EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 1, No. 3, 2008, (10-20)
ISSN 1307-5543 – www.ejpam.com

Score sequences in oriented k-hypergraphs

S. Pirzada[1,*], Zhou Guofei[2,†]

[1] *Department of Mathematics, University of Kashmir, Srinagar, 190006, INDIA*
[2] *Department of Mathematics, Nanjing University, Nanjing, 210093, P.R. CHINA*

Abstract. Given two non-negative integers n and k with $n \geq k > 1$, an oriented k-hypergraph on n vertices is a pair (V,A), where V is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains at most one of the $k!$ k-tuples whose entries belong to S.
In this paper, we define the score of a vertex in an oriented k-hypergraph and then obtain a necessary and sufficient condition for the sequence of non-negative integers $[s_1, s_2, \cdots, s_n]$ to be a score sequence of some oriented k-hypergraph.

AMS subject classifications: 05C20
Key words: Hypergraphs, Oriented k-hypergraphs, Score Sequences

---

1. Introduction

An edge of a graph is a pair of vertices and an edge of a hypergraph is a subset of the vertex set, consisting of at least two vertices. An edge in a hypergraph consisting of k vertices is called a k-edge, and a hypergraph all of whose edges are k-edges is called a k-hypergraph.

A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In other words, given two non-negative integers n and k with $n \geq k > 1$, a k-hypertournament on n vertices is a pair (V,A), where V is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. If $n < k$, $A = \phi$ and this type of hypertournament is called a null-hypertournament. Clearly, a 2-hypertournament is simply a tournament.

Instead of scores of vertices in a tournament, Zhou et al. [8] considered scores and losing scores of vertices in a k-hypertournament, and derived a result analogous to Landau's theorem [5]. The score $s(v_i)$ or s_i of a vertex v_i is the number of arcs containing v_i and in which v_i is not the last element, and the losing score $r(v_i)$ or r_i of a vertex v_i is the number of arcs containing v_i and in which v_i is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order.

*Corresponding author. *Email addresses:* sdpirzada@yahoo.co.in (S. Pirzada)
gfzhou@nju.edu.cn (Z. Guofei)
†The work of the second author was supported in part by the National Natural Science Foundation of China and The Dr. Delia Koo Grant in Michigan State University

The following characterizations of score sequences and losing score sequences in k-hypertournaments are due to Zhou et al. [8].

Theorem 1.1. Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $R = [r_1, r_2, \cdots, r_n]$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each j,

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k},$$

with equality when $j = n$.

Theorem 1.2. Given non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers is a score sequence of some k-hypertournament if and only if for each j,

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

with equality when $j = n$.

Bang and Sharp [2] proved Landau's theorem using Hall's theorem on a system of distinct representatives of a collection of sets. Based on Bang and Sharp's ideas, Koh and Ree [4] have given a different proof of Theorem 1.1 and 1.2. Some more results on scores of k-hypertournaments can be found in [3, 7].

An oriented graph is a graph with each edge endowed with an orientation. As given by Avery [1], the score $s(v_i)$ or s_i of a vertex v_i in an oriented graph with n vertices is $s(v_i) = n-1+d^+(v_i)-d^-(v_i)$, where $d^+(v_i)$ and $d^-(v_i)$ are respectively the outdegree and indegree of v_i. The score sequence of an oriented graph is formed by listing the scores in non-decreasing order.

The following result due to Avery [1] characterizes score sequences in oriented graphs, and a new proof of it is due to Pirzada et al. [6].

Theorem 1.3. A sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each j $(1 \leq j \leq n)$

$$\sum_{i=1}^{j} s_i \geq 2\binom{j}{2},$$

with equality when $j = n$.

An oriented k-hypergraph is a k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In other words, given two non-negative integers n and k with $n \geq k > 1$, an oriented k-hypergraph on n vertices is a pair (V, A), where V is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains at most one of the $k!$

k-tuples whose entries belong to S. Clearly, an oriented 2-hypergraph is simply an oriented graph.

