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A New Group Decision-Making Method Based on Fuzzy Set Operations

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Abstract. In this study we suggest a new group decision-making method which is based on some basic fuzzy set operations. By using this method, we can get two types of results. First, we can identify which alternative is the best. The second result, a crucial point of this work, is the screening of decision-makers. The decision-makers should be serious and responsible in giving their opinions otherwise the process will eliminate them because of their inappropriate evaluations to the alternatives compared with that of the other decision-makers. We also discuss an application to demonstrate the process of the method.

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Key Words and Phrases: Fuzzy sets; α-level set; group decision-making.

1. Introduction

An individual or group is frequently faced with the problem of choosing one alternative from a feasible alternative set. For an individual people, the problem is the identification of the most preferred alternative according to his/her preference structure. However, for group decision making, except the above task, another important problem is how to aggregate the experts opinion to obtain an acceptable result for the group. In general, the preference relations take the form of multiplicative preference relations [12] or fuzzy preference relations [7, 19, 20] whose elements estimate the dominance of one alternative over another and take the form of exact numerical values. It is more suitable to provide their preferences by means of linguistic variables rather than numerical ones [6, 10, 11, 15]. Xu [22] developed a practical method for group decision making with linguistic preference relations. To aggregate the preference information and to rank the given alternatives Yager [23] presented a method which is called the ordered weighted averaging operator. Kacprzyk and Nurmi [16] gave an algorithms to illustrate ways of aggregating opinions and preferences of different experts.

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After fuzzy sets theory was introduced by Zadeh [24] to deal with problems in which impreciseness was present, fuzzy decision making was suggested by Bellman and Zadeh [2]. The literature on fuzzy decision making have grown tremendously in recent years. A number of authors (among others) [1, 2, 5, 15, 16] have provided interesting results on decision making by using fuzzy sets theory. More details and historical background of fuzzy and crisp decision making can be found in, for example; [4, 8, 9, 13, 14, 21, 25].

In literature, there also have been many group decision making approaches which are based on fuzzy set theory, for example [3, 18]. In those studies, two main concepts are applied: linguistic variables and fuzzy numbers. Therefore, the corresponding methods are usually called fuzzy group decision making methods. In this paper, a new group decision making method is presented which is also based on fuzzy set theory. However the essential of the proposed method is not fuzzy since it still uses crisp numbers to score the performance of each alternative and the decision-makers/experts. That is, the key idea behind the method is to apply fuzzy set operations to solve crisp group decision making problems. This method is suitable when there is a need to seek consensus among many decision makers in crisp data situation.

This method considers two kinds of sets. The first is a collection of alternatives, called the *alternative set*. For instance, defendants in a court, participants in a competition, etc can be considered as alternative sets. The second one is a collection of decision-makers which we call the *decision-maker set*. Judges, juries, raters etc can form a decision-maker set. When the decision-makers give their opinions according to their own criteria for each alternative by selecting a value from $[0,1]$, the union of their evaluations to all the currently available alternatives can be represented in the form of a fuzzy set. That is, each decision-maker works like a membership function and each alternative is an element of the fuzzy set and is evaluated with a value in the interval $[0,1]$ from the point of view of each decision-maker. One may ask: "how can we obtain appropriate membership functions for the alternative set?" or "how can we control the decision-makers to give responsible evaluations for the alternatives?" . The answers to these questions are given next.

The presentation of the rest of the paper is organized as follows. In the next section, the basic notions of fuzzy set and fuzzy operations are introduced. In section 3, the new method is introduced step by step. In section 4, an applications of the method is described to demonstrate the process of the method. Finally, in Section 5, concluding remarks are presented.

2. Some Preliminaries

In this section, we describe some preliminary definitions of fuzzy set operations that will be used in this paper. More details and historical background of fuzzy set theory can be found in [8, 17, 25].

In a universe U, a fuzzy set $\widetilde{A}$ is defined by Zadeh[24] as

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in U, \mu_{\widetilde{A}}(x) \in [0,1]\} \tag{1}$$

where the function $\mu_{\widetilde{A}}(x)$ is called a *membership function*. The value of the membership function $\mu_{\widetilde{A}}(x)$ specifies the grade or degree to any element x in U. Larger values of $\mu_{\widetilde{A}}(x)$ indicate higher degrees of membership. We will identify any fuzzy set with its membership function and use these two concepts as interchangeable.

