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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70^{TH} BIRTHDAY

Differential Subordination and Superordination on p-Valent Meromorphic Functions Defined by Extended Multiplier Transformations

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Abstract. In this paper we derive some differential subordination and superordination results for p-valent meromorphic functions in the punctured unit disc, which are acted upon by a class of extended multiplier transformations. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $H[a,n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ with $H = H[1,1]$. If $f(z)$ and $g(z)$ are members of $H(U)$, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

$$f \prec g \quad \text{or } f(z) \prec g(z) \ (z \in U),$$

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if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g(w(z))$ $(z\in U)$. Indeed it is known that $f(z)\prec g(z)$ $(z\in U)\Rightarrow f(0)=g(0)$ and $f(U)\subset g(U)$. Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (cf., e.g., [13]; see also [14, p.4])

$$f(z)\prec g(z)\ \Leftrightarrow f(0)=g(0) \text{ and } f(U)\subset g(U).$$

Denote by D the set of all functions $q(z)$ that are analytic and injective on $\overline{U}\backslash E(q)$, where

$$E(q)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\right\},$$

and are such that $q^{'}(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(q)$. Further let the subclass of D for which $q(0)=a$ be denoted by $D(a)$, and $D(1)=D_1$.

The following classes of admissible functions will be required.

Definition 1 (14, Definition 2.3a, p. 27). *Let Ω be a set in $\mathbb{C}, q\in D$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consists of these functions $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z)\notin\Omega$ whenever $r=q(\zeta)$, $s=k\zeta q^{'}(\zeta)$ and*

$$Re\left\{\frac{t}{s}+1\right\}\geq kRe\left\{1+\frac{\zeta q^{''}(\zeta)}{q^{'}(\zeta)}\right\},$$

where $z\in U$, $\zeta\in\partial U\backslash E(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

In particular when $q(z)=M\dfrac{Mz+a}{M+\overline{a}z}$, with $M>0$ and $|a|<M$, then $q(U)=U_M=\{w:|w|<M\}$, $q(0)=a$, $E(q)=\phi$ and $q\in D(a)$. In this case, we set $\Psi_n[\Omega,M,a]=\Psi_n[\Omega,q]$, and in the special case when the set $\Omega=U_M$, the class is simply denoted by $\Psi_n[M,a]$.

Definition 2 (15, Definition 3, p. 817). *Let Ω be a set in $\mathbb{C}, q\in H[a,n]$ with $q^{'}(z)\neq 0$. The class of admissible functions $\Psi^{'}_n[\Omega,q]$ consists of these functions $\psi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\zeta)\in\Omega$ whenever $r=q(z)$, $s=\frac{zq^{'}(z)}{m}$, and*

$$Re\left\{\frac{t}{s}+1\right\}\leq\frac{1}{m}Re\left\{1+\frac{zq^{''}(z)}{q^{'}(z)}\right\},$$

where $z\in U$, $\zeta\in\partial U$ and $m\geq n\geq 1$. In particular, we write $\Psi^{'}_1[\Omega,q]$ as $\Psi^{'}[\Omega,q]$.

In our investigations we shall need the following lemmas.

Lemma 1 (14, Theorem 2.3b, p. 28). *Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If the analytic function $p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$ satisfies*

$$\psi(p(z),zp^{'}(z),z^2p^{''}(z);z)\in\Omega,$$

then $p(z)\prec q(z)$.

Lemma 2 (15, Theorem 1, p. 818). *Let* $\psi \in \Psi_n^{'}[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) \in D(a)$ *and* $\psi(p(z),\ zp^{'}(z),\ z^2p^{''}(z); z)$ *is univalent in* U *then*

$$\Omega \subset \left\{\psi(p(z), zp^{'}(z), z^2p^{''}(z); z) : z \in U\right\}$$

implies $q(z) \prec p(z)$.