Let $D = (V,A)$ denote an oriented k-hypergraph with n vertices and let $1 < k \leq n$. Clearly, there can or cannot be an arc among any k distinct vertices $v_1, v_2, \cdots, v_k$ of V. If there is an arc among $v_1, v_2, \cdots, v_k$, we denote it by $e = (v_1, v_2, \cdots, v_k)$ and if there is not an arc among $v_1, v_2, \cdots, v_k$, it is denoted by $\langle v_1, v_2, \cdots, v_k \rangle$, and we call it a non arc. We note that D contains at most $\binom{n}{k}$ arcs, that is $|A| \leq \binom{n}{k}$, and a vertex v_i in D can be in at most $\binom{n-1}{k-1}$ arcs. We denote by $d^+(v_i)$ $\left(d^-(v_i)\right)$, the number of arcs in which v_i is not the last element ((v_i is the last element), furthermore, we denote by $d_i^+(U)$ ($d_i^-(U)$) the number of arcs that are contained in U and in which v_i is not the last element ((v_i is the last element).

Now, let $V_1 = \{v_1, v_2, \cdots, v_j\} \subset V$ and $V_2 = V - V_1$. If q is the number of those arcs which contain at least one vertex from V_1 and at least one vertex from V_2, then

$$q \leq \sum_{i=1}^{k-1} \binom{j}{i} \binom{n-j}{k-i}.$$

The set of those arcs having at least one vertex in V_1 and at least one vertex in V_2 is denoted by $V_1 * V_2$.

Let $e = (v_1, v_2, \cdots, v_k)$ be an arc in D and $i < j \leq k$.

We denote by $e(v_i, v_j) = (v_1, \cdots, v_{i-1}, v_j, v_{i+1}, \cdots, v_{j-1}, v_i, v_{j+1}, \cdots, v_k)$, that is, the new arc obtained from e by interchanging v_i and v_j in e. Similarly, we denote by $f\left\langle v_i, v_j \right\rangle$ the new non arc obtained from the non arc $f = \left\langle v_1, v_2, \cdots, v_j \right\rangle$ by interchanging v_i and v_j in f.

Define the score $s(v_i)$ or s_i of a vertex v_i in oriented k-hypergraph D as

$$s(v_i) = (k-1)\binom{n-1}{k-1} + d^+(v_i) - (k-1)d^-(v_i).$$

Clearly, $0 \leq s_i \leq k\binom{n-1}{k-1}$. The score sequence $S = [s_1, s_2, \cdots, s_n]$ of D is formed by listing the scores in non-decreasing order.

Let $R = [s_1, s_2, \cdots, s_n]$ be an integer sequence. For $1 \leq i < j \leq n$, we define $S(s_i^+, s_j^-) = [s_1, s_2, \cdots, s_i + 1, \cdots, s_j - 1, \cdots, s_n]$, and $S^+(s_i^+, s_j^-) = (s_1', s_2', \cdots, s_n')$ denotes an arrangement of $S(s_i^+, s_j^-)$ such that $s_1' \leq s_2' \leq \cdots \leq s_n'$.

Let $S = [s_1, s_2, \cdots, s_n]$ be a non-decreasing sequence of non-negative integers with each s_i having the form $s_i = x_i k + y_i(k-1)$, where x_i and y_i are nonnegative integers and satisfy $0 \leq x_i, y_i \leq \binom{n-1}{k-1}$, S is called to be strict if for all $s_i < s_j$, we have $y_i > y_j$.

2. Main results

Our main result is the following theorem.

Theorem 2.1. Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing strict sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers with $s_i = x_i k + y_i(k-1)$, where x_i, y_i are nonnegative integers and satisfies $0 \leq x_i, y_i \leq \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$, is a score sequence of some oriented k-hypergraph if and only if

$$\sum_{i=1}^{j} s_i \geq j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(i-k)\binom{j}{i}\binom{n-j}{k-i} \tag{2.1}$$

with equality for $j = n$.

In order to prove this theorem, we need some lemmas.

Lemma 2.1. If D is an oriented k-hypergraph of order n, then $s(v_i) = xk + y(k-1)$, where x and y are non-negative integers.

Proof. Let $d^*(v_i)$ be the number of non arcs in which vertex v_i is contained. Then, $d^+(v_i) + d^-(v_i) + d^*(v_i) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$,

or $d^-(v_i) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - d^+(v_i) - d^*(v_i)$.

