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Approximation of Conjugate of Functions Belonging to $Lip\alpha$ Class and $W\left(L_r, \xi(t)\right)$ Class by Product Means of Conjugate Fourier Series

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Abstract. In this paper, two quite new theorems on degree of approximation of conjugate of functions $f \in Lip\alpha$ class and $f \in W\left(L_r, \xi(t)\right)$ class using $(E,1)(C,1)$ product summability means of conjugate Fourier series have been established.

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1. Introduction

A good amount of work to determine the degree of approximation of functions belonging to the classes $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W\left(L_r, \xi(t)\right)$ using Cesàro, Nörlund and generalized Nörlund single summability methods has been done by several researchers like Alexits [1], Sahney and Goel [12], Qureshi and Neha [8], Qureshi [9, 10], Chandra [2], Khan [4], Leindler [6] and Rhoades [11] . But nothing seems to have been done so far to obtain degree of approximation using different class of functions by product summability method. Therefore, in present work, two theorems on degree of approximation of the conjugate of functions $f \in Lip\alpha$ and $f \in W\left(L_r, \xi(t)\right)$, $(r \geq 1)$ using $(E,1)(C,1)$ summability means of conjugate Fourier series have been proved.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$.

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If $(E,1)$ transform is defined as the n^{th} partial sum of $(E,1)$ summability and it can be denoted by E_n^1, which is given by

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k \to s \text{ as } n \to \infty \tag{1}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable $(E,1)$ to a definite number s (Hardy [3]).

If

$$\begin{aligned} t_n &= \frac{s_0 + s_1 + s_2 + \ldots + s_n}{n+1} \\ &= \frac{1}{n+1}\sum_{k=0}^{n} s_n \to s \text{ as } n \to \infty \end{aligned} \tag{2}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method.

The $(E,1)$ transform of $(C,1)$ transform defines $(E,1)(C,1)$ product transform and we denote it by $(EC)_n^1$.

Thus if

$$(EC)_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} C_k^1 \to s \text{ as } n \to \infty \tag{3}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,1)(C,1)$ method or summable $(E,1)(C,1)$ to a definite number s.

Let $f(x)$ be a 2π-periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{4}$$

with n^{th} partial sum $s_n(f;x)$.

The conjugate series of Fourier series (4) is given by

$$\sum_{n=1}^{\infty}\left(a_n \cos nx - b_n \sin nx\right), \tag{5}$$

and we shall call it as conjugate Fourier series. A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right) \text{ for } 0 < \alpha \leq 1. \tag{6}$$

$f \in Lip(\alpha, r)$ for $0 \le x \le 2\pi$, if [definition 5.38 of McFadden, 7]

$$\left(\int_0^{2\pi} \left| f(x+t) - f(x) \right|^r dx \right)^{\frac{1}{r}} = O\left(|t|^\alpha \right), 0 < \alpha \le 1,\ r \ge 1. \tag{7}$$

Given a positive increasing function $\xi(t)$ and an integer $r \ge 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} \left| f(x+t) - f(x) \right|^r dx \right)^{\frac{1}{r}} = O(\xi(t)) \tag{8}$$

and that $f \in W\left(L_r, \xi(t)\right)$ if

$$\left(\int_0^{2\pi} \left| \{ f(x+t) - f(x) \} \sin^\beta x \right|^r dx \right)^{\frac{1}{r}} = O(\xi(t)), \beta \ge 0. \tag{9}$$

where $\xi(t)$ is a positive increasing function of t.

If $\beta = 0$ then $W\left(L_r, \xi(t)\right)$ reduces to the class $Lip(\xi(t), r)$ and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$.

L_∞-norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = sup\left\{ |f(x)| : x \in R \right\} \tag{10}$$

L_r-norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{\frac{1}{r}},\ r \ge 1. \tag{11}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of degree n under sup norm $\|\cdot\|_\infty$ is defined as [Zygmund, 13]

$$\|t_n - f\|_\infty = sup\left\{ |t_n(x) - f(x)| : x \in R \right\} \tag{12}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r \tag{13}$$

We use the following notations throughout this paper:

$$\psi(t) = f(x+t) + f(x-t)$$

$$\bar{K}_n(t) = \frac{1}{\pi\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\cos\left(\nu + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}$$

$\tau = \left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$.