Let $\sum(p)$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \ (p \in \mathbb{N} = \{1, 2, \ldots.\}; z \in U^* = U \backslash \{0\}), \tag{1}$$

which are analytic and p-valent in U^*. For functions $f_j(z) \in \sum(p)$, given by

$$f_j(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_{k,j} z^k \quad (j = 1, 2), \tag{2}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{3}$$

Now, using the linear operator $I_p^m(\lambda, \ell)$ $(\lambda \geq 0, \ell > 0, m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\})$ introduced by El-Ashwah [9] for a function $f(z) \in \sum(p)$ given by (1) as follows:

$$I_p^m(\lambda, \ell) f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \left[\frac{\ell + \lambda(k+p)}{\ell}\right]^m a_k z^k, \tag{4}$$

we can write (4) in the form:

$$I_p^m(\lambda, \ell) f(z) = (\Phi_{\lambda,\ell}^{p,m} * f)(z),$$

where

$$\Phi_{\lambda,\ell}^{p,m}(z) = z^{-p} + \sum_{k=1-p}^{\infty} \left[\frac{\ell + \lambda(k+p)}{\ell}\right]^m z^k. \tag{5}$$

It is easily verified from (4) that

$$\lambda z(I_p^m(\lambda, \ell) f(z))^{'} = \ell I_p^{m+1}(\lambda, \ell) f(z) - (\lambda p + \ell) I_p^m(\lambda, \ell) f(z) \ (\lambda > 0). \tag{6}$$

We note that: $I_p^0(\lambda, \ell) f(z) = f(z)$ and $I_p^1(1, 1) f(z) = \dfrac{(z^{p+1} f(z))^{'}}{z^p} = (p+1) f(z) + z f^{'}(z)$.

Also by specializing the parameters λ, ℓ and p, we obtain the following operators studied by various authors:

(i) $I_1^m(1,\ell)f(z) = I(m,\ell)f(z)$ (see Cho et al. [7,8]);

(ii) $I_p^m(1,1)f(z) = D_p^m f(z)$ (see Aouf and Hossen [6], Liu and Owa [11], Liu and Srivastava [12] and Srivastava and Patel [16]);

(iii) $I_1^m(1,1)f(z) = I^m f(z)$ (see Uralegaddi and Somanatha [17]).

Also we note that:

(i) $I_p^m(1,\ell)f(z) = I_p(m,\ell)f(z)$, where $I_p(m,\ell)f(z)$ is defined by

$$I_p(m,\ell)f(z) = z^{-p} + \sum_{k=1-p}^{\infty}\left[\frac{\ell+k+p}{\ell}\right]^m a_k z^k \ (\ell > 0; m \in \mathbb{N}_0); \tag{7}$$

(ii) $I_p^m(\lambda,1)f(z) = D_{\lambda,p}^m f(z)$, where $D_{\lambda,p}^m f(z)$ is defined by

$$D_{\lambda,p}^m f(z) = z^{-p} + \sum_{k=1-p}^{\infty}\left[1+\lambda(k+p)\right]^m a_k z^k \ (\lambda \geq 0; m \in \mathbb{N}_0). \tag{8}$$

Aghalary et al. [1,2], Ali et al. [3,4,5], Aouf and Hossen [6] and Kim and Srivestava [10] obtained sufficient conditions for certain differential subordination implications to hold.

In the present paper, the differential subordination result of Miller and Mocanu [14, Theorem 2.3b, p. 28] is extended for functions associated with the operator $I_p^m(\lambda,\ell)$, and we obtain certain other related results. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination Results Involving the Operator $I_p^m(\lambda,\ell)$

Unless otherwise mentioned, we assume throughout this paper that $\ell > 0,\ \lambda > 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in D_1 \cap H$. The class of admissible functions $\Phi_H[\Omega,q]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + \left(\frac{\ell}{\lambda}\right)q(\zeta)}{\left(\frac{\ell}{\lambda}\right)},$$

$$Re\left\{\frac{\left(\frac{\ell}{\lambda}\right)(w-u)}{v-u} - 2\left(\frac{\ell}{\lambda}\right)\right\} \geq kRe\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U,\ \zeta \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 1. *Let* $\varphi \in \Phi_H[\Omega, q]$. *If* $f(z) \in \sum(p)$ *satisfies*

$$\left\{\varphi(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z)\ ; z) : z \in U\right\} \in \Omega, \tag{9}$$

then

$$z^p I_p^m(\lambda,\ell)f(z) \prec q(z).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = z^p I_p^m(\lambda,\ell)f(z). \tag{10}$$

