EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 34-41
ISSN 1307-5543 – www.ejpam.com

Simultaneous Generalizations of Regularity and Normality

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Abstract. A generalization of regularity called θ-regularity was earlier introduced to decompose normality and also utilised to factorize regularity. Every normal space need not be regular, but every normal space is θ-regular. In this paper three variants of θ-regular spaces is introduced and studied.

2000 Mathematics Subject Classifications: 54D10, 54D15
Key Words and Phrases: θ-open sets, θ-closed sets, almost normal space, (weakly)(functionally) θ-normal space, (weakly) θ-regular, point (weakly) θ-regular.

1. Introduction and Preliminaries

Many generalizations of regularity that exists in the mathematical literature fails to be a generalization of normality. But in order to obtain a decomposition of normality, the notion of θ-regularity was introduced in [6] which is a simultaneous generalization of regularity as well as normality. It is obvious from the definition that every regular space is θ-regular as in a regular space every closed set is θ-closed [14]. In general a normal space need not be regular, but in contrast every normal space is θ-regular [6]. Also it is observed in [5] that the notion of θ-regularity serves as a decomposition of regularity in terms of R_0 and R_1 spaces. In this paper we introduced three more variants of θ-regular spaces and studied their properties.

Let X be a topological space and let $A \subset X$. Throughout the present paper, the closure of a set A will be denoted by $\overline{A}$ or clA and the interior by $intA$. A set $U \subset X$ is said to be ***regularly open*** if $U = int\overline{U}$. The complement of a regularly open set is called ***regularly closed***. A point $x \in X$ is called a ***θ-limit point*** [14] of A if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denotes the set of all θ-limit point of A. The set A is called ***θ-closed*** if $A = cl_\theta A$. The complement of a θ-closed set will be referred to as a ***θ-open*** set. The family of θ-open sets forms a topology on X. A space X is said to be ***almost regular*** [9] if every regularly closed set and a point not in it are contained in disjoint open sets. A space is called ***almost normal*** [10] if every pair of disjoint closed sets, one of which is regularly closed, are contained in disjoint open sets and a space X is said to be ***mildly normal*** [12] (or κ-normal [13]) if every pair of disjoint regularly closed sets are contained in disjoint open sets. A space X is said to be

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nearly compact[11] if every open covering of X admits a finite subcollection the interiors of the closures of whose members cover X.

Definition 1. *A topological space X is said to be*

(i) θ***-normal****[6] if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;*

(ii) ***weakly*** θ***-normal****[6] if every pair of disjoint θ-closed sets are contained in disjoint open sets;*

(iii) ***functionally*** θ***-normal****([4, 6]) if for every pair of disjoint closed sets A and B one of which is θ-closed there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B)=1$;*

(iv) ***weakly functionally*** θ***-normal (wf*** θ***-normal)****([4, 6]) if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B)= 1$; and*

(v) Σ***-normal****[7] if for each closed set F and each open set U containing F, there exists a regular F_σ set V such that $F \subset V \subset U$.*

2. Variants of θ-regular Spaces

Definition 2. *A topological space X is said to be*

(i) θ***-regular****[6] if for each closed set F and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.*

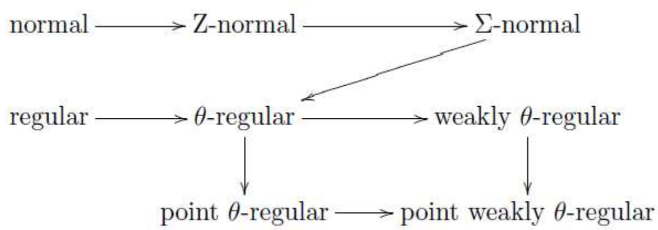
(ii) ***weakly*** θ***-regular*** *if for each θ-closed set F and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.*

(iii) ***point*** θ***-regular*** *if for each closed singleton $\{x\}$ and each open set U containing x, there exists a θ-open set V such that $x \in V \subset U$.*

(iv) ***point weakly*** θ***-regular*** *if for each θ-closed singleton $\{x\}$ and each open set U containing x, there exists a θ-open set V such that $x \in V \subset U$.*

The above notion of θ-regularity is exclusively different from the concept of θ-regularity introduced by Jankovic [3] which was utilized by Kovar [8] to study covering axioms including compactness and paracompactness. In [8], Kovar proved that Jankovic's θ-regularity coincides with the notion of point paracompactness introduced by Boyte [1]. From here onward the term "θ-regularity" will always be meant in the sense of Definition 2.