2. Main Theorems

We prove the following theorems:

Theorem 1. *If a function $\overline{f}$, conjugate to a 2π-periodic function f, belongs to $Lip\alpha$ class, then its degree of approximation by $(E,1)(C,1)$ means of conjugate Fourier series is given by*

$$\sup_{0<x<2\pi}\left|\overline{(EC)_n^1}(x)-\overline{f}\right|=\left\|\overline{(EC)_n^1}-\overline{f}\right\|_\infty=O\left(\frac{1}{(n+1)^\alpha}\right),\ 0<\alpha<1 \tag{14}$$

where $\overline{(EC)_n^1}$ denotes the $(E,1)(C,1)$ transform as defined in (3).

Theorem 2. *If $\overline{f}$, conjugate to a 2π-periodic function f, belongs to $W\left(L_r,\xi(t)\right)$ class, then its degree of approximation by $(E,1)(C,1)$ means of conjugate Fourier series is given by*

$$\left\|\overline{(EC)_n^1}-\overline{f}\right\|_r=O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right] \tag{15}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\}\text{ be a decreasing sequence,} \tag{16}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,\left|\psi(t)\right|}{\xi(t)}\right)^r\sin^{\beta r}t\,dt\right\}^{\frac{1}{r}}=o\left(\frac{1}{n+1}\right) \tag{17}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\left|\psi(t)\right|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}}=O\left\{(n+1)^\delta\right\} \tag{18}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, conditions (17) and (18) hold uniformly in x and $\overline{(EC)_n^1}$, as defined in (3), is $(E,1)(C,1)$ means of the series (5) and

$$\overline{f}(x)=-\frac{1}{2\pi}\int_0^{2\pi}\psi(t)\,cot\left(\frac{t}{2}\right)\,dt \tag{19}$$

3. Lemmas

For the proof of our theorems, following lemmas are required:

Lemma 1.

$$\left|\overline{G}_n(t)\right|=O\left(\frac{1}{t}\right),\ \text{for } 0\le t\le\frac{1}{n+1}$$

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $|\cos nt| \le 1$

$$
\begin{aligned}
\left|\overline{G}_n(t)\right| &\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\cos\left(\nu+\frac{1}{2}\right)t}{sin\frac{t}{2}} \right\} \right| \\
&\le \frac{1}{\pi\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|sin\frac{t}{2}\right|} \right\} \\
&= \frac{1}{t\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} 1 \right\} \\
&= \frac{1}{t\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \right\} \\
&= \frac{1}{t\ 2^{n+1}} 2^n \\
&= O\left(\frac{1}{t}\right) \text{ since } \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) = 2^n
\end{aligned}
$$

Lemma 2. *For $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n, we have*

$$\left|\overline{G}_n(t)\right| = O\left[\frac{1}{t}\right]$$

Proof. For $0 \le \frac{1}{n+1} \le t \le \pi$, $sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$.

$$
\begin{aligned}
\left|\overline{G}_n(t)\right| &\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right| \\
&\le \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} Re\left\{ \sum_{\nu=0}^{k} e^{i\left(\nu+\frac{1}{2}\right)t} \right\} \right] \right| \\
&\le \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} Re\left\{ \sum_{\nu=0}^{k} e^{i\nu t} \right\} \right] \right| \left|e^{i\frac{t}{2}}\right| \\
&\le \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} Re\left\{ \sum_{\nu=0}^{k} e^{i\nu t} \right\} \right] \right| \\
&\le \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{\tau-1} \left[\left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} Re\left\{ \sum_{\nu=0}^{k} e^{i\nu t} \right\} \right] \right| \\
&\quad + \frac{1}{2^{n+1}\ t} \left| \sum_{k=\tau}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(1+k)} Re\left\{ \sum_{\nu=0}^{k} e^{i\nu t} \right\} \right] \right| \qquad (20)
\end{aligned}
$$