From (6) and (10), we have

$$z^p I_p^{m+1}(\lambda,\ell)f(z) = \frac{\left(zp^{'}(z) + \left(\frac{\ell}{\lambda}\right)p(z)\right)}{\left(\frac{\ell}{\lambda}\right)}. \tag{11}$$

Further computations show that

$$z^p I_p^{m+2}(\lambda,\ell)f(z) = \frac{z^2p^{''}(z) + \left(1 + 2\left(\frac{\ell}{\lambda}\right)\right)zp^{'}(z) + \left(\frac{\ell}{\lambda}\right)^2 p(z)}{\left(\frac{\ell}{\lambda}\right)^2}. \tag{12}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t) = r,\ v(r,s,t) = \frac{s + \left(\frac{\ell}{\lambda}\right)r}{\left(\frac{\ell}{\lambda}\right)},\ w(r,s,t) = \frac{t + \left(1 + 2\left(\frac{\ell}{\lambda}\right)\right)s + \left(\frac{\ell}{\lambda}\right)^2 r}{\left(\frac{\ell}{\lambda}\right)^2}. \tag{13}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \varphi(u,v,w;z)\\ &= \varphi\left(r, \frac{s + \left(\frac{\ell}{\lambda}\right)r}{\left(\frac{\ell}{\lambda}\right)},\ \frac{t + \left(1 + 2\left(\frac{\ell}{\lambda}\right)\right)s + \left(\frac{\ell}{\lambda}\right)^2 r}{\left(\frac{\ell}{\lambda}\right)^2}; z\right).\end{aligned} \tag{14}$$

The proof will make use of Lemma 1. Using (10), (11) and (12), from (14), we obtain

$$\begin{aligned}&\psi(p(z), zp^{'}(z), z^2p^{''}(z); z)\\ &= \varphi\left(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z), z^p I_p^{m+2}(\lambda,\ell)f(z); z\right).\end{aligned} \tag{15}$$

Hence (9) becomes

$$\psi(p(z), zp^{'}(z), z^2p^{''}(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{\left(\frac{\ell}{\lambda}\right)(w - u)}{v - u} - 2\left(\frac{\ell}{\lambda}\right),$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1, $p(z) \prec q(z)$ or $z^p I_p^m(\lambda, \ell) f(z) \prec q(z)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$.

The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let $\varphi \in \Phi_H[h, q]$ with $q(0) = 1$. If $f(z) \in \sum(p)$ satisfies*

$$\varphi(z^p I_p^m(\lambda, \ell) f(z), z^p I_p^{m+1}(\lambda, \ell) f(z), z^p I_p^{m+2}(\lambda, \ell) f(z); z) \prec h(z), \tag{16}$$

then

$$z^p I_p^m(\lambda, \ell) f(z) \prec q(z).$$

Our next result is an extension of Theorem 1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U, $q(0) = 1$. Let $\varphi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where, $q_\rho(z) = q(\rho z)$. If $f \in \sum(p)$ and*

$$\varphi(z^p I_p^m(\lambda, \ell) f(z), z^p I_p^{m+1}(\lambda, \ell) f(z), z^p I_p^{m+2}(\lambda, \ell) f(z); z) \in \Omega,$$

then

$$z^p I_p^m(\lambda, \ell) f(z) \prec q(z).$$

Proof. Theorem 1 yields $z^p I_p^m(\lambda, \ell) f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 3. *Let $h(z)$ and $q(z)$ be univalent in U, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) $\varphi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or

(2) there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in \sum(p)$ satisfies (16), then

$$z^p I_p^m(\lambda, \ell) f(z) \prec q(z).$$

Proof. The proof is similar to [14, Theorem 2.3d, p. 30] and is therefore omitted.