The following implications are obvious, but none of them are reversible.



Example 1 (A point θ-regular space which is not θ-regular.). *Let $X = \{a, b, c, d, e\}$ and $T = \{\{a, b, c\}, \{c, d, e\}, \{c\}, \varphi, X\}$. Here X is vacuously point θ-regular, but not θ-regular as $\{a, b\} \subset \{a, b, c\}$ but there is no θ-open set containing $\{a, b\}$ and contained in $\{a, b, c\}$.*

Example 2 (A point weakly θ-regular space which is not point θ-regular.). *Co-finite topology is point weakly θ-regular but not point θ-regular.*

Example 3 (A point weakly θ-regular space which is not point θ-regular.). *Let $X = \{a, b, c\}$ and $T = \{\{a, b\}, \{b, c\}, \{b\}, \varphi, X\}$. Here X is vacuously point weakly θ-regular, but not point θ-regular as $\{a\} \subset \{a, b\}$ but there is no θ-open set containing $\{a\}$ and contained in $\{a, b\}$.*

Question 1. *Does there exists a point weakly θ-regular space which is not weakly θ-regular?*

It is obvious from the definitions that, a R_0-space is regular if and only if it is θ-regular and a T_1-space is T_3 if and only if it is point θ-regular. Similarly, a Hausdroff space is T_3 if and only if it is point weakly θ-regular.

Theorem 1. *For a point θ-regular space, the following statements are equivalent.*

(i) For every pair of distinct points x and y in X, there exist θ-open sets P and Q such that $x \in U, y \in V$ and $\overline{P} \cap \overline{Q} = \varphi$.

(ii) X is θT_2.

(iii) X is Urysohn.

(iv) X is T_2.

(v) X is T_1.

Proof. Let x and y be two disjoint points in X. Since X is T_1, the closed set $\{x\}$ is contained in an open set $X - \{y\}$. Thus by point θ-regularity of X, there exists a θ-open set V such that $x \in V \subset X - \{y\}$. Since V is θ-open there exists a open set U such that $x \in U \subset \overline{U} \subset V \subset X - \{y\}$. i.e.; $x \in U$ and $y \in X - \overline{U}$. Again by point θ-regularity, there exist θ-open sets P and Q such that $x \in P$, $y \in Q$ and $\overline{P} \cap \overline{Q} = \varphi$.

Theorem 2. *For a T_1 space, the following statements are equivalent.*

(i) X *is* T_3.

(ii) X *is regular.*

(iii) X *is* θ*-regular.*

(iv) X *is point* θ*-regular.*

Proof. Let X be a T_1 point θ-regular space. Let $x \notin A$, where A is a closed set in X. Since X is a T_1 space, the singleton $\{x\}$ is closed and contained in $X - A$. By Point θ-regularity of X, there exists a θ-open set V such that $x \in V \subset X - A$. Since V is θ-open there exists an open set U such that $x \in U \subset \overline{U} \subset V \subset X - A$. Therefore X is regular and thus T_3.

Theorem 3. *Every* T_1 *point* θ*-regular space is Hausdorff.*

Proof. Let X be a T_1 point θ-regular space and let x, y be two distinct points in X. Since X is T_1, $\{x\}$ is a closed singleton contained in the open set $X - \{y\}$. By point θ-regularity of X, there exists a θ-open set U such that $x \in U \subset X - \{y\}$. Thus there exists an open set V such that $x \in V \subset \overline{V} \subset U \subset X - \{y\}$. So V and $X - \overline{V}$ are two disjoint open sets containing x and y respectively.

Theorem 4. *For a T_2 space, the following statements are equivalent.*

(i) X *is* T_3.

(ii) X *is regular*

(iii) X *is* θ*-regular*

(iv) X *is weakly* θ*-regular*

(v) X *is point* θ*-regular*

(vi) X *is point weakly* θ*-regular*

Proof. Obvious.

Theorem 5. *Every functionally* θ*-normal space is weakly* θ*-regular.*

Proof. Let A be a θ-closed set contained in an open set U. Let $B = X - U$. Then A and B are disjoint closed sets in X. By functional θ-normality of X, there exists a continuous function $f : X \rightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $V = f^{-1}[0,1/2)$. Then $A \subset V \subset U$. We claim that V is a θ-open set. Let $x \in V$. Then $f(x) \in [0,1/2)$. So there is a closed neighbourhood N of $f(x)$ contained in $[0,1/2) \subset [0,1]$. Let $U_x = int f^{-1}(N)$. Then $x \in U_x \subset \overline{U}_x \subset f^{-1}(N) \subset V$. Hence V is θ-open. Therefore X is θ-regular.

