On the Solution of Fractional Order Nonlinear Boundary Value Problems By Using Differential Transformation Method

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April 14, 2010

Abstract

In this research, we study about fractional order for nonlinear of fifth-order boundary value problems and create a theorem for higher order of fractional of nth-order boundary value problems. The aim of this study was to evaluate and validate the theorem and provide several numerical examples to test the performance of our theorem. We also make comparison between exact solutions and differential transformation method(DTM) by calculating the error between them. It is shown that DTM has very small error and suitable in several numerical solutions since it is effective and provide high accuracy.

Keywords: Differential transformation method, Taylor series, fractional order of nonlinear boundary value problems.

2000 Mathematics Subject Classifications: 35C10, 74S30, 65L10, 34B05, 34B15.

1 Introduction

Recently, there has been an increasing interest in differential transformation method(DTM) to solve ordinary differential equations, partial differential equations as well as the integral equations. For details, see [1], [2], [3], [4] and [5]. It is also known that the DTM concept was introduced by Zhou in [6] in order to solve the related problems in electrical circuit analysis for linear and nonlinear initial value problems. Since then DTM was applied to several different problems in linear and nonlinear boundary value problems and also seems that the method is easy to perform to solve problem numerically, for example, see [7], [8],[9], [10].

The DTM is approximation to exact solutions which are differentiable and it has very high accuracy with minor error. This method is different with the traditional high order Taylor series since high order of the Taylor series needs a long time in computation and it requires the computation of the necessary derivatives [11]. The DTM can be applied in the high order differential equations and it is an alternative way to get Taylor series solution for the given differential equations. This method finally gives series solution but truncated series solution in practice. In addition, the series of the method always coincides with the Taylor expansion of true solution because it has very small error. Ayaz [12], was studied in application of two-dimensional DTM in partial differential equations and Borhanifar and Abazari [13] was also studied for two-dimensional and three-dimensional DTM in partial differential equations.

The basic definitions and operations of differential transformation method is discussed in Section 2. The proposed theorems and methodology will be described in Section 3. In Section 4, we provide several numerical examples to prove that the DTM has high accuracy.In addition, result is displayed in Section 5 and finally the conclusion had made in Section 6. We introduced new theorem and proved the theorem. This theorem is about fractional order of nonlinear function. By using this theorem, we can solve the higher order of fractional order for nonlinear function easily, more efficient and the result is more accurate because we generate general form of high fractional order for nth order boundary value problems.

2 The Differential Transformation Method (DTM)

It is necessary here to clarify exactly what is meant by the differential transform of the function $y(x)$ for the kth derivative. It is defined like the following [7]:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{1}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function. The inverse differential transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty}\left[\frac{(x-x_0)^k}{k!}\right]Y(k) \tag{2}$$

Substitute (1) into (2), we will get

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^k\,\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{3}$$

which is the Taylor's series for $y(x)$ at $x = x_0$.

The following theorems are easy to prove and considered the fundamental operations of differential transforms Method (DTM).

Theorem 1 *If* $t(x) = r(x) \pm p(x)$ *then* $T(k) = R(k) \pm P(k)$.

Theorem 2 *If* $t(x) = \alpha r(x)$ *then,* $T(k) = \alpha R(k)$.

Theorem 3 *If* $t(x) = \frac{dr(x)}{dx}$ *then,* $T(k) = (k+1)\,R\,(k+1)$.

Theorem 4 *If* $t(x) = \frac{d^2 r(x)}{dx^2}$ *then,* $T(k) = (k+1)\,(k+2)\,R\,(k+2)$.

Theorem 5 *If* $t(x) = \frac{d^b r(x)}{dx^b}$ *then,* $T(k) = (k+1)\,(k+2)\ldots(k+b)\,R\,(k+b)$.

Theorem 6 *If* $t(x) = r(x)p(x)$ *then* $T(k) = \sum_{l=0}^{k} P\,(l)\,R\,(k-l)$.

Theorem 7 *If* $t(x) = x^b$ *then* $T(k) = \delta\,(k-b)$ *where,* $\delta\,(k-b) = \begin{cases} 1 & \text{if } k = b \\ 0 & \text{if } k \neq b \end{cases}$

Theorem 8 *If* $t(x) = exp\,(\lambda x)$ *then,* $T(k) = \frac{\lambda^k}{k!}$

Theorem 9 *If* $t(x) = (1+x)^b$ *then,* $T(k) = \frac{b(b-1)\ldots(b-k+1)}{k!}$.

Theorem 10 *If* $t(x) = \sin\,(jx+\alpha)$ *then,* $T(k) = \frac{j^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$.

Theorem 11 *If* $t(x) = \cos\,(jx+\alpha)$ *then,* $T(k) = \frac{j^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$.

2.1 Two-dimensional DTM

We note that the differential transform methods can easily be extended to the multiple dimensional cases, For example if we take a function with two variables, for instance $y(x,t)$ having a transform $Y(k,j)$ then two-dimensional differential transformation method can be applied several partial differential equations. Thus the two dimensional form of the differential transform methods is defined as the following:

$$Y(k,j) = \frac{1}{k!j!}\left[\frac{\partial^{k+j}}{\partial x^k \partial y^j} y(x,t)\right]_{\substack{x=0\\y=0}} \tag{4}$$

Differential equation in form of $y(x,t)$ is like the following:

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} Y(k,j)x^k y^j. \tag{5}$$

From Eq. (4) and (5) we can demonstrate as follows:

$$Y(k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{k!j!}\left[\frac{\partial^{k+j}}{\partial x^k \partial y^j} y(x,t)\right]_{\substack{x=0\\y=0}}. \tag{6}$$

It is clear that Eq (6) implies the two dimensional of Taylor series expansion. One easily deduce several similar results as in the Section 1.

