EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 20-33
ISSN 1307-5543 – www.ejpam.com

On $\beta^* g-$closed Sets and New Separation Axioms

Ahu Açıkgöz

Department of Mathematics, Aksaray University, 68100 Aksaray

Abstract. In this paper, by using β^*-set [24] we introduce a new class of sets called $\beta^* g-$closed sets, which is stronger than $g-$closed sets and weaker than closed sets. We define two new separation axioms called $\beta^* T_{1/2}$ and $\beta^{**} T_{1/2}$ spaces as applications of $\beta^* g-$closed sets. The notions $\beta^* g-$continuity and $\beta^* g-$irresoluteness are also introduced.

2000 Mathematics Subject Classifications: 54C08, 54A05

Key Words and Phrases: β^*-set, $\beta^* g-$closed set, $\beta^* g-$continuous,$\beta^* T_{1/2}$ space

1. Introduction and Preliminaries

To date, many studies have been made on closed sets and set concepts derived from this set. The concept of $g-$closed sets was introduced by Levine [17] and was used to obtain a $T_{1/2}$ space in which the closed sets and $g-$closed sets coincide. This natural generalization of a closed set concept has made it possible to use the concept in many areas, especially in quantum physics [13] and computer graphics [13-15]. The notion has been studied extensively in recent years by many topologists. More importantly several new separations which are between T_0 and T_1 such as $T_{1/2}$, T_{gs}, $\pi gp - T_{1/2}$ and $T_{3/4}$ are suggested. Some of these have been found to be useful in computer science and digital topology (see [7, 12-15], for example). As a brief literature review, related studies of $g-$closed sets can be summarized as follows. Dontchev and Noiri [8] introduced the notion of $rg-$closed sets which are weaker than that of $g-$closed sets. Kumar [16] defined the notion of g^*-closed sets that are generalizations of $g-$closed sets and introduced $T^*_{1/2}$ and ${}^*T_{1/2}$ spaces as applications of g^*-closed sets. Devi et al. [6] introduced and studied $gs-$closed and $sg-$closed sets which are weaker than $g-$closed sets. Arya and Nour [1] gave some properties of s-normal spaces by using $gs-$open sets. The notion of s-normal space was studied extensively by Noiri [20]. Zaitsev [25] introduced the notions of π-closed sets and guasi normal spaces.

Dontchev and Noiri [8] introduced the notion of πg-closed sets and obtained some theorems in quasi normal spaces by using this notion. Of course, these studies of general topology

Email address: ahuacikgoz@aksaray.edu.tr

are not limited with these [4,9-10].More recently, several topologists have defined new separation axioms at topological space by giving some convenient definitions of variety. Park [21] has introduced the class of $\pi gp-$closed sets which is weaker than $gp-$closed and stronger than gpr-closed sets. Park and Park [22] further studied the class of $\pi gp-$closed sets and defined the concepts πGP-compactness and πGP-connectedness. Aslım et al. [2] have introduced the notions of $\pi gs-$closed sets which are implied by that of $gs-$closed sets and $\pi gs - T_{1/2}$-spaces. On the other hand, recently Yuksel and Beceren [24] have defined the notion of β^*-set and established a decomposition of continuity. At this point, we shall introduce and study the notions of $\beta^* g-$closed sets which are situated between the class of closed sets and $g-$closed sets. Using these sets, we introduce two new separation axioms called $\beta^* T_{1/2}$ and $\beta^{**} T_{1/2}$. (Both $\beta^* T_{1/2}$ and $\beta^{**} T_{1/2}$ contain the class of $T_{1/2}$ spaces.) We show that the class of $\beta^{**} T_{1/2}$ spaces is the dual of the class of $\beta^* T_{1/2}$ spaces to the class of $T_{1/2}$ spaces. We also introduce $\beta^* g-$continuity and $\beta^* g-$irresolute functions for preservation theorems. It should be mentioned that the present work may be found relevant to work of Witten [23].

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be locally closed (briefly, LC-set) [3] if $A = U \cap V$, where U is open and V is closed. A subset A is said to be regular open (resp. regular closed) if $A = Int(Cl(A)$ (resp. $A = Cl(Int(A))$. The finite union of regular open sets is said to be π-open. The complement of a π-open set is said to be π-closed. A subset A is said to be semiopen [5] if $A \subset Cl(Int(A))$ and the complement of a semiopen set is called semiclosed. The intersection of all semiclosed sets containing A is called the semiclosure [5] of A and is denoted by $sCl(A)$. Dually the semiinterior [5] of A is defined to be the union of all semiopen sets contained in A and is denoted by $sInt(A)$. A subset A is said to be pre open [19] if $A \subset Int(Cl(A))$ and the complement of a pre open set is called pre closed. The intersection of all preclosed sets containing A is called the preclosure [19] of A and is denoted by $pCl(A)$. Dually the preinterior [19] of A is defined to be the union of all pre open sets contained in A and is denoted by $pInt(A)$. Note that $sCl(A) = A \cup Int(Cl(A))$, $sInt(A) = A \cap Cl(Int(A))$, $pCl(A) = A \cup Cl(Int(A))$ and $pInt(A) = A \cap Int(Cl(A))$.