Therefore, $s(v_i) = (k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + d^+(v_i) - (k-1)d^-(v_i)$

gives

$$\begin{aligned} s(v_i) &= (k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + d^+(v_i) - (k-1)\left[\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - d^+(v_i) - d^*(v_i)\right] \\ &= kd^+(v_i) + (k-1)d^*(v_i) \end{aligned}$$

As $d^+(v_i)$ and $d^*(v_i)$ are non-negative integers, the proof follows.□

It follows from Lemma 2.1 that the score of a vertex v_i besides satisfying $0 \leq s_i \leq k\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$ should also satisfy $s_i = xk + y(k-1)$, where x and y are non-negative integers. A vertex v_i if belonging to an arc and not the last element contributes k to the score of v_i, and if not belonging to an arc contributes $k-1$ to the score of v_i.

For $k = 2$, D is simply an oriented graph and the score of a vertex in that case becomes $s(v_i) = \begin{pmatrix} n-1 \\ 2 \end{pmatrix} + d^+(v_i) - d^-(v_i)$,

which is same as defined by Avery.

Lemma 2.2. If $[s_1, s_2, \cdots, s_n]$ is a score sequence of an oriented k-hypergraph of order n, then $\sum_{i=1}^{n} s_i = n(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$.

Proof. In the following, d_i^+ and d_i^- denote $d(v_i)^+$ and $d(v_i)^-$ respectively. Let D be an oriented k-hypergraph with score sequence $[s_1, s_2, \cdots, s_n]$. We have,

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n}\left[(k-1)\binom{n-1}{k-1} + d_i^+ - (k-1)d_i^-\right]$$
$$= n(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{n} d_i^+ - (k-1)\sum_{i=1}^{n} d_i^-.$$

Let D contains p k-arcs. Then, $\sum_{i=1}^{n} d_i^+ = (k-1)p$ and $\sum_{i=1}^{n} d_i^- = p$.

Therefore,

$$\sum_{i=1}^{n} s_i = n(k-1)\binom{n-1}{k-1} + (k-1)p - (k-1)p$$
$$= n(k-1)\binom{n-1}{k-1}$$

□

Lemma 2.3. If $S = [s_1, s_2, \cdots, s_n]$ is a score sequence of an oriented k-hypergraph D with $s_i < s_j$ and $s_i = xk + y(k-1)$, $s_j = \alpha k + \beta(k-1)$, where x, y, α and β are non-negative integers. If $y > \beta$, then $S^+(s_i^+, s_j^-)$ is a score sequence of an oriented k-hypergraph D'.

Proof. For simplicity, $A(D)$ denotes the set of arcs in D; $A^*(D)$ denotes the set of non arcs in D.

Since $d(v)^* = y > \beta \geq 0$, we have $A^*(D) \neq \emptyset$.

Case 1. There exists a non arc $e^* = \langle u_1, u_2, \cdots, u_{k-1}, v_i \rangle \in A^*(D)$ which does not contain v_j and such that $e = (u'_1, u'_2, \cdots, u'_{k-1}, v_j) \in A(D)$, where $(u'_1, u'_2, \cdots, u'_{k-1})$ is a permutation of $(u_1, u_2, \cdots, u_{k-1})$.

If there exists an arc e_1 that contains both v_i and v_j and that v_i is the last entry. Then by exchanging v_i and v_j in e_1, adding the arc $e' = (u_1, u_2, \cdots, u_{k-1}, v_i)$ to D, and deleting e from D, we get an oriented k-hypergraph D' with $S^+(s_i^+, s_j^-)$ as its score sequence. So in the following, we assume that for each arc containing both v_i and v_j, v_i is not the last entry.