Remark 1. *Functionally θ-normal spaces need not be θ-regular. i.e.; Let $X = \{a, b, c\}$, $\tau = \{\{a, b\}, \{b\}, \{b, c\}, \phi, X\}$ is a functionally θ-normal space which is not θ-regular.*

Theorem 6. *Every nearly compact weakly θ-regular space is θ-normal.*

Proof. Let A and B be two disjoint closed sets of X where A is θ-closed. Then $A \subset X - B$. Thus by θ-regularity of X there exist an θ-open set V such that $A \subset V \subset X - B$. Since V is θ-open, for every $x \in A$ there exist an open set U_x such that $x \in U_x \subset \overline{U}_x \subset V \subset X - B$. Then $\mathscr{U} = \{u_x : x \in A\}$ is an open cover of A. Since A is θ-closed, by [2, Proposition 2.1], A is N-closed relative to X. Hence $\mathscr{U}$ has finite subcollection such that $A \subset \bigcup_{i=1}^{n} int\overline{U_{x_i}}$. Thus $B \subset \bigcap_{i=1}^{n}(X - \overline{U_{x_i}})$. therefore X is θ-normal.

Corollary 1. *Every nearly compact θ-regular space is normal.*

Proof. The above result is obvious, since every θ-regular θ-normal space is normal.

Remark 2. *The following example shows that the hypothesis of θ-regularity in the above Corollary cannot be weakened to "weak θ-regularity" as nearly compact weakly θ-regular spaces need not be almost normal. e.g.; The set $X = \{a, b, c, d\}$ with topology $\tau = \{\{a, b\}, \{b\}, \{b, c\}, \{c\}, \{b, c, d\}, \{a, b, c\}, X, \emptyset\}$ is compact and weakly θ-regular but not almost normal as the regularly closed set $\{c, d\}$ and closed set $\{a\}$ cannot be separated by disjoint open sets.*

It is well known that every compact Hausdorff space is normal. However, in the absence of Hausdorffness or regularity a compact space may fail to be normal. Thus it is useful to know which topological property weaker than Hausdorffness with compactness implies normality. The property of being a T_1-space fails to do the job since the cofinite topology on an infinite set is a compact T_1 space which is not normal. However, it is well known that Every compact R_1-space is normal

The following result of [6] is an improvement of well known results such as every compact Hausdorff space is normal and every compact (or Lindelöf) regular space is normal.

Theorem 7. *Every paracompact θ-regular space is normal.*

Theorem 8. *Every Lindelöf θ-regular space is normal.*

Remark 3. *The condition of θ-regularity in the above theorem cannot be weakened as the example cited in Remark 2 is a paracompact weakly θ-regular space which fails to be almost normal.*

Although every compact θ-regular space is normal, but it is in the absence of T_1 property, as every T_1 θ-regular space is regular. Thus it is very natural to ask the following Question.

Question 2. *Which non-regular, non-Hausdorff, T_1-compact spaces are normal ?*

Let us recall that a space X is seminormal if for every closed set F contained in an open set U there exists a regularly open set V such that $F \subset V \subset U$. A space is said to be θ-seminormal [15] if for every θ-closed set F contained in an open set U there exists a regularly open set V such that $F \subset V \subset U$.

Example 4. *A seminormal space which is not θ-regular. Let X be the set of positive integers. Define a topology on X by taking every odd integer to be open and a set $U \subset X$ is open if for every even integer $p \in U$, the predecessor and the successor of p are also in U. Since every open set is regularly open in this topology, the space is seminormal but the space is not θ-regular.*

Theorem 9. *Every almost regular seminormal space is θ-regular.*

Proof. Let F be a closed set contained in an open set U. Since X is seminormal there exists a regularly open set V such that $F \subset V \subset U$. Since in an almost regular space every regularly open set is θ-open, the space is θ-regular.

Corollary 2. *An almost regular space is normal if and only if it is seminormal and weakly θ-normal.*

Proof. Proof is obvious, since every θ-regular weakly θ-normal space is normal.