2. $\beta^* g-$-closed Sets

Definition 1. *A subset A of a space (X,τ) is called a generalized closed set (briefly, $g-$closed) [17] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a $g-$closed set is called a $g-$open set.*

(a) a regular generalized closed set (for short, $rg-$closed) [8] if $Cl(A) \subset U$ whenever $A \subset U$ and U is regular open in X;

(b) g^*-closed [16] if $Cl(A) \subset U$ whenever $A \subset U$ and U is $g-$open;

(c) $\pi g-$closed [8] if $Cl(A) \subset U$ whenever $A \subset U$ and U is π-open in X;

(d) $gp-$closed [18] if $pCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(e) $gs-$*closed [1] if* $sCl(A) \subset U$ *whenever* $A \subset U$ *and* U *is open in X;*

(f) $\pi gp-$*closed [21] if* $pCl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* π*-open inX;*

(g) $\pi gs-$*closed [2] if* $sCl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* π*-open in X;*

(h) $\pi gs-$*open (resp.* g^**-open,* $\pi g-$*open,* $gp-$*open,* $\pi gp-$*open,* $gs-$*open) if the complement of A is* $\pi gs-$*closed (resp.* g^**-closed,* $\pi g-$*closed,* $gp-$*closed,* $\pi gp-$*closed,* $gs-$*closed).*

Definition 2. *A subset A of a space* (X,τ) *is called*

(a) a β^**-set [24] if* $A = U \cap V$*, where* U *is open and* $Int(V) = Cl(Int(V))$.

(b) $\beta^* g-$*closed if* $Cl(A) \subset U$ *whenever* $A \subset U$ *and* U *is a* β^**-set.*

(c) $\beta^* sg-$*closed if* $sCl(A) \subset U$ *whenever* $A \subset U$ *and* U *is a* β^**-set.*

(d) $\beta^* pg-$*closed if* $pCl(A) \subset U$ *whenever* $A \subset U$ *and* U *is a* β^**-set.*

(e) $\beta^* pg-$*open (resp.* $\beta^* g-$*open,* $\beta^* sg-$*open) if the complement of A is* $\beta^* pg-$*closed (resp.* $\beta^* g-$*closed,* $\beta^* sg-$*closed).*

The class of all $\beta^* g-$closed subsets of (X,τ) is denoted by $\beta^* GC(X,\tau)$.

Levine [17] and Kumar [16] gave the following diagrams using some of the expressions, respectively.

DIAGRAM I.

closed set $\longrightarrow$ $g-$closed set $\longrightarrow$ $rg-$closed set

DIAGRAM II.

closed set $\longrightarrow$ g^*-closed set $\longrightarrow$ $g-$closed s

Furhermore, Aslım et al. [2] indicated that every $gs-$closed set is a $\pi gs-$closed set and every $\pi g-$closed set is a $\pi gs-$closed set. They gave the following diagram using these properties.

DIAGRAM III.

$$\begin{array}{ccccccc}
 & & \text{pre-closed} & \longrightarrow & gp-\text{closed} & \longrightarrow & \pi gp-\text{closed} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\pi - closed & \longrightarrow & \text{closed} & \longrightarrow & g-\text{closed} & \longrightarrow & \pi g-\text{closed} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{semi-closed} & \longrightarrow & gs-\text{closed} & \longrightarrow & \pi gs-\text{closed}
\end{array}$$

Remark 1. *A LC-set is independent from a* $g-$*closed set as it can be seen from the next two examples.*

Remark 2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then $\{a\}$ is a LC-set, but it is not a $g-$closed set.*

Remark 3. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then $\{a, b\}$ is a $g-$closed set, but it is not a LC-set.*

Theorem 1. *For a subset A of a topological space (X, τ), the following are equivalent:*

(a) A is a LC-set.

(b) $A = U \cap Cl(A)$ for some U open set.

Proof.

(a) $\rightarrow (b)$: Since A is a LC-set, then $A = U \cap V$, where U is open and V is closed. So, $A \subset U$ and $A \subset V$. Hence, $Cl(A) \subset Cl(V)$. Therefore, $A \subset U \cap Cl(A) \subset U \cap Cl(V) = U \cap V = A$. Thus, $A = U \cap Cl(A)$.

(b) $\rightarrow (a)$: It is obvious because $Cl(A)$ is closed.

Theorem 2. *For a subset A of a topological space (X, τ), the following are equivalent:*

(a) A is closed.

(b) A is a LC-set and $g-$closed.

Proof.

(a) $\rightarrow (b)$: This is obvious.

(b) $\rightarrow (a)$: Since A is a LC-set, then $A = U \cap Cl(A)$, where U is an open set in X. So, $A \subset U$ and since A is $g-$closed, then $Cl(A) \subset U$. Therefore, $Cl(A) \subset U \cap Cl(A) = A$. Hence, A is closed.