Now considering the first term of (20),

$$\begin{aligned}
&\frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k}e^{i\nu t}\right\}\right]\right| \\
&\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\frac{1}{(1+k)}\ \left\{\sum_{\nu=0}^{k}1\right\}\right]\right|\left|e^{i\nu t}\right| \\
&\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\right]\right| \qquad (21)
\end{aligned}$$

Now considering the second term of (20) and using Abel's lemma

$$\begin{aligned}
&\frac{1}{2^{n+1}\ t}\left|\sum_{k=\tau}^{n}\left[\binom{n}{k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k}e^{i\nu t}\right\}\right]\right| \\
&\leq \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\binom{n}{k}\frac{1}{(1+k)}\ \underset{0\leq m\leq k}{max}\ \left|\sum_{\nu=0}^{m}e^{i\nu t}\right| \\
&\leq \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\binom{n}{k}\frac{1}{(1+k)}(1+k) \\
&= \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\binom{n}{k} \qquad (22)
\end{aligned}$$

Combining (20), (21) and (22), we get

$$\begin{aligned}
\left|\overline{G}_n(t)\right| &\leq \frac{1}{2^{n+1}\ t}\sum_{k=0}^{\tau-1}\binom{n}{k}+\frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\binom{n}{k} \\
&= O\left[\frac{1}{t}\right]
\end{aligned}$$

4. Proof of Theorems

4.1. Proof of Theorem 1

Let $\overline{s}_n\left(f;x\right)$ denote the partial sum of series (5). Then following Lal [5], we have

$$\overline{s}_n(x)-\overline{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\ \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\ dt$$

Using (5) the $(C,1)$ transform C_n^1 of $\overline{s}_n\left(f;x\right)$ is given by

$$\overline{C_n^1}-\overline{f}(x)=\frac{1}{2\pi\ (n+1)}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt \qquad (23)$$

Now denoting $(E,1)(C,1)$ transform of $\overline{s}_n$ by $\overline{(EC)_n^1}$, we write

$$\begin{aligned}
\overline{(EC)_n^1} - \overline{f}(x) &= \frac{1}{2^{n+1}\,\pi}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\left(\frac{1}{k+1}\right)\left\{\sum_{\nu=0}^{k}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right] \\
&= \int_0^{\pi}\psi(t)\;\overline{G}_n(t)\,dt \\
&= \left[\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\;\overline{G}_n(t)\;dt \\
&= I_{1.1}+I_{1.2}\ \text{(say)} \qquad (24)
\end{aligned}$$

Now using Lemma 1, we have

$$\begin{aligned}
\left|I_{1.1}\right| &\le \int_0^{\frac{1}{n+1}}\left|\psi(t)\right|\;\left|\overline{G}_n(t)\right|dt \\
&= \int_0^{\frac{1}{n+1}}\frac{|t^{\alpha}|}{t}dt \\
&= \int_0^{\frac{1}{n+1}}t^{\alpha-1}dt \\
&= \left(\frac{t^{\alpha}}{\alpha}\right)_0^{\frac{1}{n+1}} \\
&= O\left\{\frac{1}{(n+1)^{\alpha}}\right\} \qquad (25)
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
\left|I_{1.2}\right| &= \int_{\frac{1}{n+1}}^{\pi}\left|\psi(t)\right|\;\left|\overline{G}_n(t)\right|dt \\
&= \int_{\frac{1}{n+1}}^{\pi}\frac{|t^{\alpha}|}{|t|}dt \\
&= \int_{\frac{1}{n+1}}^{\pi}t^{\alpha-1}dt \\
&= \left(\frac{t^{\alpha}}{\alpha}\right)_{\frac{1}{n+1}}^{\pi} \\
&= O\left\{\frac{1}{(n+1)^{\alpha}}\right\} \qquad (26)
\end{aligned}$$

Combining (24), (25) and (26), we get

$$\left\| \overline{(EC)_n^1} - \overline{f} \right\|_\infty = \left\{ \left| \overline{(EC)_n^1} - \overline{f} \right| : x \in [0, 2\pi] \right\} = O\left(\frac{1}{(n+1)^\alpha} \right)$$

This completes the proof of Theorem 1.