The next theorem yields the best dominant of the differential subordination (16).

Theorem 4. *Let $h(z)$ be univalent in U, and$\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi\left(p(z), \frac{zp'(z) + \left(\frac{\ell}{\lambda}\right)p(z)}{\left(\frac{\ell}{\lambda}\right)}, \frac{z^2p''(z) + \left(1 + 2\left(\frac{\ell}{\lambda}\right)\right)zp'(z) + \left(\frac{\ell}{\lambda}\right)^2 p(z)}{\left(\frac{\ell}{\lambda}\right)^2}; z\right) = h(z) \tag{17}$$

has a solution $q(z)$ with $q(0) = 1$ and satisfy one of the following conditions:

(1) $q(z) \in D_1$ *and* $\varphi \in \Phi_H[h, q]$,

(2) $q(z)$ *is univalent in* U *and* $\varphi \in \Phi_H[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*

(3) $q(z)$ *is univalent in* U *and there exists* $\rho_0 \in (0,1)$ *such that* $\varphi \in \Phi_H[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

If $f(z) \in \sum(p)$ *satisfies (16), then*

$$z^p I_p^m(\lambda,\ell) f(z) \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. Following the same arguments in [14, Theorem 2.3e, p. 31], we deduce that $q(z)$ is a dominant from Theorems 2 and 3. Since $q(z)$ satisfies (17) it is also a solution of (16) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the particular case $q(z) = 1 + Mz,\ M > 0$, and in view of Definition 3, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, is described below.

Definition 4. *Let* Ω *be a set in* $\mathbb{C}$ *and* $M > 0$. *The class of admissible functions* $\Phi_H[\Omega, M]$ *consists of those functions* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ *such that*

$$\varphi\left(1 + Me^{i\theta}, 1 + \frac{k + \left(\frac{\ell}{\lambda}\right)}{\left(\frac{\ell}{\lambda}\right)} Me^{i\theta}, 1 + \frac{L + \left[\left(1 + 2\left(\frac{\ell}{\lambda}\right)\right)k + \left(\frac{\ell}{\lambda}\right)^2\right] Me^{i\theta}}{\left(\frac{\ell}{\lambda}\right)^2}; z\right) \notin \Omega \qquad (18)$$

whenever $z \in U,\ \theta \in \mathbb{R}$, *Re* $\left(Le^{-i\theta}\right) \geq (k-1)kM$ *for all real* θ *and* $k \geq 1$.

Corollary 2. *Let* $\varphi \in \Phi_H[\Omega, M]$. *If* $f(z) \in \sum(p)$ *satisfies*

$$\varphi(z^p I_p^m(\lambda,\ell) f(z),\ z^p I_p^{m+1}(\lambda,\ell) f(z),\ z^p I_p^{m+2}(\lambda,\ell) f(z)\ ; z) \in \Omega,$$

then

$$\left| z^p I_p^m(\lambda,\ell) f(z) - 1 \right| < M.$$

In the special case $\Omega = q(U) = \{w : |w - 1| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$. Corollary 2 can be written as:

Corollary 3. *Let* $\varphi \in \Phi_H[M]$. *If* $f(z) \in \sum(p)$ *satisfies*

$$\left| \varphi(z^p I_p^m(\lambda,\ell) f(z),\ z^p I_p^{m+1}(\lambda,\ell) f(z),\ z^p I_p^{m+2}(\lambda,\ell) f(z);\ z) - 1 \right| < M,$$

then

$$\left| z^p I_p^m(\lambda,\ell) f(z) - 1 \right| < M.$$

Corollary 4. *If $M > 0$ and $f(z) \in \sum(p)$ satisfies*

$$\left| z^p I_p^{m+1}(\lambda,\ell)f(z) - z^p I_p^m(\lambda,\ell)f(z) \right| < \frac{M}{\left(\frac{\ell}{\lambda}\right)},$$