Theorem 3. *Let (X, τ) be a topological space. Then we have*

(a) Every closed set is a β^*g-closed set.

(b) Every β^*g-closed set is a $g-$closed set.

Proof.

(a) This is obvious.

(b) Let A be a β^*g-closed set of (X, τ) and $A \subset U$ where $U \in \tau$. Since every open set is a β^*-set, so U is a β^*-set of (X, τ). Since A is a β^*g-closed set, we obtain that $Cl(A) \subset U$, hence A is a $g-$closed set of (X, τ).

Remark 4. *The converses of Theorem 3 need not be true as shown in the following examples.*

Example 1. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. *Then* $\{a, b\}$ *is a* $\beta^* g-$*closed set, but it is not a closed set.*

Example 2. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Then* $\{c\}$ *is a* $g-$*closed set, but it is not a* $\beta^* g-$*closed set.*

Theorem 4. *Let* (X, τ) *be a topological space. Then we have*

(a) Every $\beta^* g-$*closed set is a* $\beta^* pg-$*closed set.*

(b) Every $\beta^* g-$*closed set is a* $\beta^* sg-$*closed set.*

Proof. This is obvious.

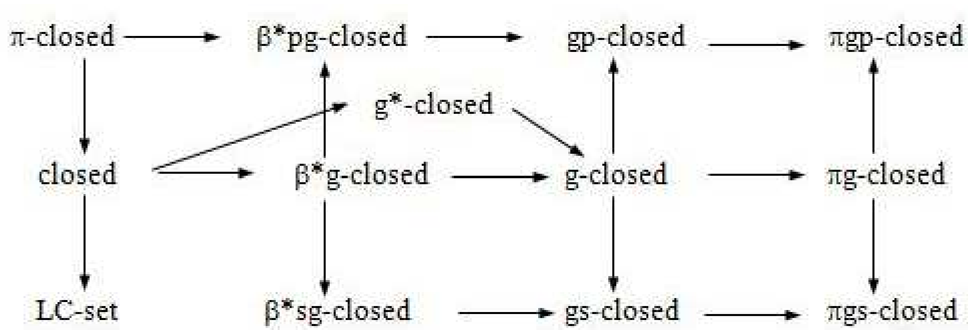
Remark 5. *The converses of Theorem 4 need not be true as shown in the following examples.*

Example 3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Then* $\{a, b\}$ *is a* $\beta^* pg-$*closed set which is not a* $\beta^* g-$*closed set.*

Example 4. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. *Then* $\{b, c\}$ *is a* $\beta^* sg-$*closed set which is not a* $\beta^* g-$*closed set.*

It can be expanded to the following diagram using Diagrams I, II and III

DIAGRAM IV.



Remark 6. *By the two examples stated below, we show that* $\beta^* g-$*closed and* g^**-closed are independent of each other.*

Example 5. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. *Then* $\{a, b\}$ *is a* $\beta^* g-$*closed set, but it is not a* g^**-closed set.*

Example 6. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Then* $\{c\}$ *is a* g^**-closed set, but it is not a* $\beta^* g-$*closed set.*

Remark 7. *A β^*-set is independent from $\beta^* g-$closed as it can be seen from the next two examples.*

Example 7. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then $\{a\}$ is a β^*-set, but it is not a $\beta^* g-$closed set.*

Example 8. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then $\{a, b\}$ is a $\beta^* g-$closed set, but it is not a β^*-set.*

Theorem 5. *If A is both β^*-set and $\beta^* g-$closed set of (X, τ), then A is closed.*

Proof. Let A be both β^*-set and $\beta^* g-$closed set of (X, τ). Then $Cl(A) \subset A$, whenever A is a β^*-set and $A \subset A$. So we obtain that $A = Cl(A)$ and hence A is closed.

Proposition 1. *If A and B are $\beta^* g-$closed sets, then $A \cup B$ is $\beta^* g-$closed.*

Proof. Let $A \cup B \subseteq U$, where U is a β^*-set. Since A, B are $\beta^* g-$closed sets, $Cl(A) \subseteq U$ and $Cl(B) \subseteq U$, whenever $A \subseteq U$, $B \subseteq U$ and U is a β^*-set. Therefore,
$Cl(A \cup B) = Cl(A) \cup Cl(B) \subseteq U$. Hence we obtain that $A \cup B$ is a $\beta^* g-$closed set of (X, τ).

Remark 8. *The intersection of two $\beta^* g-$closed sets are not always a $\beta^* g-$closed set.*

Example 9. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\{a, b\}$ and $\{b, c\}$ are $\beta^* g-$closed sets, but $\{a, b\} \bigcap \{b, c\} = \{b\}$ is not $\beta^* g-$closed.*

Theorem 6. *If A is a $\beta^* g-$closed set of (X, τ) such that $A \subset B \subset Cl(A)$, then B is also a $\beta^* g-$closed set of (X, τ).*

Proof. Let U be a β^*-set of (X, τ) such that $B \subset U$. Then $A \subset U$. Since A is $\beta^* g-$closed, we have $Cl(A) \subset U$. Now $Cl(B) \subset Cl(Cl(A)) = Cl(A) \subset U$. Therefore, B is also a $\beta^* g-$closed set of (X, τ).