Let $\widetilde{A}$ be a fuzzy set in the universe U as in (1). Then the support of $\widetilde{A}$ is defined as

$$supp\widetilde{A} = \{x : x \in U, \mu_{\widetilde{A}}(x) > 0\} \tag{2}$$

The cardinality of a crisp set A, denoted as $|A|$, is the number of elements of the set A, and the cardinality of a fuzzy set $\widetilde{A}$ is defined as

$$card\widetilde{A} = \sum_{x \in U} \mu_{\widetilde{A}}(x) \tag{3}$$

The mean relative cardinality of $\widetilde{A}$ is defined as

$$mrc\widetilde{A} = \frac{card\widetilde{A}}{|supp\widetilde{A}|} \tag{4}$$

and then the *α-level set (α-cut)* of $\widetilde{A}$ is defined as

$$\widetilde{A}_\alpha = \{x : x \in U, \mu_{\widetilde{A}}(x) \geq \alpha\} \tag{5}$$

where $\alpha \in [0,1]$. The concept of α-cut is very important in the relationship between fuzzy sets and crisp sets.

3. The Proposed Method

When using this method, first the decision-makers give their evaluations according to their own opinions for all the considered alternatives in the form of a fuzzy set. That is, each alternative is evaluated with a value in the interval $[0,1]$ from the point of view of each decision-maker.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an alternative set and let $B = \{b_1, b_2, \ldots, b_m\}$ be a decision-maker set in a finite universe U_a and U_b, respectively. Then, this method can be described by the following steps in k-cycles.

Step (k.1) Let an evaluation of decision-maker $b_i \in B_k$ for an alternative $a \in A$ be a value $b_i(a) \in [0,1]$. Where B_k is a set in k-cycle and $B_1 = B$. Then, for all elements of A, each decision-maker b_i gives his/her evaluations separately and independently according to his/her own preference by a fuzzy set as

$$\widetilde{A}_{b_i} = \{(a, \mu_{\widetilde{A}_{b_i}}(a)) : a \in A, \mu_{\widetilde{A}_{b_i}}(a) = b_i(a)\} \tag{6}$$

which is called the *b_i-fuzzy set* for $b_i \in B_k$. In order to apply the fuzzy set formula (1), we have to deal with the universal set, that is the alternatives set A, ($U_a = A$).

In this way, each decision-maker b_i presents a fuzzy set $\widetilde{A}_{b_i}$ where the elements are the considered alternatives. Here we assume that the majority of the experts can offer fair and proper evaluations for the alternatives.

Step (k.2) In the fuzzy sets $\widetilde{A}_{b_i}$, an alternative a is given evaluations $b_i(a)$ by the decision-makers $b_i \in B_k$. By using the arithmetic mean concept we can obtain a fuzzy set as follows:

$$\widetilde{A}_{B_k} = \{(a, \mu_{\widetilde{A}_{B_k}}(a)) : a \in A, \mu_{\widetilde{A}_{B_k}}(a) = \frac{1}{|B_k|} \sum_{b_i \in B_k} \mu_{\widetilde{A}_{b_i}}(a)\} \tag{7}$$

which is called the *fuzzy mean set* of the sets $\widetilde{A}_{b_i}$ in the k-cycle.

Step (k.3) The distances between the sets $\widetilde{A}_{b_i}$ and the set $\widetilde{A}_{B_k}$ for all $b_i \in B_k$ can be characterized by fuzzy sets as

$$\widetilde{A}_k(b_i) = \{(a, \mu_{\widetilde{A}_k(b_i)}(a)) : a \in A, \mu_{\widetilde{A}_k(b_i)}(a) = |\mu_{\widetilde{A}_{b_i}}(a) - \mu_{\widetilde{A}_{B_k}}(a)|\} \tag{8}$$

which are called the *fuzzy distance sets* in k-cycle.

In this step, we shall investigate how close each b_i-fuzzy decision set $\widetilde{A}_{b_i}$, $b_i \in B_k$, to the fuzzy mean set $\widetilde{A}_{B_k}$ can be.

Step (k.4) By using the mean relative cardinality (4) of each b_i-fuzzy distance sets $\widetilde{A}_k(b_i)$, $b_i \in B_k$, we can evaluate the decision-makers' performance by the following fuzzy set as

$$\widetilde{B}_k = \{(b, \mu_{\widetilde{B}_k}(b)) : b \in B_k, \mu_{\widetilde{B}_k}(b) = 1 - mrc\widetilde{A}_k(b)\} \tag{9}$$

which is called the *decision-makers performance fuzzy set* in the k-cycle. In order to apply the fuzzy set formula (1), we have to deal with the universal set, that is the decision-maker set B, $(U_b = B_k)$.