If there exists a pair of arcs $f = (w_1, w_2, \cdots, w_{k-1}, v_i)$, and $f' = (w'_1, w'_2, \cdots, v_j, \cdots, w'_{k-1})$, where $(w'_1, w'_2, \cdots, w'_{k-1})$ is a permutation of $(w_1, w_2, \cdots, w_{k-1})$. Then by exchanging v_i and v_j between f and f', adding the arc $e' = (u_1, u_2, \cdots, u_{k-1}, v_i)$ to D, and deleting e from D, we get an oriented k-hypergraph D' with $S^+(s_i^+, s_j^-)$ as its score sequence. So in the following, we assume that no such pair of arcs exist. Furthermore, for each $f' = (w'_1, w'_2, \cdots, v_j, \cdots, w'_{k-1})$, $f = (w_1, w_2, \cdots, v_i, \cdots, w_{k-1})$ must be an arc, where $(w'_1, w'_2, \cdots, w'_{k-1})$ is a permutation of $(w_1, w_2, \cdots, w_{k-1})$, otherwise, by adding $(w_1, w_2, \cdots, v_i, \cdots, w_{k-1})$ to D and deleting f' from D, we get an oriented k-hypergraph D' with $S^+(s_i^+, s_j^-)$ as its score sequence. And therefore, we have $d^+(v_j) \leq d^+(v_i)$. Meanwhile, since $y > \beta$, $s_i < s_j$ and by the proof of Lemma 2.1, $s_i = kd^+(v_i) + (k-1)d^*(v_i) < s_j = kd^+(v_j) + (k-1)d^*(v_j)$, which implies that $k(d^+(v_j) - d^+(v_i)) > (k-1)(d^*(v_i) - d^*(v_j)) > 0$, thus we have $(d^+(v_j) - d^+(v_i) > 0$, which contradicts the fact that $d^+(v_j) \leq d^+(v_i)$.

Case 2. For each non arc $e^* = \langle u_1, u_2, \cdots, u_{k-1}, v_i \rangle$, either $f^* = \langle u_1, u_2, \cdots, u_{k-1}, v_j \rangle$ is a non arc, or $\{u_1, u_2, \cdots, u_{k-1}, v_j\}$ forms an arc, but v_j is not the last entry. Note that the later case will deduce that result is valid, so we assume that for each non arc $e^* = \langle u_1, u_2, \cdots, u_{k-1}, v_i \rangle$, $f^* = \langle u_1, u_2, \cdots, u_{k-1}, v_j \rangle$ is also a non arc. This implies that $d^*(v_i) \leq d^*(v_j)$, which contradicts that $y > \beta$.□

We note when $y \leq \beta$, Lemma 2.3 need not be true. To see this consider a 3-hypergraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2, 3), (3, 4, 1)\}$ it is easy to check that [5,5,7,7] is the score sequence of D. But the sequence [5,6,6,7], which is just $S^+(s_i^+, s_j^-)$, where $s_i = 5$ and $s_j = 7$, is not a score sequence of any 3-hypergraph.

Lemma 2.4. If $S = [s_1, s_2, \cdots, s_n]$ with $s_1 \leq s_2 \leq \cdots \leq s_n$ is a non-negative integer sequence satisfying (1), and if $s_n < k \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$, then there exists p $(1 \leq p \leq n-1)$ such that $S(s_n^+, s_p^-)$ is non-decreasing and satisfies (2.1).

Proof. Let p be the maximum integer such that

$$s_{p-1} < s_p = s_{p+1} = \cdots = s_{n-1} \text{ with } s_0 = 0 \text{ if } p = 1.$$

To see $S(s_n^+, s_p^-)$ satisfies (2.1), we only need to show for each j $(p \leq j \leq n-1)$,

$$\sum_{i=1}^{j} s_i > j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(i-k)\binom{j}{i}\binom{n-j}{k-i}. \tag{2.2}$$

Since $s_n < k \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$, therefore

$$\begin{aligned}
\sum_{i=1}^{n-1} s_i &= \sum_{i=1}^{n} s_i - s_n \\
&= n(k-1)\binom{n-1}{k-1} - s_n \\
&> n(k-1)\binom{n-1}{k-1} - k\binom{n-1}{k-1} \\
&= (n-1)(k-1)\binom{n-1}{k-1} - \binom{n-1}{k-1}.
\end{aligned}$$

As $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \leq \sum_{i=1}^{k-1}(k-i) \begin{pmatrix} n-1 \\ i \end{pmatrix} \begin{pmatrix} n-(n-1) \\ k-i \end{pmatrix}$,
so

$$\begin{aligned}
-\binom{n-1}{k-1} &\geq -\sum_{i=1}^{k-1}(k-i)\binom{n-1}{i}\binom{1}{k-i} \\
&= \sum_{i=1}^{k-1}(i-k)\binom{n-1}{i}\binom{1}{k-i}.
\end{aligned}$$

Therefore, $\sum_{i=1}^{n-1} s_i > (n-1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} n-1 \\ i \end{pmatrix}\begin{pmatrix} 1 \\ k-i \end{pmatrix}$.