then

$$\left| z^p I_p^m(\lambda,\ell)f(z) - 1 \right| < M. \tag{19}$$

Proof. The proof follows from Corollary 2 by taking $\varphi(u,v,w;z) = v - u$ and $\Omega = h(U)$, where $h(z) = \frac{Mz}{\left(\frac{\ell}{\lambda}\right)}$, $M > 0$. To use Corollary 2, we need to show that $\varphi \in \Phi_H[\Omega, M]$, that is, the admissible condition (18) is satisfied. This follows since

$$\left| \varphi\left(1 + Me^{i\theta}, 1 + \frac{k+\left(\frac{\ell}{\lambda}\right)}{\left(\frac{\ell}{\lambda}\right)} Me^{i\theta}, 1 + \frac{L + \left\{\left(2\left(\frac{\ell}{\lambda}\right)+1\right)k + \left(\frac{\ell}{\lambda}\right)^2\right\} Me^{i\theta}}{\left(\frac{\ell}{\lambda}\right)^2}; z\right) \right| = \frac{kM}{\left(\frac{\ell}{\lambda}\right)} \geq \frac{M}{\left(\frac{\ell}{\lambda}\right)},$$

where $z \in U$, $\theta \in \mathbb{R}$, and $k \geq 1$. Hence by Corollary 2, we deduce the required result.

Theorem 4 shows that the result is sharp. The differential equation

$$\frac{zq'(z)}{\left(\frac{\ell}{\lambda}\right)} = \frac{M}{\left(\frac{\ell}{\lambda}\right)} z \quad (\ell < \lambda M)$$

has a univalent solution $q(z) = 1 + Mz$. It follows from Theorem 4 that $q(z) = 1 + Mz$ is the best dominant.

Definition 5. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in D_1 \cap H$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left(\left(\frac{\ell}{\lambda}\right) q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right) (q(\zeta) \neq 0),$$

$$Re\left\{\frac{\left(\frac{\ell}{\lambda}\right) v(w-v)}{v-u} - \left(\frac{\ell}{\lambda}\right)(2u - v)\right\} \geq kRe\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 5. *Let $\varphi \in \Phi_{H,1}[\Omega, q]$. If $f(z) \in \sum(p)$ satisfies*

$$\left\{\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{20}$$

then

$$\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \prec q(z).$$

Proof. Define an analytic function $p(z)$ in U by

$$p(z) = \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}. \tag{21}$$

By making use of (6) and (21), we obtain

$$\frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} = p(z) + \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left[\frac{zp^{'}(z)}{p(z)}\right]. \tag{22}$$

Further computations show that

$$\frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)} = p(z) + \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left[\frac{zp^{'}(z)}{p(z)} + \frac{\left(\frac{\ell}{\lambda}\right)zp^{'}(z) + \frac{zp^{'}(z)}{p(z)} - \left(\frac{zp^{'}(z)}{p(z)}\right)^2 + \frac{z^2p^{''}(z)}{p(z)}}{\left(\frac{\ell}{\lambda}\right)p(z) + \frac{zp^{'}(z)}{p(z)}}\right]. \tag{23}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = r + \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left(\frac{s}{r}\right), w = r + \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left[\frac{s}{r} + \frac{\left(\frac{\ell}{\lambda}\right)s + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + \frac{t}{r}}{\left(\frac{\ell}{\lambda}\right)r + \frac{s}{r}}\right]. \tag{24}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \varphi(u,v,w;z)\\ &= \varphi\left(r, \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left[\left(\frac{\ell}{\lambda}\right)r + \frac{s}{r}\right], \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left[\left(\frac{\ell}{\lambda}\right)r + \frac{s}{r} + \frac{\left(\frac{\ell}{\lambda}\right)s + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + \frac{t}{r}}{\left(\frac{\ell}{\lambda}\right)r + \frac{s}{r}}\right]; z\right).\end{aligned} \tag{25}$$

Using equations (21), (22) and (23), from (25), we obtain

$$\psi(p(z), zp^{'}(z), z^2p^{''}(z); z) = \varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right). \tag{26}$$

Hence (20) implies

$$\psi(p(z), zp^{'}(z), z^2p^{''}(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{\left(\frac{\ell}{\lambda}\right)v(w-v)}{v-u} - \left(\frac{\ell}{\lambda}\right)(2u - v),$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1, $p(z) \prec q(z)$ or

$$\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, with $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$.

