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Ore Extensions over Weak σ-rigid Rings and $\sigma(*)$-rings

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Abstract. Let R be a ring and σ an endomorphism of a ring R. Recall that R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R. We also recall that R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R.
In this paper we give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring. We also give a necessary and sufficient condition for a Noetherian ring to be a weak σ-rigid ring. Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ and δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$. With this we show that if R is a 2-primal commutative Noetherian ring which is also an algebra over $\mathbb{Q}$ (where $\mathbb{Q}$ is the field of rational numbers), σ is an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$, then R is a weak σ-rigid ring implies that $R[x;\sigma,\delta]$ is a weak $\overline{\sigma}$-rigid ring.

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1. Introduction

Throughout this paper R will denote an associative ring with identity $1 \neq 0$. Let now σ be an endomorphism of a ring R. The field of complex numbers is denoted by $\mathbb{C}$, the field of rational numbers is denoted by $\mathbb{Q}$, the ring of integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The set of prime ideals of R is denoted by $Spec(R)$. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. We note that for a commutative ring $P(R)$ and $N(R)$.

Now let R be a ring and σ an endomorphism of R and δ is a σ-derivation of R. Recall that the skew polynomial ring $R[x;\sigma,\delta]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i, a_i \in R, n \in \mathbb{N}\}$$

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with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$, $a_i \in R$ as in McConnell and Robson [14]. We denote $R[x;\sigma,\delta]$ by $O(R)$. In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$ and in case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. For more details on Ore extensions (the skew polynomial rings), we refer the reader to Chapter (1) of McConnell and Robson [14].

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. For example [1, 3, 6, 8, 11, 12, 15].

The classical study of any commutative Noetherian ring is done by studying its primary decomposition and this forms the fundamental edifice on which any such ring is studied. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of a Noetherian ring. The notion of the quotient ring of a ring appears in chapter (9) of [8].

In [6] it is shown that if R is embeddable in a right Artinian ring and if characteristic of R is zero, then the differential operator ring $R[x;\delta]$ embeds in a right Artinian ring. It is also shown in [6] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x;\sigma]$ embeds in an Artinian ring.

A non commutative analogue of associated prime ideals of a Noetherian ring has also been also discussed. We would like to note that a considerable work has been done in the investigation of prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings (K. R. Goodearl and E. S. Letzter [9], C. Faith [7], S. Annin [1], Leroy and Matczuk [12], Nordstrom [15]) and Bhat [3].

Another related area of interest since recent past has been the study of 2-primal rings. This involves the notions of prime radical and the set of nilpotent elements of a ring. Further more the concept of completely prime ideals and the completely semiprime ideals are also studied in this area. Recall that a ring R is 2-primal if $N(R) = P(R)$; i.e. if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that a reduced is 2-primal and a commutative ring is also 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [13], Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Marks [13], it has been shown that for a local ring R with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Krempa in [10] introduced σ-rigid rings; Kwak in [11] introduced $\sigma(*)$-rings and established a relation between a 2-primal ring and a $\sigma(*)$-ring. Ouyang in [16] introduced weak σ-rigid rings, where σ is an endomorphism of ring R. These rings are related to 2-primal rings. In this paper we study these rings and find a relation between these rings. Towards this we prove the following:

Theorem 1. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring. (This is proved in Theorem (5)).*

We also discuss skew polynomial rings over weak σ-rigid rings. Towards this we have the following:

Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.
We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}.$$

With this we prove the following:

Theorem 2. *Let R be a commutative Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then R is a weak σ-rigid ring if and only if $O(R) = R[x;\sigma,\delta]$ is a weak $\overline{\sigma}$-rigid ring. (This is proved in Theorem (7)).*

2. Preliminaries

2.1. General

We begin with the following definitions:

Definition 1 (Krempa[10])**.** *Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$. The ring R is said to be a σ-rigid ring if there exists a σ-rigid endomorphism R.*

For example let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a + ib) = a - ib$, a, $b \in \mathbb{R}$. Then it can be seen that σ is a rigid endomorphism of R.

In Theorem 3.3 of [10], Krempa has proved the following:
Let R be a ring, let σ be an endomorphism and δ a σ-derivation of R. If σ is a monomorphism, then the skew polynomial ring $R[x;\sigma,\delta]$ is reduced if and only if R is reduced and σ is rigid. Under this conditions any minimal prime ideal (annihilator) of $R[x;\sigma;\delta]$ is of the form $P[x;\sigma;\delta]$ where P is a minimal prime ideal (annihilator) in R.

Definition 2 (Kwak [11]). *Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.*

Example 1 (Example 2 of [11]). *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.*

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The ring in Example (1) is also 2-primal. We also note that a ring R is a $I(*)$-ring if and only if R is a 2-primal ring, where I is the identity map on R.