Thus for $p = 1$, (2.2) holds. Now, we assume $p \le n-2$. Clearly, (2.2) holds for $j = n-1$.

If there exists j_0 ($p \le j_0 \le n-2$) such that $\sum_{i=1}^{j_0} s_i = j_0(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(i-k)\binom{j_0}{i}\binom{n-j_0}{k-i}$, choose j_0 as large as possible.

Since

$\sum_{i=1}^{j_0+1} s_i > (j_0+1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0+1 \\ i \end{pmatrix}\begin{pmatrix} n-(j_0+1) \\ k-i \end{pmatrix}$,

therefore

$$\begin{aligned} s_{j_0} &= s_{j_0+1} \\ &= \sum_{i=1}^{j_0+1} s_i - \sum_{i=1}^{j_0} s_i \\ &> (j_0+1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0+1 \\ i \end{pmatrix}\begin{pmatrix} n-j_0-1 \\ k-i \end{pmatrix} - j_0(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \\ &= (k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0+1 \\ i \end{pmatrix}\begin{pmatrix} n-j_0-1 \\ k-i \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{j_0-1} s_i = \sum_{i=1}^{j_0} s_i - s_{j_0} & \\ &< j_0(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - \left[(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0+1 \\ i \end{pmatrix}\begin{pmatrix} n-j_0-1 \\ k-i \end{pmatrix}\right] \\ &= (j_0-1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0+1 \\ i \end{pmatrix}\begin{pmatrix} n-j_0-1 \\ k-i \end{pmatrix} \end{aligned}$$

Now, $\begin{pmatrix} j_0+1 \\ i \end{pmatrix} = \frac{j_0(j_0+1)}{(j_0-i+1)(j_0-i)}\begin{pmatrix} j_0-1 \\ i \end{pmatrix}$

and $\begin{pmatrix} n-j_0-1 \\ k-i \end{pmatrix} = \frac{(n-j_0-k+i+1)(n-j_0-k+i)}{(n-j_0+1)(n-j_0)}\begin{pmatrix} n-(j_0-1) \\ k-i \end{pmatrix}$.

So, $\sum_{i=1}^{j_0-1} s_i < (j_0-1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$

$$-\sum_{i=1}^{k-1}\frac{(i-k)j_0(j_0+1)(n-j_0-k+i+1)(n-j_0-k+i)}{(j_0-i+1)(j_0-i)(n-j_0+1)(n-j_0)}\begin{pmatrix} j_0-1 \\ i \end{pmatrix}\begin{pmatrix} n-(j_0-1) \\ k-i \end{pmatrix},$$

or $\sum_{i=1}^{j_0-1} s_i < (j_0-1)(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j_0-1 \\ i \end{pmatrix}\begin{pmatrix} n-(j_0-1) \\ k-i \end{pmatrix}$, a contradiction to the hypothesis on S. Hence, (2.2) holds.□

Proof of Theorem 2.1. Necessity. Let $S = [s_1, s_2, \cdots, s_n]$ be the score sequence of an oriented k-hypergraph D. Further, let $V_1 = [v_1, v_2, \cdots, v_j]$ and $V_2 = V - V_1$. Clearly, $\left|V_1\right| = j$, $\left|V_2\right| = n - j$.

Now,

$$\sum_{i=1}^{j} s_i = \sum_{i=1}^{j} (k-1)\binom{n-1}{k-1} + d_i^+(D) - (k-1)d_i^-(D)$$
$$= j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{j} d_i^+(D) - (k-1)\sum_{i=1}^{j} d_i^-(D)$$
$$= j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{j}\left[d_i^+(V_1) + d_i^+(V_1 * V_2)\right] - (k-1)\sum_{i=1}^{j}\left[d_i^-(V_1) + d_i^-(V_1 * V_2)\right]$$

If there are α arcs in V, then $\sum_{i=1}^{j} d_i^+(V_1) = (k-1)\alpha$ and $\sum_{i=1}^{j} d_i^-(V_1) = \alpha$,

so that $\sum_{i=1}^{j} d_i^+(V_1) - (k-1)\sum_{i=1}^{j} d_i^-(V_1) = (k-1)\alpha - (k-1)\alpha = 0$.