The following theorem is an immediate consequence of Theorem 5.

Theorem 6. *Let $\varphi \in \Phi_{H,1}[h, q]$ with $q(0) = 1$. If $f(z) \in \sum(p)$ satisfies*

$$\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) \prec h(z), \tag{27}$$

then

$$\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)} \prec q(z).$$

In the particular case $q(z) = 1 + Mz$, $M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$ becomes the class $\Phi_{H,1}[\Omega, M]$.

Definition 6. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$\varphi\left(1 + Me^{i\theta}, 1 + \frac{k + \left(\frac{\ell}{\lambda}\right)(1 + Me^{i\theta})}{\left(\frac{\ell}{\lambda}\right)(1 + Me^{i\theta})}Me^{i\theta}, 1 + \frac{k + \left(\frac{\ell}{\lambda}\right)(1 + Me^{i\theta})}{\left(\frac{\ell}{\lambda}\right)(1 + Me^{i\theta})}Me^{i\theta} + \right.$$
$$\left.\frac{(M + e^{-i\theta})\left\{Le^{-i\theta} + \left(\left(\frac{\ell}{\lambda}\right) + 1\right)kM + \left(\frac{\ell}{\lambda}\right)kM^2e^{i\theta}\right\} - k^2M^2}{\left(\frac{\ell}{\lambda}\right)(M + e^{-i\theta})\left\{\left(\frac{\ell}{\lambda}\right)e^{-i\theta} + \left(2\left(\frac{\ell}{\lambda}\right) + k\right)M + \left(\frac{\ell}{\lambda}\right)M^2e^{i\theta}\right\}}; z\right) \notin \Omega, \tag{28}$$

where $z \in U$, $\theta \in \mathbb{R}$, Re $\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real θ and $k \geq 1$.

Corollary 5. *Let $\varphi \in \Phi_{H,1}[\Omega, M]$. If $f(z) \in \sum(p)$ satisfies*

$$\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)} - 1\right| < M.$$

In the special case $\Omega = q(U) = \{w : |w - 1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$, and Corollary 5 takes the following form:

Corollary 6. *Let $\varphi \in \Phi_{H,1}[M]$. If $f(z) \in \sum(p)$ satisfies*

$$\left|\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)} - 1\right| < M.$$

Corollary 7. *If* $M > 0$ *and* $f(z) \in \sum(p)$ *satisfies*

$$\left| \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} - \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)} \right| < \frac{M}{\left(\frac{\ell}{\lambda}\right)(1+M)},$$

then

$$\left| \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)} - 1 \right| < M.$$

Proof. This follows from Corollary 6 by taking $\varphi(u,v,w;z) = v - u$ and $\Omega = h(U)$, where $h(z) = \frac{M}{\left(\frac{\ell}{\lambda}\right)(1+M)} z$, $M > 0$. To use Corollary 6, we need to show that $\varphi \in \Phi_{H,1}[M]$, that is, the admissible condition (28) is satisfied. This follows since

$$\begin{aligned} \left|\varphi(u,v,w;z)\right| &= \left| -1 - Me^{i\theta} + 1 + \frac{k + \left(\frac{\ell}{\lambda}\right)(1+Me^{i\theta})}{\left(\frac{\ell}{\lambda}\right)(1+Me^{i\theta})} Me^{i\theta} \right| \\ &= \left| \frac{kMe^{i\theta}}{\left(\frac{\ell}{\lambda}\right)(1+Me^{i\theta})} \right| \geq \frac{M}{\left(\frac{\ell}{\lambda}\right)(1+M)}, \end{aligned}$$

for $z \in U$, $\theta \in \mathbb{R}$, $\lambda > 0$, $\ell > 0$ and $k \geq 1$. Hence by Corollary 6, we deduce the required result.