In [11], the 2-primal property has also been extended to the skew-polynomial ring $R[x;\sigma]$. Kwak in [11] also establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. It has been proved in Theorem 5 of [11] that if R is a 2-primal ring and σ is an automorphism of R, then R is a $\sigma(*)$-ring if and only if $\sigma(P) = P$ for all $P \in Min.Spec(R)$. In Theorem 12 of [11] it has been proved that if R is a $\sigma(*)$-ring with $\sigma(P(R)) = P(R)$, then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.

We now give an example of a ring R, and an endomorphism σ of R such that R is not a $\sigma(*)$-ring, however R is 2-primal.

Example 2 (Example 4 of [11]). *Let $R = F[x]$ be the polynomial ring over a field F. Then R is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then R is not a $\sigma(*)$-ring. For example consider $f(x) = xa$, $a \neq 0$.*

Let R be a ring and σ an automorphism of R. We now give a necessary and sufficient condition for R to be a $\sigma(*)$-ring in the following Theorem:

Proposition 1. *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies that $P(R)$ is completely semiprime.*

Proof. Proposition (2.2) of Bhat and Neetu [4].

Proposition 2. *Let R be a Noetharian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies that R is 2-primal.*

Proof. By Proposition (1) $P(R)$ is completely semiprime. Therefore, R is 2-primal.

Recall that a completely prime ideal in a ring R is any (prime) ideal such that R/P is a domain (Chapter 9 of Goodearl and Warfield [8]). By definition we note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

Example 3. *Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. *If* p *is a prime number, then the ideal* $P = M_2(p\mathbb{Z})$ *is a prime ideal of R, but is not completely prime, since for* $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, *we have* $ab \in P$, *even though* $a \notin P$ *and* $b \notin P$.

A necessary and sufficient condition for a Noetherian ring R to be a $\sigma(*)$-ring (where σ is an automorphism of R) has been given in Theorem (2.4) of [4]:

Theorem 3. *Let R be a Noetherian ring. Let* σ *be an automorphism of R. Then R is a* $\sigma(*)$*-ring if and only if for each minimal prime* U *of R,* $\sigma(U) = U$ *and* U *is completely prime ideal of R.*

Proof. See Theorem (2.4) of [4].

Proposition 3. *Let R be a Noetherian ring which is also an algebra over* $\mathbb{Q}$. *Let* σ *be an automorphism of R such that R is a* $\sigma(*)$*-ring and* δ *a* σ*-derivation of R. Then* $\delta(U) \subseteq U$ *for all* $U \in Min.Spec(R)$.

Proof. We note that Proposition (1) implies that $P(R)$ is completely semiprime. Let $U \in Min.Spec(R)$. Then Theorem (3) implies that $\sigma(U) = U$.

Let now $T = \{a \in U \mid$ such that $\delta^k(a) \in U$ for all integers $k \geq 1\}$. First of all, we will show that T is an ideal of R. Let $a,\ b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1\}$. Now $\delta^k(a-b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in T$. Therefore T is a δ-invariant ideal of R.

We will now show that $T \in Spec(R)$. Suppose $T \notin Spec(R)$. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$.

Theorem 4. *Let R be a ring. Let* σ *be an automorphism of R and* δ *be a* σ*-derivation of R. Then:*

1. *For any completely prime ideal P of R with* $\delta(P) \subseteq P$ *and* $\sigma(P) = P$, $O(P) = P[x;\sigma,\delta]$ *is a completely prime ideal of* $O(R)$.

2. *For any completely prime ideal* U *of* $O(R)$, $U \cap R$ *is a completely prime ideal of R.*

Proof. See Proposition (2.2) of Bhat [2].

Proposition 4. *Let R be a Noetherian ring which is also an algebra over* $\mathbb{Q}$. *Let* σ *be an automorphism of R such that R is a* $\sigma(*)$*-ring. Then* $U \in Min.Spec(R)$ *implies that* $UO(R) = U[x;\sigma,\delta]$ *is a completely prime ideal of* $O(R) = R[x;\sigma,\delta]$.

Proof. Proposition (1) implies that $P(R)$ is completely semiprime ideal of R. Let $U \in Min.Spec(R)$. Then Theorem (3) implies that $\sigma(U) = U$ and U is completely prime. Also by Proposition (3) $\delta(U) \subseteq U$. Now Theorem (4) implies that $UO(R) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$.

3. Skew polynomial Rings over Weak σ-rigid Rings

Definition 3 (Ouyang [16]). *Let R be a ring. Then R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.*

Example 4 (Example (2.1) of Ouyang [16]). *Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let*

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. The it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Ouyang has proved in [16] that if σ is an endomorphism of a ring R, then R is σ-rigid if and only if R is weak σ-rigid and reduced.

We now give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring in the following Theorem:

Theorem 5. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring.*

Proof. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Now Proposition (2) implies that R is 2-primal, i.e. $N(R) = P(R)$. Thus $a\sigma(a) \in N(R) = P(R)$ implies that $a \in P(R) = N(R)$. Hence R is weak σ-rigid ring.