Also, $\sum_{i=1}^{j} d_i^-(V_1 * V_2) \le \sum_{i=1}^{k-1}\binom{j}{i}\binom{n-j}{k-i}$,

and $\sum_{i=1}^{j} d_i^+(V_1 * V_2) \ge \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i}$.

Therefore,

$$\sum_{i=1}^{j} s_i \ge j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i} - (k-1)\sum_{i=1}^{k-1}\binom{j}{i}\binom{n-j}{k-i}$$
$$= j(k-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(i-k)\binom{j}{i}\binom{n-j}{k-i}$$

Sufficiency. Induct on n. If $n = k$, there is only one arc (or one non arc) in which case the scores are $n, n, \cdots, n, 0(n-1, n-1, \cdots, n-1)$, and the result is true.

Assume $n > k$. Now,

$$s_n = \sum_{i=1}^{n} s_i - \sum_{i=1}^{n-1} s_i$$
$$\le n(k-1)\binom{n-1}{k-1} - (n-1)(k-1)\binom{n-1}{k-1} - \sum_{i=1}^{k-1}(i-k)\binom{n-1}{i}\binom{1}{k-i}$$
$$= k\binom{n-1}{k-1}.$$

Case 1. If $s_n = k \binom{n-1}{k-1}$.

Let $s_i^{\prime} = s_i - \frac{k(k-2)}{n-1} \binom{n-1}{k-1}$, $\quad 1 \leq i \leq n-1$. Clearly, $s_i^{\prime}$ is of the form $xk + y(k-1)$.

Then,

$$\begin{aligned}\sum_{i=1}^{n-1} s_i^{/} &= \sum_{i=1}^{n-1} \left[s_i - \frac{k(k-2)}{n-1} \binom{n-1}{k-1} \right] \\ &= (n(k-1)-k) \binom{n-1}{k-1} - k(k-2) \binom{n-1}{k-1},\end{aligned}$$

since

$$\begin{aligned}\sum_{i=1}^{n-1} s_i &= \sum_{i=1}^{n} s_i - s_n \\ &= n(k-1) \binom{n-1}{k-1} - k \binom{n-1}{k-1} \\ &= (n(k-1)-k) \binom{n-1}{k-1}.\end{aligned}$$

So,

$$\begin{aligned}\sum_{i=1}^{n-1} s_i^{/} &= (n(k-1)-k(k-2)) \binom{n-1}{k-1} \\ &= (n(k-1)-k(k-2)) \left(\frac{n-1}{n-k} \right) \binom{n-2}{k-1} \\ &= (n-1)(k-1) \binom{n-2}{k-1}.\end{aligned}$$

Also, for $1 \leq j < n-1$,

$$\begin{aligned}
\sum_{i=1}^{j} s_i^{/} &= \sum_{i=1}^{j}\left[s_i - \frac{k(k-2)}{n-1}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\right] \\
&\geq j(k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} n-j \\ k-i \end{pmatrix} - \frac{jk(k-2)}{n-1}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \\
&= \left[j(k-1) - \frac{jk(k-2)}{n-1}\right]\left(\frac{n-1}{n-k}\right)\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} n-j-1 \\ k-i \end{pmatrix} \\
&\geq \frac{j(n-1)(k-1) - jk(k-2)}{n-k}\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} n-1-j \\ k-i \end{pmatrix} \\
&= \frac{j\left[(k-1)(n-k)+1\right]}{n-k}\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} n-1-j \\ k-i \end{pmatrix} \\
&\geq \frac{j(k-1)(n-k)}{n-k}\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} + \sum_{i=1}^{k-1}(i-k)\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} n-1-j \\ k-i \end{pmatrix}.
\end{aligned}$$

Thus, the sequence $[s_1', s_2', \cdots, s_{n-1}']$ satisfies (2.1) and by induction hypothesis is a score sequence of some oriented k-hypergraph D'. Now, construct the oriented k-hypergraph D as follows.