3. Superordination Results Involving the Operator $I_p^m(\lambda,\ell)$

In this section we obtain differential superordination for the operator $I_p^m(\lambda,\ell)$. For this purpose the class of admissible functions is given in the following definition.

Definition 7. *Let* Ω *be a set in* $\mathbb{C}$ *and* $q(z) \in H$ *with* $zq^{'}(z) \neq 0$*. The class of admissible functions* $\Phi_H^{'}[\Omega, q]$ *consists of those functions* $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\varphi(u,v,w;\zeta) \in \Omega$$

whenever

$$u = q(z), v = \frac{zq^{'}(z) + m\left(\frac{\ell}{\lambda}\right)q(z)}{m\left(\frac{\ell}{\lambda}\right)},$$

$$Re\left\{ \frac{\left(\frac{\ell}{\lambda}\right)(w-u)}{v-u} - 2\left(\frac{\ell}{\lambda}\right) \right\} \leq \frac{1}{m} Re\left\{ 1 + \frac{zq^{''}(z)}{q^{'}(z)} \right\},$$

where $z \in U$*,* $\zeta \in \partial U$ *and* $m \geq 1$.

Theorem 7. *Let* $\varphi \in \Phi_H^{'}[\Omega, q]$. *If* $f(z) \in \sum(p)$, $z^p I_p^m(\lambda,\ell)f(z) \in D_1$ *and*

$$\varphi\left(z^p I_p^m(\lambda,\ell)f(z)\,, z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z)\,; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\varphi\left(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z);\ z\right) : z \in U\right\} \tag{29}$$

implies

$$q(z) \prec z^p I_p^m(\lambda,\ell)f(z).$$

Proof. Let $p(z)$ defined by (10) and $\psi(z)$ defined by (15). Since $\varphi \in \Phi_H^{'}[\Omega, q]$, from (15) and (29), we have

$$\Omega \subset \left\{\psi(p(z),\ zp^{'}(z),\ z^2 p^{''}(z);\ z) : z \in U\right\}.$$

From (14), we see that the admissibility condition for $\varphi \in \Phi_H^{'}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi^{'}[\Omega, q]$, and by Lemma 2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p I_p^m(\lambda,\ell)f(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ for U onto Ω. In this case the class $\Phi_H^{'}[h(U), q]$ is written as $\Phi_H^{'}[h, q]$.

Proceeding similarly as in Section 2, the following result is an immediate consequence of Theorem 7.

Theorem 8. *Let* $q(z) \in H$, $h(z)$ *is analytic on U and* $\varphi \in \Phi_H^{'}[h, q]$. *If* $f(z) \in \sum(p)$, $z^p I_p^m(\lambda,\ell)f(z) \in D_1$ *and* $\varphi(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z); z)$*is univalent in U, then*

$$h(z) \prec \varphi(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z)\,; z) \tag{30}$$

implies

$$q(z) \prec z^p I_p^m(\lambda,\ell)f(z).$$

Theorem 7 and Theorem 8 can only be used to obtain subordinants of differential superordination of the form (29) or (30).

The following theorem proves the existence of the best subordinant of (30) for certain φ.

Theorem 9. *Let* $h(z)$ *be analytic in U and* $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\varphi\left(p(z),\ \frac{zp^{'}(z) + \left(\frac{\ell}{\lambda}\right)p(z)}{\left(\frac{\ell}{\lambda}\right)}, \frac{z^2 p^{''}(z) + \left(2\left(\frac{\ell}{\lambda}\right) + 1\right)zp^{'}(z) + \left(\frac{\ell}{\lambda}\right)^2 p(z)}{\left(\frac{\ell}{\lambda}\right)^2}; z\right) = h(z) \tag{31}$$

has a solution $q(z) \in D_1$. *If* $\varphi \in \Phi_H^{'}[h,q]$, $f(z) \in \sum(p)$, $z^p I_p^m(\lambda,\ell)f(z) \in D_1$ *and*

$$\varphi\left(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z);\ z\right)$$

is univalent in U, then

$$h(z) \prec \varphi\left(z^p I_p^m(\lambda,\ell)f(z),\ z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z);\ z\right)$$

implies

$$q(z) \prec z^p I_p^m(\lambda,\ell)f(z)$$

and $q(z)$ *is the best subordinant.*

Proof. The proof is similar to the proof of Theorem 4 and is therefore omitted.