In this step, we shall investigate the performance of the decision-makers. The $\mu_{\widetilde{B}_k}(b)$ is an evaluation of the performance of decision-maker b in k-cycle. It is the higher the better.

Step (k.5) Let $s_k^2 = \frac{1}{n}\sum_{b_i \in B_k}(\mu_{\widetilde{B}_k}(b_i) - mrc\widetilde{B}_k)^2$ be sample variance where s_k is the sample standard deviation and n is the cardinality of $supp\widetilde{B}_k$. Then we can get statistically an α_k as

$$\alpha_k = mrc\widetilde{B}_k - s_k \tag{10}$$

By using α_k we find a subset of the set B_k as

$$\widetilde{B}_{\alpha_k} = \{b : b \in B_k, \mu_{\widetilde{B}_k}(b) \geq \alpha_k\} \tag{11}$$

which is call α_k*-level set*. Where if $\widetilde{B}_{\alpha_k} \subset B_k$, then the procedure has to start $(k+1)$-cycle with $B_{k+1} = \widetilde{B}_{\alpha_k}$. If $\widetilde{B}_{\alpha_k} = B_k$, then the procedure is finished. That is;

$$k\text{-cycle} \begin{cases} \text{goes to } (k+1)\text{-cycle}, & \text{if } \widetilde{B}_{\alpha_k} \subset B_k \\ \text{stops}, & \text{if } \widetilde{B}_{\alpha_k} = B_k \end{cases}$$

4. An Illustrative Example

Assume that there is a company which has decision-makers who are working as experts to choose one or more desirable alternatives among a set of alternatives for their own or other companies. The company applies this method because when choosing the most desirable alternatives, the experts are also evaluated by the method.

Let the procedure stops in the k-cycle. Then we can get the following results in the applications:

1. The fuzzy mean set $\widetilde{A}_{B_k}$ gives us the average evaluations of the alternatives by the experts. Therefore, one can use this method as a process of selecting the most preferred alternative from available alternatives.

2. The decision-makers performance fuzzy set $\widetilde{B}_k$ gives us the performance evaluation of the decision-makers in the k-cycle. Therefore, one can use this method as a process of choosing reliable decision-makers.

3. The decision-makers in the set $(B - B_{\alpha_k})$ are called outlier decision-makers whose evaluations to the alternatives are inconsistent with the evaluations from the other experts. Therefore, the decision-makers have to be responsible in giving their opinions otherwise the process will eliminate them. That is, this system can stimulate the decision-makers to give as proper as possible evaluations for the alternatives.

Numerical Data

Let $B = \{b_1, b_2, b_3, b_4, b_5\}$ be a decision-maker set of company X, and $A = \{a_1, a_2, a_3, a_4\}$ be an alternative set of company Y. Then the proposed method works as follows:

(1.1) Assume that $B_1 = B$ and the experts $b_i \in B_1$ evaluate the degree of suitability of the alternatives by the b_i-fuzzy sets as follows:

$$\begin{aligned}
\widetilde{A}_{b_1} &= \{(a_1, 0.50), (a_2, 0.75), (a_3, 0.55), (a_4, 0.85)\} \\
\widetilde{A}_{b_2} &= \{(a_1, 0.55), (a_2, 0.75), (a_3, 0.50), (a_4, 0.75)\} \\
\widetilde{A}_{b_3} &= \{(a_1, 0.35), (a_2, 0.90), (a_3, 0.70), (a_4, 0.65)\} \\
\widetilde{A}_{b_4} &= \{(a_1, 0.55), (a_2, 0.70), (a_3, 0.40), (a_4, 0.75)\} \\
\widetilde{A}_{b_5} &= \{(a_1, 0.50), (a_2, 0.70), (a_3, 0.55), (a_4, 0.75)\}
\end{aligned}$$