Theorem 10. *Every almost regular θ-seminormal space is weakly θ-regular.*

3. Subspaces

Lemma 1. *If $Y \subset X$ and A is any θ-open set in X then $A \cap Y$ is θ-open in Y.*

Theorem 11. *If Y is a closed subspace of X and X is θ-regular then Y is θ-regular.*

Proof. Let X be a θ-regular space and $Y \subset X$. Let F be a closed set in Y which is contained in an open set U of Y. Since F is closed in Y and Y is a closed subspace of X, F is closed in X. Since U is open in Y, there exists an open set V in X such that $U = V \cap Y$. Thus $F \subset V$. By θ-regularity of X, there exists a θ-open set W in X such that $F \subset W \subset V$, i.e,;

$$F \cap Y \subset W \cap Y \subset V \cap Y \Rightarrow F \subset W \cap Y \subset U.$$

By the previous lemma $W \cap Y$ is θ-open in Y. Hence Y is θ-regular.

Theorem 12. *If Y is a closed subspace of X and X is point θ-regular, then Y is point θ-regular.*

Lemma 2. *If Y is θ-open in X and A is θ-open in Y, then A is θ-open in X.*

Lemma 3. *If Y is θ-open in X and A is θ-closed in Y then A is θ-closed in X.*

Proof. Let Y be a θ-open set in X and let A be θ-closed in Y. Then $(Y - A)$ is θ-open in Y.Thus by previous lemma $(Y - A)$ is θ-open in X. Therefore $X - (Y - A)$ is θ-closed in X. Hence A is θ-closed in X.

Theorem 13. *If Y is a θ-open subspace of X and X is weakly θ-regular, then Y is weakly θ-regular.*

Proof. Let Y be a θ-open subspace of X and X is weakly θ-regular. Let F be a θ-closed set in Y and contained in an open set U of Y. Since Y is θ-open in X, F is θ-closed in X. Since U is open in Y, there exists a open set V in X such that $U = V \cap Y$. So $F \subset V$. By weak θ-regularity of X, there exists a θ-open set W in X such that $F \subset W \subset V$. Thus $F \subset W \cap Y \subset V$, where $W \cap Y$ is θ-open in Y. Hence Y is weakly θ-regular.

Theorem 14. *If Y is a θ-open subspace of X and X is point weakly θ-regular, then Y is point weakly θ-regular.*

References

[1] J. M. Boyte, *Point (countable) paracompactness*, J. Austr. Math. Soc. **15**, 138-144. 1973.

[2] J. Dontchev and T. Noiri, *N-closed subsets of nearly compact spaces*, Acta. Math. Hungar., **86(1-2)**, 117–125. 2000.

[3] D. Jankovic, *θ-regular spaces*, Inter. J. Math. Math. Sc., **8, no.3**, 615-619. 1985.

[4] J.K. Kohli and A.K. Das, *On functionally θ-normal spaces*, Appl. Gen. Topol., **6(1)**, 1–14. 2005.

[5] J. K. Kohli and A. K. Das, *Characterizations of certain sub(super)-classes of Hausdorff spaces and a factorization of regularity*, Indian J. pure appl. Math., **35(4)**, 463-470. 2004.

[6] J.K. Kohli and A.K. Das, *New normality axioms and decompositions of normality*, Glas. Mat. Ser. III **37(57)**, no- 1, 163–173. 2002.

[7] J.K. Kohli and D. Singh, *Weak normality properties and factorizations of normality*, Acta. Math. Hungar. **110(1-2)**, 67–80. 2006.

[8] M.M. Kovar, *On θ-regular spaces*, Inter. J. Math. Math. Sc., **17 (4)** 687-692. 1994.

[9] M.K. Singal and S.P. Arya, *On almost regular spaces* , Glasnik Mat. **4(24)**, 89–99. 1969.

[10] M.K. Singal and S.P. Arya, *On almost normal and almost completely regular spaces*, Glasnik Mat. **5(25)** ,141–152. 1970.

[11] M.K.Singal and A. Mathur, *On nearly compact spaces*, Boll. U.M.I. **4**, 702–710. 1969.

[12] M.K.Singal and A.R. Singal, *Mildly normal spaces*, Kyungpook Math J. **13**, 27–31. 1973.

[13] E.V. Stchepin, *Real valued functions and spaces close to normal*, Sib. J. Math. **13:5**, 1182–1196. 1972

[14] N.V. Veličko *H-closed topological spaces*, Amer. Math. Soc. Transl. **78(2)**,, 103–118. 1968.

[15] G. Vigilino, *Seminormal and C-compact spaces*, Duke J. Math. **38**, 57–61. 1971.