^*$ satisfies the condition (25). For all functions $f\in\mathscr{S}$ with $I(m,\lambda,l)f(z)\neq 0$, $z\in U^*$ $\left(\lambda>0,\ l\geq 0,\ m\in N_0\right)$, suppose that* $\left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\delta}\in H[q(0),1]\cap Q$, *and that $\chi_2(m,\lambda,l;\delta,\eta;f)$ is univalent in* U*, where $\chi_2(m,\lambda,l;\delta,\eta;f)$ is given by (22).*

Then,

$$q(z)+\frac{\eta}{\alpha}zq'(z)\prec\ \chi_2(m,\lambda,l;\alpha,\eta;f)(z), \tag{28}$$

implies

$$q(z)\prec\left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\alpha},$$

and q is the best subordinant of (28).

Combining Theorem 1 and Theorem 3, we deduce the following sandwich theorem:

Theorem 4. *Let q_1 and q_2 be convex functions in* U. *Suppose that $\delta,\eta \in C^*$ satisfies (25) and q_2 satisfies (10). Let $f, g \in \mathscr{S}$, with $(f*g)(z) \neq 0$, $z \in U^*$, suppose that $\left(\frac{z}{(f*g)(z)}\right)^{\delta} \in H[q(0),1] \cap Q$, and that $\chi_g(\delta,\eta;f)$ is univalent in* U, *where $\chi_g(\delta,\eta;f)$ is given by (12). Then,*

$$q_1(z) + \frac{\eta}{\delta} z q_1'(z) \prec \chi_g(\delta,\eta;f)(z) \prec q_2(z) + \frac{\eta}{\delta} z q_2'(z), \tag{29}$$

implies

$$q_1(z) \prec \left(\frac{z}{(f*g)(z)}\right)^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively, the best subordinant and the best dominant.

Combining Corollary 2 and Corollary 6, we get the sandwich result:

Corollary 8. *Let q_1 and q_2 be convex functions in* U. *Suppose that $\delta,\eta \in C^*$ satisfies (25) and q_2 satisfies (10). Let $f \in \mathscr{S}$, with $H_{l,s}(\alpha_1)f(z) \neq 0$, $z \in U^*$, suppose that $\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} \in H[q(0),1] \cap Q$, and that $\chi_1(\alpha_1;\delta,\eta;f)$ is univalent in* U, *where $\chi_1(\alpha_1;\delta,\eta;f)$ is given by (20). Then,*

$$q_1(z) + \frac{\eta}{\delta} z q_1'(z) \prec \chi_1(\alpha_1;\delta,\eta;f) \prec q_2(z) + \frac{\eta}{\delta} z q_2'(z),$$

implies

$$q_1(z) \prec \left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively, the best subordinant and the best dominant.

Combining Corollary 3 and Corollary 7, we get the sandwich result:

Corollary 9. *Let q_1 and q_2 be convex functions in* U. *Suppose that $\delta,\eta \in C^*$ satisfies (25) and q_2 satisfies (10). Let $f \in \mathscr{S}$, with $I(m,\lambda,l)f(z) \neq 0$, $z \in U^*$, suppose that $\left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\delta} \in H[q(0),1] \cap Q$, and that $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is univalent in* U, *where $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is given by (22). Then,*

$$q_1(z) + \frac{\eta}{\delta} z q_1'(z) \prec \chi_2(m,\lambda,l;\alpha,\eta;f)(z) \prec q_2(z) + \frac{\eta}{\delta} z q_2'(z),$$

implies

$$q_1(z) \prec \left(\frac{z}{I(m,\lambda,l)f(z)}\right)^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively, the best subordinant and the best dominant.

Remark 2. *Taking g in the form (4) in Theorems 1, 3 and 4, respectively, we obtain the results obtained by Shanmugam et al. [24, Theorems, 3.1, 4.1 and 5.1, respectively].*

Specializing the parameters $\alpha_j (j = 1,2,...,s+1)$, $\beta_j (j = 1,2,...,s)$, λ, l and m, in Corollaries 8 and 9, we obtain the sandwich results for the corresponding operators.

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