Theorem 7. *For any topological space (X, τ), every singleton $\{x\}$ of X is a β^*-set.*

Proof. Let $x \in X$, If $\{x\} \in \tau$, then $\{x\}$ is a β^*-set [3]. If $\{x\} \notin \tau$,
then $Int(\{x\}) = \emptyset = Cl(Int(\{x\}))$, so $\{x\}$ is a β^*-set.

Corollary 1. *For every $x \in X$, $\{x\}$ is a $\beta^* g-$closed set of (X, τ) if and only if $\{x\}$ is a closed set of X.*

Proof.

Necessity: Let $\{x\}$ be $\beta^* g-$closed. Then, by Theorem 7 $\{x\}$ is closed.

Sufficiency: Let $\{x\}$ be a closed set. By Theorem 3 $\{x\}$ is $\beta^* g-$closed.

Theorem 8. *Let A be $\beta^* g-$closed in (X, τ). Then $Cl(A) - A$ does not contain any non-empty complement of a β^*-set.*

Proof. Let A be a $\beta^* g-$closed set. Suppose that F is the complement of a β^*-set and $F \subset Cl(A) - A$. Since $F \subset Cl(A) - A \subset X - A$, $A \subset X - F$ and $X - F$ is a β^*-set. Therefore, $Cl(A) \subset X - F$ and $F \subset X - Cl(A)$. However, since $F \subset Cl(A) - A$, $F = \emptyset$.

3. β^*g-closures

In this section, the notion of the β^*g-closure is defined and some of its basic properties are studied.

Definition 3. *For a subset A of (X,τ), we define the β^*g-closure of as follows:*

$$\beta^*g - Cl(A) = \bigcap \; F \text{ is } \beta^*g\text{−closed in } X,\; A \subset F\}.$$

Lemma 1. *Let A be a subset of (X,τ) and $x \in X$. Then $x \in \beta^*g - Cl(A)$ if and only if $V \cap A \neq \emptyset$ for every β^*g-open set V containing x.*

Proof. Suppose that there exists a β^*g-open set V containing x such that $V \cap A = \emptyset$. Since $A \subset X - V$, $\beta^*g - Cl(A) \subset X - V$ and then $x \notin \beta^*g - Cl(A)$. Conversely, suppose that $x \notin \beta^*g - Cl(A)$. Then there exists a β^*g-closed set F containing A such that $x \notin F$. Since $x \in X - F$ and $X - F$ is β^*g-open, $(X - F) \cap A = \emptyset$.

Lemma 2. *Let A and B be subsets of (X,τ). Then we have*

(a) $\beta^*g - Cl(\emptyset) = \emptyset$ *and* $\beta^*g - Cl(X) = X$.

(b) *If* $A \subset B$, *then* $\beta^*g - Cl(A) \subset \beta^*g - Cl(B)$.

(c) $\beta^*g - Cl(A) = \beta^*g - Cl(\beta^*g - Cl(A))$.

(d) $\beta^*g - Cl(A \cup B) = \beta^*g - Cl(A) \cup \beta^*g - Cl(B)$.

(e) $\beta^*g - Cl(A \cap B) \subset \beta^*g - Cl(A) \cap \beta^*g - Cl(B)$.

Proof. Straightforward.

Remark 9.

(a) *If A is β^*g-closed in (X,τ), then $\beta^*g - Cl(A) = A$. but the converse is not true as seen by the following example:*

(b) *In general, $\beta^*g - Cl(A) \cap \beta^*g - Cl(B) \not\subset \beta^*g - Cl(A \cap B)$. for example,*

Example 10. *Let $X = \{a,b,c,d\}$ and $\tau = \{X,\emptyset,\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$. Let $A = \{b\}$ then $\beta^*g - Cl(A) = \beta^*g - Cl(\{b\}) = \{b\}$ but $\{b\}$ is not β^*g-closed set.*

Example 11. *Let $X = \{a,b,c,d\}$ and $\tau = \{X,\emptyset,\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$. Let $A = \{a,c\}$ and $B = \{a,b\}$. Then $\beta^*g - Cl(A) \cap \beta^*g - Cl(B) = \{a,d\} \not\subset \{a\} = \beta^*g - Cl(A \cap B)$.*

Definition 4. *For a subset A of (X,τ),*

(a) $c^*(A) = \bigcap\{F\colon F \text{ is } g\text{−closed}, A \subset F\}$ *: g-closure of A [18];*

(b) $\pi g - cl(A) = \bigcap\{F : F \text{ is } \pi g\text{−closed}, A \subset F\}$ *: $\pi g-$closure of A [11].*

Definition 5. *For a topological space* (X,τ),

(a) $c\tau^* = \{U \subset X : c^*(X-U) = (X-U)\}$ *[18];*

(b) $\beta\tau^* = \{U \subset X : \beta^* g - cl(X-U) = (X-U)\}$*;*

(c) $\pi g\tau^* = \{U \subset X : \pi g - cl(X-U) = (X-U)\}$ *[11];*

Proposition 2. *For a subset A of* (X,τ)*, the following statements hold:*

(a) $A \subset \pi g - cl(A) \subset c^*(A) \subset \beta^* g - cl(A)$.