Combining Theorems 2 and 8, we obtain the following sandwich theorem.

Corollary 8. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in U,* $h_2(z)$ *be univalent function in U,* $q_2(z) \in D_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\varphi \in \Phi_H[h_2,q_2] \cap \Phi_H^{'}[h_1,q_1]$. *If* $f(z) \in \sum(p)$, $z^p I_p^m(\lambda,\ell)f(z) \in H \cap D_1$ *and*

$$\varphi\left(z^p I_p^m(\lambda,\ell)f(z)\ , z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z);\ z\right)$$

is univalent in U, then

$$h_1(z) \prec \varphi\left(z^p I_p^m(\lambda,\ell)f(z)\ , z^p I_p^{m+1}(\lambda,\ell)f(z),\ z^p I_p^{m+2}(\lambda,\ell)f(z);\ z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec z^p I_p^m(\lambda,\ell)f(z) \prec q_2(z).$$

Definition 8. *Let* Ω *be a set in* $\mathbb{C}$ *with* $q(z) \in H$ *and* $zq^{'}(z) \neq 0$. *The class of admissible functions* $\Phi_{H,1}^{'}[\Omega,q]$ *consists of those functions* $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\varphi(u,v,w;\zeta) \in \Omega$$

whenever

$$u = q(z), v = q(z) + \frac{1}{\left(\frac{\ell}{\lambda}\right)}\left(\frac{zq^{'}(z)}{mq(z)}\right)\ (q(z) \neq 0)$$

$$Re\left\{\frac{\left(\frac{\ell}{\lambda}\right)v(w-v)}{v-u} - \left(\frac{\ell}{\lambda}\right)(2u-v)\right\} \leq \frac{1}{m}Re\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\},$$

where $z \in U$, $\zeta \in \partial U$ *and* $m \geq 1$.

Now we will give the dual result of Theorem 5 for differential superordination.

Theorem 10. *Let* $\varphi \in \Phi'_{H,1}[\Omega, q]$. *If* $f(z) \in \sum(p)$, $\dfrac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \in D_1$ *and*

$$\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) : z \in U\right\}. \tag{32}$$

implies

$$q(z) \prec \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}.$$

Proof. Let $p(z)$ defined by (21) and ψ defined by (25). Since $\varphi \in \Phi'_{H,1}[\Omega, q]$, from (26) and (32), we have $\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\right\}$. From (25), we see that the admissibility condition for $\varphi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{H,1}[h(U), q]$ is written as $\Phi'_{H,1}[h, q]$.

The following result is an immediate consequence of Theorem 10.

Theorem 11. *Let* $q(z) \in H$, $h(z)$ *be analytic in U and* $\varphi \in \Phi'_{H,1}[h, q]$. *If* $f(z) \in \sum(p)$, $\dfrac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \in D_1$ *and*

$$\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right)$$

is univalent in U, then

$$h(z) \prec \varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right), \tag{33}$$

implies

$$q(z) \prec \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}.$$

Combining Theorems 6 and 11, we obtain the following sandwich-type theorem.

Corollary 9. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in D_1$ with $q_1(0) = q_2(0) = 1$ and $\varphi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f(z) \in \sum(p)$, $\dfrac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \in H \cap D_1$ and*

$$\varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \varphi\left(\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)}, \frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}, \frac{I_p^{m+3}(\lambda,\ell)f(z)}{I_p^{m+2}(\lambda,\ell)f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} \prec q_2(z).$$

Remark 1.

(i) Putting $\lambda = 1$ in the above results we obtain results associated with the operator $I_p(m,\ell)$ which defined by (7);

(ii) Putting $\ell = 1$ in the above results we obtain results associated with the operator $D^m_{\lambda,p}$ which defined by (8).

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