4.2. Proof of Theorem 2

Following the proof of Theorem 1,

$$\begin{aligned} \overline{(EC)_n^1} - \overline{f}(x) &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \psi(t)\ \overline{G}_n(t)\ dt \\ &= I_{2.1} + I_{2.2} \text{ (say)} \end{aligned} \tag{27}$$

Applying Hölder's inequality and the fact that $\psi(t) \in W\left(L_r, \xi(t)\right)$, condition (17), Lemma 1 and second mean value theorem for integrals, we have

$$\begin{aligned} \left| I_{2.1} \right| &\leq \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t\ \left| \psi(t) \right| \sin^\beta t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t) \left| \overline{G}_n(t) \right|}{t\ \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1} \right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)}{t^{2+\beta}} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left\{ \left(\frac{1}{n+1} \right) \xi\left(\frac{1}{n+1} \right) \right\} \left[\int_\in^{\frac{1}{n+1}} \frac{dt}{t^{(2+\beta)s}} \right]^{\frac{1}{s}} \quad \text{for some } 0 < \in < \frac{1}{n+1} \\ &= O\left[\left(\frac{1}{n+1} \right) \xi\left(\frac{1}{n+1} \right) \left\{ \frac{t^{-(2+\beta)s+1}}{-(2+\beta)s+1} \right\}_\in^{\frac{1}{n+1}} \right]^{\frac{1}{s}} \\ &= O\left[\left(\frac{1}{n+1} \right) \xi\left(\frac{1}{n+1} \right) (n+1)^{2+\beta-\frac{1}{s}} \right] \\ &= O\left\{ (n+1)^{\beta+1-\frac{1}{s}}\, \xi\left(\frac{1}{n+1} \right) \right] \\ &= O\left\{ (n+1)^{\beta+\frac{1}{r}}\, \xi\left(\frac{1}{n+1} \right) \right\} \text{ since } \frac{1}{r} + \frac{1}{s} = 1, 1 \leq r \leq \infty. \end{aligned} \tag{28}$$

Now using Hölder's inequality, $|\sin t| < 1$, $\sin t \geq \left(\frac{2t}{\pi}\right)$, conditions (16) and (18), Lemma 2 and second mean value theorem for integrals, we have

$$\left| I_{2.2} \right| \leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} \left| \psi(t) \right| \sin^\beta t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t) \left| \overline{G}_n(t) \right|}{t^{-\delta}\ \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(1+\beta-\delta)-1}-\pi^{s(\delta-1-\beta)+1}}{s(1+\beta-\delta)-1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1 \tag{29}$$

Now combining (27) to (29), we get

$$\left|\overline{(EC)_n^1}-\overline{f}\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

$$\left\|\overline{(EC)_n^1}-\overline{f}\right\|_r = \left\{\int_0^{2\pi}\left|\overline{(EC)_n^1}-\overline{f}\right|^r dx\right\}^{\frac{1}{r}}$$

$$= \left\{\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= \left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the Theorem 2.

5. Applications

Following corollaries can be derived from our main theorem:

Corollary 1. *If $\beta = 0$ and $\xi(t) = t^{\alpha}$, then the degree of approximation of a function $\overline{f}$, conjugate to 2π-periodic function $f \in Lip(\alpha, r)$, $\frac{1}{r} \leq \alpha \leq 1$, is given by*

$$\left\| \overline{(EC)_n^1} - \overline{f} \right\|_r = O\left\{ \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right\}$$

Corollary 2. *If $r \to \infty$ in Corollary 1, then $Lip(\alpha, r)$ reduces to $Lip\alpha$ for $0 < \alpha < 1$, and we have*

$$\left\| \overline{(EC)_n^1} - \overline{f} \right\|_r = O\left\{ \frac{1}{(n+1)^{\alpha}} \right\}$$

Remark 1. *An independent proof of Corollary 1 can be obtained along the same lines of our Theorem 2.*

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