(b) $\tau \subset \beta\tau^* \subset c\tau^* \subset \pi g\tau^*$.

Proof. The proof follows from definitions.

Definition 6. *A topological space* (X,τ) *is said to be*

(a) $T_{1/2}$ *space [17] if every* $g-$*closed set is closed.*

(b) $T^*_{1/2}$ *space [16] if every* g^**-closed set is closed.*

(c) $^*T_{1/2}$ *space [16] if every* $g-$*closed set is* g^**-closed.*

Theorem 9. *Let* (X,τ) *be a space. Then*

(a) Every $g-$*closed set is closed (i.e.* (X,τ) *is* $T_{1/2}$*) if and only if* $c\tau^* = \tau$.

(b) Every $\beta^* g-$*closed set is closed (i.e.* (X,τ) *is* $\beta^* T_{1/2}$*) if and only if* $\beta\tau^* = \tau$.

(c) Every $g-$*closed set is* $\beta^* g-$*closed (i.e.* (X,τ) *is* $\beta^{**} T_{1/2}$*) if and only if* $c\tau^* = \beta\tau^*$.

Proof.

(a) Let $A \in c\tau^*$. Then $c^*(X-A) = (X-A)$. By hypothesis, $Cl(X-A) = c^*(X-A) = X-A$ and hence $A \in \tau$. Conversely, let A be a $g-$closed set. Then $c^*(A) = A$ and hence $X-A \in c\tau^* = \tau$, i.e. A is closed.

(b) Let $A \in \beta\tau^*$. Then $\beta^* g - cl(X-A) = X-A$ and by hypothesis, $Cl(X-A) = \beta^* g - cl(X-A) = X-A$. Hence $A \in \tau$.

(c) Similar to (a).

Definition 7. *A topological space* (X,τ) *is called a* $\beta^* T_{1/2}$ *space if every* $\beta^* g-$*closed set is closed.*

Theorem 10. *A topological space* (X,τ) *is* $\beta^* T_{1/2}$ *if and only if each singleton of X is open or* $X-\{x\}$ *is a* β^**-set for each* $x \in X$.

Proof.

Necessity: Let x be a point of X. Suppose that $X-\{x\}$ is not a β^*-set. Then $X-\{x\}$ is β^*g-closed. Since (X,τ) is $\beta^*T_{1/2}$, $X-\{x\}$ is closed and thus $\{x\}$ is open in (X,τ).

Sufficiency: Suppose that A is β^*g-closed. We shall show that $Cl(A)\subset A$. Let x be any point of $Cl(A)$. Then $\{x\}$ is open in (X,τ) or $X-\beta^*$-set.

(i) In case $\{x\}$ is open: Since $x\in Cl(A)$, $\{x\}\cap A\neq\emptyset$ and hence $x\in A$.

(ii) In case $X-\{x\}$ is a β^*-set: By Theorem 8, $Cl(A)-A$ does not contain any nonempty complement of a β^*-set. Therefore, $x\notin Cl(A)-A$ but $x\in Cl(A)$. Thus, $x\in A$.

By (i) and (ii), we obtain $Cl(A)\subset A$ and hence A is closed.

Theorem 11. *Every $T_{1/2}$ space is a $\beta^*T_{1/2}$ space.*

Proof. Let (X,τ) be a $T_{1/2}$ space and A a β^*g-closed set of (X,τ). By Theorem 3, A is a $g-$closed set of (X,τ). Since X is a $T_{1/2}$ space, A is closed. Therefore, X is a $\beta^*T_{1/2}$ space.

Definition 8. *A topological space (X,τ) is called a $\beta^{**}T_{1/2}$ space if every $g-$closed set is β^*g-closed.*

Theorem 12. *Every $T_{1/2}$ space is a $\beta^{**}T_{1/2}$ space.*

Proof. Let (X,τ) be a $T_{1/2}$ space and A a $g-$closed set of (X,τ). Since X is a $T_{1/2}$ space, A is closed. By Theorem 3, A is a β^*g-closed set of (X,τ). Therefore, X is a $\beta^{**}T_{1/2}$ space.

Theorem 13. *A space (X,τ) is $T_{1/2}$ space if and only if it is $\beta^*T_{1/2}$ and $\beta^{**}T_{1/2}$.*

Proof.

Necessity: It follows from the Theorems 11 and 12.