(1.2) Hence one can get the fuzzy mean set of the sets $\widetilde{A}_{b_i}$ in the 1-cycle by using (7) as follows:

$$\widetilde{A}_{B_1} = \{(a_1, 0.49), (a_2, 0.76), (a_3, 0.54), (a_4, 0.75)\}$$

(1.3) The fuzzy distance sets in 1-cycle are obtained by using (8) as follows:

$$\begin{aligned}
\widetilde{A}_1(b_1) &= \{(a_1,0.01),(a_2,0.01),(a_3,0.01),(a_4,0.10)\} \\
\widetilde{A}_1(b_2) &= \{(a_1,0.06),(a_2,0.01),(a_3,0.04),(a_4,0.00)\} \\
\widetilde{A}_1(b_3) &= \{(a_1,0.14),(a_2,0.14),(a_3,0.16),(a_4,0.10)\} \\
\widetilde{A}_1(b_4) &= \{(a_1,0.06),(a_2,0.06),(a_3,0.14),(a_4,0.00)\} \\
\widetilde{A}_1(b_5) &= \{(a_1,0.01),(a_2,0.06),(a_3,0.01),(a_4,0.00)\}
\end{aligned}$$

(1.4) The decision-maker performance fuzzy set is calculated by using (9) as follows:

$$\widetilde{B}_1 = \{(b_1,0.9675),(b_2,0.9725),(b_3,0.8650),(b_4,0.9350),(b_5,0.9800)\}$$

(1.5) The $\alpha_1 = 0.9016$ is obtained by using (10). Then we get the α_1-level set by using (11) as follows:

$$\widetilde{B}_{\alpha_1} = \{b_1, b_2, b_4, b_5\}$$

where the decision-maker b_3 is eliminated since the value of $\mu_{\widetilde{B}^1}(b_3) = 0.8650$ is less then α_1, and then the procedure has to start 2-cycle with $B_2 = \widetilde{B}_{\alpha_1}$ since $\widetilde{B}_{\alpha_1} \subset B_1$.

Now we have to continue to the 2-cycle to do the same procedure with $B_2 = \{b_1, b_2, b_4, b_5\}$ and the same alternative set A.

(2.1) Here we have $\widetilde{A}_{b_1}, \widetilde{A}_{b_2}, \widetilde{A}_{b_4}, \widetilde{A}_{b_4}$ which are given in **(1.1)**.

(2.2) $\widetilde{A}_{B_2} = \{(a_1,0.525),(a_2,0.725),(a_3,0.500),(a_4,0.775)\}$

(2.3)

$$\begin{aligned}
\widetilde{A}_2(b_1) &= \{(a_1,0.025),(a_2,0.025),(a_3,0.050),(a_4,0.075)\} \\
\widetilde{A}_2(b_2) &= \{(a_1,0.025),(a_2,0.025),(a_3,0.000),(a_4,0.025)\} \\
\widetilde{A}_2(b_4) &= \{(a_1,0.025),(a_2,0.025),(a_3,0.100),(a_4,0.025)\} \\
\widetilde{A}_2(b_5) &= \{(a_1,0.025),(a_2,0.025),(a_3,0.050),(a_4,0.025)\}
\end{aligned}$$

(2.4) $\widetilde{B}_2 = \{(b_1,0.95625),(b_2,0.98125),(b_4,0.95625),(b_5,0.96875)\}$

(2.5) In this 2-cycle, the $\alpha_2 = 0.955260$, then we get the α_2-level set as

$$\widetilde{B}_{\alpha_2} = \{b_1, b_2, b_4, b_5\}$$

where the procedure is finished since $\widetilde{B}_{\alpha_2} = B_2$.

It can be seen that the alternative a_4 has the largest membership grade 0.775 hence it is selected as the best alternative. It can also be seen that the decision-maker b_2 is well working one, since the value 0.98125 is the largest one. Here the b_3 is an outlier decision-maker and his evaluations for the alternatives have been discarded in the 2-cycle. Please note that in the 1-cycle (with b_3) the best alternative was a_2 with membership grade 0.76. After the 2-cycle, it turns out that a_4 should be the best alternative. So company Y will select a_4 instead of a_2.

5. Concluding Remarks

The decision-making is a process of choosing the most desirable alternative among a set of alternatives and is, therefore, important in many disciplines including social, physical, medical and engineering sciences. When choosing a preferable alternative, people must set a priority for each available alternative. This process is not easy for an individual. It is even more difficult when there are more than one decision-maker involved in the process. How to aggregate the individual choices into a group preference has always been a hot topic.

In this study, the proposed method applies fuzzy set operations to solve crisp group decision making problems. It can achieve two goals simultaneously. One is the usual identification of the best alternative among the considered alternatives. The second one is the screening of decision-makers. Decision-makers have to be responsible in giving their opinions otherwise the process will eliminate them. This method is suitable for crisp data decision problems in many area, especially in situations where multiple decision-makers are involved and some of the decision-makers may not be reliable enough.

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