Let $V(D') = \{v_1, v_2, \cdots, v_{n-1}\}$ with $s(v_i) = s_i'$. Adding a new vertex v_n, and taking all $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ arcs with v_n not in the last entry in any of these arcs, we get an oriented k-hypergraph D of order n with score sequence

$$\left[s_1^{/} + \tfrac{k(k-2)}{n-1}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}, \cdots, \tfrac{k(k-2)}{n-1}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}, \; k\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\right] = [s_1, s_2, \cdots, s_n].$$

Case 2. If $s_n < k\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$. By (2.1), we get that $s_n \geq (k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$. Let $x_n = s_n - (k-1)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$, and $y_n = k\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - s_n$, then $s_n = kx_n + (k-1)y_n$. Now applying Lemma 2.4 repeatedly until we obtain a new non-decreasing sequence $S' = [s_1', s_2', \cdots, s_n']$ such that $s_n' = k\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$. It is obvious that Lemma 2.4 is applied y_n times. We denote by P_1 the operation that makes S become some $S_1 = S(s_n^+, s_{p_1}^-)$ and P_2 the operation that makes S_1 become some $S_2 = S_1(s_n^+, s_{p_2}^-)$, and so on. Furthermore will denote by P_i^{-1} the corresponding reversal operation. Note that since $s_i - 1 = (x_i - 1)k + (y_i + 1)(k-1)$ and $s_n + 1 = (x_n + 1)k + (y_n - 1)(k-1)$, the resulting sequence $S_{y_n} = S'$ is still strict.

So by case 1, S' is a score sequence of some oriented k-hypergraph. Now, we make the operations $P_{y_n}^{-1}, \cdots, P_2^{-1}, P_1^{-1}$, applying Lemma 2.3 on each operation, we finally get the original non-decreasing sequence $S = [s_1, s_2, \cdots, s_n]$. Note that after each operation P_i^{-1}, the corresponding integer sequence remains strict, so by Lemma 2.3, S is a score sequence of an oriented k-hypergraph.

Remark. If $k = 2$ in Theorem 2.1, then the necessary and sufficient condition for the sequence of non-negative integers $[s_1, s_2, \cdots, s_n]$ in non-decreasing order becomes

$$\begin{aligned}\sum_{i=1}^{j} s_i &\geq j\binom{n-1}{1} + \sum_{i=1}^{1}(i-2)\binom{j}{i}\binom{n-j}{2-i} \\ &= j\binom{n-1}{1} - \binom{j}{1}\binom{n-j}{1} \\ &= j(n-1) - j(n-j) = j^2 - j = j(j-1)\end{aligned}$$

with $\sum_{i=1}^{n} s_i = n(2-1)\binom{n-1}{1} = n(n-1)$,
which is Avery's theorem for oriented graphs.

References

[1] P. Avery, Score sequences of oriented graphs, J. Graph Theory, Vol. 15, No. 3(1991) 251-257.

[2] C. M. Bang and H. Sharp Jr., Score vectors of tournaments, J. Combin.Theory Ser. B 26 (1) (1979) 81-84.

[3] Y. Koh and S. Ree, Score sequences of hypertournament matrices, J. Korea Soc. Math. Educ. Ser. B: Pure and Appl. Math. 8 (2) (2001) 185-191.

[4] Y. Koh and S. Ree, On k-hypertournament matrices, Linear Algebra and its Applications 373 (2003) 183-195.

[5] H. G. Landau, On dominance relations and the structure of animal societies. III. The condition for a score structure, Bull. Math. Biophys. 15 (1953) 143-148.

[6] S. Pirzada, T. A. Naikoo and N. A. Shah, Score sequences in oriented graphs, J. Applied Mathematics and Computing (2006), 22(1-2) (2007) 257-268.

[7] C. Wang and G. Zhou, Note on the degree sequences of k-hypertournaments, Discrete Mathematics, To appear.

[8] G. Zhou, T. Yao and K. Zhang, On score sequences of k-hypertournaments, European J. Combin. 21 (8) (2000) 993-1000.