Sufficiency: Suppose that X is both $\beta^*T_{1/2}$ and $\beta^{**}T_{1/2}$. Let A be a $g-$closed set of X. Since X is $\beta^{**}T_{1/2}$, then A is β^*g-closed. Since X is a $\beta^*T_{1/2}$ space, then A is a closed set of X. Thus X is a $T_{1/2}$ space.

4. β^*g-open Sets

Theorem 14. *Let (X,τ) be a topological space. $A\subset X$ is β^*g-open if and only if $F\subset Int(A)$ whenever $X-F$ is a β^*-set and $F\subset A$.*

Proof.

Necessity: Let A be β^*g-open. Let $X-F$ be a β^*-set and $F\subset A$. Then $X-A\subset X-F$ where $X-F$ is a β^*-set. β^*g-closedness of $X-A$ implies $Cl(X-A)\subset X-F$. So $F\subset Int(A)$.

Sufficiency: Suppose $X - F$ is a β^*-set and $F \subset A$ imply $F \subset Int(A)$. Let $X - A \subset U$ where U is a β^*-set. Then $X - U \subset A$ and $X - (X - U)$ is a β^*-set. By hypothesis $X - U \subset Int(A)$, That is $X - Int(A) \subset U$ and $Cl(X - A) \subset U$. So, $X - A$ is $\beta^* g-$closed and A is $\beta^* g-$open.

Theorem 15. *If A is a $\beta^* g-$open set of (X, τ) such that $Int(A) \subset B \subset A$, then B is also a $\beta^* g-$open set of (X, τ).*

Proof. This is an immediate consequence of Theorem 6.

Definition 9. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(a) $\beta^* g-$open if $f(V)$ is $\beta^* g-$open in Y for every open set V of X.

(b) $\beta^* g-$closed if $f(F)$ is $\beta^* g-$closed in Y for every closed set F of X.

(c) $\beta^* g-$preserving (resp. contra $\beta^* g-$open) if $f(F)$ is $\beta^* g-$closed (resp. $\beta^* g-$closed) in Y for every $\beta^* g-$closed (resp. open) set F of X.

5. $\beta^* g-$continuity and $\beta^* g-$irresoluteness

Definition 10. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(a) π-continuous [9] (resp. $\pi g-$continuous [8], πgp-continuous [22], πgs-continuous [2]) if $f^{-1}(F)$ is π-closed (resp. $\pi g-$closed, $\pi gp-$closed, $\pi gs-$closed) in (X, τ) for every closed set F of (Y, σ);

(b) LC-continuous [4] if $f^{-1}(F)$ is a LC-set in (X, τ) for every closed set F of (Y, σ);

(c) g^*-continuous [17] if $f^{-1}(F)$ is g^*-closed in (X, τ) for every closed set F of (Y, σ);

(d) $g-$continuous [18] (resp. gp-continuous [19], gs-continuous [1]) if $f^{-1}(F)$ is $g-$closed (resp. $gp-$closed, $gs-$closed) in (X, τ) for every closed set F of (Y, σ).

Definition 11. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta^* g-$continuous (resp. β^*gp-continuous, β^*gs-continuous) if $f^{-1}(F)$ is $\beta^* g-$closed (resp. $\beta^* gp-$closed, $\beta^* gs-$closed) in (X, τ) for every closed set F of (Y, σ).*

Definition 12. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta^* g-$irresolute (resp. β^*-irresolute) if $f^{-1}(V)$ is $\beta^* g-$closed (resp. β^*-set) in X for every $\beta^* g-$closed (resp. β^*-set) set V of Y.*

Definition 13. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be perfectly $\beta^* g-$continuous (resp. strongly $\beta^* g-$continuous) if $f^{-1}(V)$ is clopen (resp. open) in (X, τ) for every $\beta^* g-$open set V of (Y, σ).*

Definition 14. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost $\beta^* g-$continuous if $f^{-1}(V)$ is $\beta^* g-$open in (X, τ) for every regular open set V of (Y, σ).*

Definition 15. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be contra β^*g-continuous if $f^{-1}(V)$ is β^*g-closed in (X,τ) for every open set V of (Y,σ).*

Theorem 16. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(a) f is continuous,

(b) f is LC-continuous and $g-$continuous.

Theorem 17. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties are hold:*

(a) If f is continuous, then f is β^*g-continuous.

(b) If β^*g-continuous, then f is $g-$continuous.

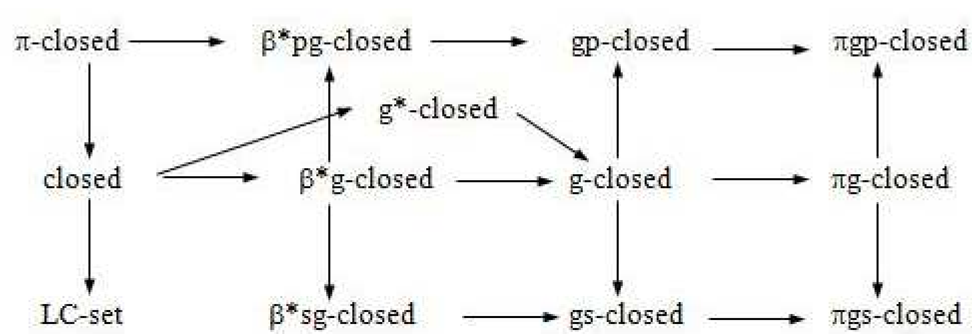
Theorem 18. *Let (X,τ) be a topological space. Then we have*

(a) If f is β^*g-continuous, then f is β^*pg-continuous.