Conversely let R be 2-primal weak σ-rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Let R be a Noetherian ring and σ an automorphism of R. We now give a characterization for R to be a weak σ-rigid ring. (An analog of Proposition (1) for weak σ-rigid rings)

Theorem 6. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring implies that $N(R)$ is completely semiprime.*

Proof. First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R$ such that $\sigma(a) = n \in N(R)\}$ is an ideal of R. Now I is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then
$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R)$.
Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Corollary 1. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. Combine Theorem (3) and Theorem (6).

Proposition 5. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then $O(N(R)) = N(O(R))$.*

Proof. Proposition (2) implies that R is 2-primal. Now it is easy to see that $O(N(R)) \subseteq N(O(R))$. We will show that $N(O(R)) \subseteq O(N(R))$. Let $f = \sum_{i=0}^{m} x^i a_i \in N(O(R))$. Then $(f)(O(R)) \subseteq N(O(R))$, and $(f)(R) \subseteq N(O(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get

$$(x^m a_m R)^k = 0.$$

After simplification equating leading term to zero, we get

$$x^{km}\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)\dots a_m R = 0.$$

Therefore,

$$\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)...a_m R = 0 \subseteq P,$$

for all $P \in Min.Spec(R)$. This implies that $\sigma^{(k-j)m}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$. Therefore, $a_m R \subseteq \sigma^{-(k-j)m}(P)$. But $\sigma^{-(k-j)m}(P) = P$ by Theorem (3). So we have $a_m R \subseteq P$, for all $P \in Min.Spec(R)$. Therefore, $a_m \in P(R)$, and R being 2-primal implies that $a_m \in N(R)$. Now $x^m a_m \in O(N(R)) \subseteq N(O(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(O(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore, $f \in O(N(R))$. Hence $N(O(R)) \subseteq O(N(R))$ and the result follows.

As mentioned earlier, we note that if σ is an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$ derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$. We now prove the following:

Theorem 7. *Let R be a 2-primal commutative Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then R is a weak σ-rigid ring implies that $O(R) = R[x;\sigma,\delta]$ is a weak $\overline{\sigma}$-rigid ring.*

Proof. Let R be a weak σ-rigid ring. Then Theorem (5) implies that R is a $\sigma(*)$-ring. Also Proposition (5) implies that $O(N(R)) = N(O(R))$. We show that $R[x;\sigma,\delta]$ is a weak $\overline{\sigma}$-rigid ring.

Let $f \in O(R)$ (say $f = \sum_{i=0}^{m} x^i a_i$) be such that $f\overline{\sigma}(f) \in N(O(R))$. We use induction on m to prove the result. For $m = 1$, $f = xa_1 + a_0$. Now $f\overline{\sigma}(f) \in N(O(R))$ implies that $(xa_1 + a_0)(x\sigma(a_1) + \sigma(a_0)) \in N(O(R)) = O(N(R))$, i.e.

$$x^2\sigma^2(a_1) + x\delta(a_1)\sigma(a_1) + x\sigma(a_0)\sigma(a_1)$$

$$+\delta(a_0)\sigma(a_1)+xa_1\sigma(a_0)+a_0\sigma(a_0)\in O(N(R)) \qquad (1)$$

Therefore, $\sigma^2(a_1)\in N(R)$. Now $\sigma(N(R))=N(R)$ implies that $\sigma^i(a_1)\in N(R)$ for all $i\geq 1$. So (1) implies that $a_0\sigma(a_0)\in N(R)$, and R being a weak σ-rigid ring implies that $a_0\in N(R)$. Therefore, $f\in O(N(R))=N(O(R))$.

Suppose the result is true for $m=k$. We prove for $m=k+1$. Now $f\overline{\sigma}(f)\in N(O(R))$ implies that

$$(x^{k+1}a_{k+1}+\ldots+a_0)(x^{k+1}\sigma(a_{k+1})+\ldots+\sigma(a_0))\in N(O(R))=O(N(R)),$$

i.e.

$$x^{2k+2}\sigma^{k+2}(a_{k+1})+x^{2k+1}(\sigma^k(a_{k+1})\sigma(a_k)+\sigma^{k+1}(a_k)\sigma(a_{k+1}))+g\overline{\sigma}(g)\in O(N(R)),$$

where $g=\sum_{i=0}^{k}x^ia_i$. Therefore, $\sigma^{k+2}(a_{k+1})\in N(R)$ implies that $a_{k+1}\in N(R)$. Also $\sigma^k(a_{k+1})\sigma(a_k)+\sigma^{k+1}(a_k)\sigma(a_{k+1})\in N(R)$ implies that $g\overline{\sigma}(g)\in N(O(R))$, but degree of g is k, therefore, by induction hypothesis, the result is true for all m.

Question: Let R be a commutative Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a))=\delta(\sigma(a))$ for all $a\in R$. Let R be a weak σ-rigid ring. Is $O(R)=R[x;\sigma,\delta]$ a weak $\overline{\sigma}$-rigid ring?

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