(b) If f is β^*g-continuous, then f is β^*sg-continuous.

Theorem 19. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties hold:*

Figure 1: *DIAGRAM IV (repeated).



Proof. Obvious by Diagram 4.

Theorem 20. *If $f:(X,\tau)\to(Y,\sigma)$ is a β^*-irresolute and closed function, then $f(A)$ is β^*g-closed in Y for every β^*g-closed set A of X.*

Proof. Let A be any β^*g-closed set of X and U be any β^*-set of Y containing $f(A)$. Since f is β^*-irresolute, $f^{-1}(U)$ is a β^*-set in X and $A\subset f^{-1}(U)$. Therefore, we have $Cl(A)\subset f^{-1}(U)$ and hence $f(Cl(A))\subset U$. Since f is closed, $Cl(f(A))\subset f(Cl(A))\subset U$. Hence $f(A)$ is β^*g-closed in Y.

The composition of two β^*g-continuous functions need not be β^*g-continuous. For, consider the following example:

Example 12. *Let* $X = \{a,b,c,d\}$*,* $\tau = \{X,\emptyset,\{b\},\{c\},\{b,c\}\}, \sigma = \{X,\emptyset,\{a,b,d\}\}$*,* $\eta = \{X,\emptyset,\{a,d\}\}$*. Define* $f:(X,\tau)\to(X,\sigma)$ *by* $f(a)=a$*,* $f(b)=c$*,* $f(c)=b$*,* $f(d)=d$ *and* $g:(X,\sigma)\to(X,\eta)$ *by* $g(a)=d$*,* $g(b)=c$*,* $g(c)=b$*,* $g(d)=a$*. Then* f *and* g *are* β^*g-*continuous.* $\{b,c\}$ *is closed in* (X,η)*.*
$(g\circ f)^{-1}(\{b,c\}) = f^{-1}(g^{-1}(\{b,c\})) = f^{-1}(\{b,c\}) = \{b,c\}$ *which is not* β^*g-*closed in* (X,τ)*. Hence* $g\circ f$ *is not* β^*g-*continuous.*

Theorem 21. *Let* $f:(X,\tau)\to(Y,\sigma)$ *and* $g:(Y,\sigma)\to(Z,\eta)$ *be any two functions. Then*

(a) $g\circ f$ *is* β^*g-*continuous, if* g *is continuous and* f *is* β^*g-*continuous.*

(b) $g\circ f$ *is* β^*g-*irresolute, if* g *is* β^*g-*irresolute and* f *is* β^*g-*irresolute.*

(c) $g\circ f$ *is* β^*g-*continuous, if* g *is* β^*g-*continuous and* f *is* β^*g-*irresolute.*

(d) $g\circ f$ *is* β^*g-*continuous, if* f *is* β^*g-*continuous and* g *is* β^*g-*continuous and* Y *is a* $\beta^*T_{1/2}$*-space.*

Proof.

(a) Let V be closed in (Z,η). Then $g^{-1}(V)$ is closed in (Y,σ), since g is continuous. β^*g-continuity of f implies that $f^{-1}(g^{-1}(V))$ is β^*g-closed in (X,τ). Hence $g\circ f$ is β^*g-continuous.

(b) Let V be β^*g-closed in (Z,η). Then $g^{-1}(V)$ is β^*g-closed in (Y,σ), since g is β^*g-irresolute. Since f is β^*g-irresolute, $f^{-1}(g^{-1}(V))$ is β^*g-closed in (X,τ). Hence $g\circ f$ is β^*g-irresolute.

(c) Let V be closed in (Z,η). Since g is β^*g-continuous, $g^{-1}(V)$ is β^*g-closed in (Y,σ). As f is β^*g-irresolute, $f^{-1}(g^{-1}(V))$ is β^*g-closed in (X,τ). Hence $g\circ f$ is β^*g-continuous.

(d) Let V be closed in (Z,η). Then $g^{-1}(V)$ is β^*g-closed in (Y,σ), since g is β^*g-continuous. As (Y,σ) is a $\beta^*T_{1/2}$ space, $g^{-1}(V)$ is closed in (Y,σ). β^*g-continuity of f implies that $f^{-1}(g^{-1}(V))$ is β^*g-closed in (X,τ). Hence $g\circ f$ is β^*g-continuous.

Theorem 22. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a* β^*g-*continuous function. If* (X,τ) *is a* β^*-$T_{1/2}$ *space, then* f *is continuous.*

Proof. Let f be a β^*g-continuous function. Then $f^{-1}(V)$ is a β^*g-closed set of X for every closed set V of Y. Since X is a $\beta^*T_{1/2}$ space, $\beta^*GC(X,\tau) = C(X,\tau)$. Hence, for every closed set V of Y, $f^{-1}(V)$ is a closed set of X and so f is continuous.

Theorem 23. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be onto,* β^*g-*irresolute and closed. If* (X,τ) *is a* $\beta^*T_{1/2}$ *space, then* (Y,σ) *is also a* $\beta^*T_{1/2}$ *space.*

Proof. Let F be any β^*g-closed set of Y. Since f is β^*g-irresolute, $f^{-1}(F)$ is β^*g-closed in X. Since X is $\beta^*T_{1/2}$, $f^{-1}(F)$ is closed in X and hence $f(f^{-1}(F)) = F$ is closed in Y. This shows that (Y,σ) is also a $\beta^*T_{1/2}$ space.

References

[1] S.P. Arya, T. Nour. Characterizations of s-normal spaces. Indian J. Pure Appl. Math., 21, 717-719, 1990.

[2] G. Aslım, C. Guler and T. Noiri. On $\pi gs-$closed sets in topological spaces. Acta Math. Hungar., 112, 4, 275-283, 2006.

[3] N. Bourbaki. General Topology. Part I. Addison-Wesley. Reading Mass., 1966.

[4] M. Caldas, S. Jafari and T. Noiri and M. Simoes. A new generalization of contra-continuity via Levine's $g-$closed sets. Chaos Solitions Fractals, in press, doi: 10.1016/j.chaos.2005.12.032.

[5] S.G. Crossley and S. K. Hildebrand. Semi topological properties. Fund Math., 74, 233-254, 1972.

[6] R. Devi, H. Maki and K.Balachandran. Semi-generalized closed maps and generalized semi-closed maps. Mem. Fac. Sci. Kochi Univ. Ser. A Math., 14, 41-54, 1993.

[7] J. Dontchev and M. Ganster. On δ-generalized closed sets and T3/4-spaces. Mem. Fac. Sci. Kochi Univ. Ser. A Math., 17, 15-31, 1996.

[8] J. Dontchev and T. Noiri. Quasi-normal spaces and $\pi g-$closed sets. Acta Math Hungar., 89, 211-219, 2000.

[9] E. Ekici. On almost πgp-continuous functions, Chaos Solitions Fractals, doi:10.1016/j.Chaos.2005.12.056., 2006.

[10] E. Ekici. On contra $\pi g-$continuous functions. Chaos Solitions Fractals, doi :10.1016/j.chaos.2006.05.016., 2006

[11] E. Ekici and C.W. Baker. On $\pi g-$closed sets and continuity. Kochi J. Math., in press.

[12] E.D. Khalimsky. Applications of connected ordered topological spaces in topology. In: Conference of Math Department of Povolsia, 1970.

[13] E.D. Khalimsky, R. Kopperman and P.R. Meyer. Computer graphics and connected ordered topological topologies on finite ordered sets. Topol. Apply., 36, 1-17, 1990.

[14] T.Y. Kong, R. Kopperman and P.R. Meyer. A topological approach to digital topology. Am. Math. Monthly, 98, 901-917, 1991.

[15] V. Kovalevsky and R. Kopperman. Some topology-based image processing algorithims. Ann NY Acad. Sci., 728, 174-182, 1994.

[16] M. K. R. S. V. Kumar. Between closed sets and $g-$closed sets. Mem. Fac. Sci. Kochi Univ. Ser A Math.,. 21, 1-19, 2000.

[17] N. Levine. Generalized closed sets in topology. Rend. Circ. Mat. Palermo, 19, 89-96, 1970.

[18] H. Maki, J. Umehara and T. Noiri. Every topological space is pre-$T_{1/2}$. Mem. Fac. Sci. Kochi Univ. Ser. A (Math)., 17, 33-42, 1996.

[19] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deep. On precontinuous and weak precontinuous mappings. Proc. Math. Phys. Soc. Egypt, 53, 47-53, 1982.

[20] T. Noiri. On s-normal spaces and pre $gs-$closed functions. Acta Math. Hungar., 80, 105-113, 1998.

[21] J.H. Park. On p$gp-$closed sets in topological spaces. Indian J. Pure Apply Math., (to appear)

[22] J.H. Park and J.K. Park. On pgp-continuous functions in topological spaces. Chaos Solitions Fractals, 20, 467–77, 2004.

[23] W. Witten. Phys. Today, 24–30, 1996.

[24] Yuksel, S. and Beceren, Y.: A Decomposition of Continuity. Selcuk Univ. Fac. of Arts Science J., 14, 1, 79 – 83, 1997.

[25] V. Zaitsev. On certain classes of topological spaces and their bicompactifications. Dokl. Akad. Nauk. SSSR., 178, 778–789, 1968.