3 General solution for nth-order boundary value problems for mth-order nonlinear functions

Consider the following problem. If $y(x)$ is transformable then on using the equation (1) we consider the solution to the high order differential equation $y^{(n)}(x) = e^{-x}y^{\frac{1}{m}}(x)$. Now we can consider several cases as follows:

If $n=5$ and $p=\frac{1}{m}=\frac{1}{2}$ then on using the equation (1) one can easily prove that,

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(-\frac{1}{2})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\ldots}$ and

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{1}{2})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\ldots}$.

Now, if $p=\frac{1}{3}$ then,

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(-\frac{2}{3})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\ldots}$ and

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{2}{3})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\ldots}$

Similarly, if $p=\frac{1}{m}$ then,

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{1}{m}-1)^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\ldots}$ and

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(1-\frac{1}{m})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\ldots}$

If $p=\frac{1}{m+1}$ then,

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{1}{m+1}-1)^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\ldots}$ and

$Y(k+5) = \frac{k!}{(k+5)!}\left[\left(\frac{1}{(1-\frac{1}{m+1})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\ldots}$

Thus, if $n=1$ and $p=\frac{1}{m}$ then,
$Y(k+1)=\frac{k!}{(k+1)!}\left[\left(\frac{1}{(\frac{1}{m}-1)}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,...}$ and
$Y(k+1)=\frac{k!}{(k+1)!}\left[\left(\frac{1}{(1-\frac{1}{m})}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,...}$

If $n=2$ then,
$Y(k+2)=\frac{k!}{(k+2)!}\left[\left(\frac{1}{(\frac{1}{m}-1)^2}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,...}$ and
$Y(k+2)=\frac{k!}{(k+2)!}\left[\left(\frac{1}{(1-\frac{1}{m})^2}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,...}$

If $n=q$ then,
$Y(k+q)=\frac{k!}{(k+q)!}\left[\left(\frac{1}{(\frac{1}{m}-1)^q}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,...}$ and
$Y(k+q)=\frac{k!}{(k+q)!}\left[\left(\frac{1}{(1-\frac{1}{m})^q}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,...}$

If $n=q+1$ then,
$Y(k+q+1)=\frac{k!}{(k+q+1)!}\left[\left(\frac{1}{(\frac{1}{m}-1)^{q+1}}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,...}$ and
$Y(k+q+1)=\frac{k!}{(k+q+1)!}\left[\left(\frac{1}{(1-\frac{1}{m})^{q+1}}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,...}$
The above calculations suggest that the general case by using the induction method can be proved and is given as the following theorem.

Theorem 12 *Let $y(x)$ is transformable then solution to the high order differential equation $y^{(n)}(x)=e^{-x}y^{\frac{1}{m}}(x)$ is given by*

$$\begin{aligned} Y(k+n) &= \frac{k!}{(k+n)!}\left[\left(\frac{1}{(\frac{1}{m}-1)^n}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,...} \\ Y(k+n) &= \frac{k!}{(k+n)!}\left[\left(\frac{1}{(1-\frac{1}{m})^n}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,...}. \end{aligned}$$

In this research, we study nth-order boundary value problems having fractional-order with non-linear functions. We also compute error between exact solution and differential transformation method (DTM).

4 Numerical examples

Example 1

In this section, we provide three examples to make better understanding to the theorem. First, we take the case of fractional in form $\frac{1}{2}$ order for fifth order BVP.

$$y^(5)=e^{-x}\sqrt{y\,(x)} \qquad 0<x<1 \tag{7}$$

subject to the boundary conditions

$$y(0)=1, \quad y'(0)=-2, \quad y''(0)=4, \quad y(1)=e^{-2}, \quad y'(1)=-2e^{-2}. \tag{8}$$

Applying Eq. (8) to Eq. (1) at $x=0$, the following transformed boundary conditions can be obtained

$$Y(0)=1, \quad Y(1)=-2, \quad Y(2)=2. \tag{9}$$

By using differential transform properties in theorem (13) to Eq. (7) then the transformed equation is given by

$$
\begin{aligned}
Y(k+5) &= \frac{k!}{(k+5)!}\left[\left(\frac{1}{(-\frac{1}{2})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\dots} \qquad \textit{and} \\
Y(k+5) &= \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{1}{2})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\dots}
\end{aligned} \tag{10}
$$

which is based on Eq. (1), $t = \dfrac{y'''(0)}{3!} = Y(3)$ and $w = \dfrac{y^{(4)}(0)}{4!} = Y(4)$. Using the transformed boundary conditions in Eq. (9) and transformed equation in Eq. (10), we can get the solution for $Y(k),\ k \geq 5$ easily. The values of t and w can be evaluated by using boundary conditions in Eq. (8) at $x = 1$ for $N = 20$ by solving two equations such as:

$$\sum_{k=0}^{21} Y(k) = e^{-2} \quad \text{and} \quad \sum_{k=0}^{21} kY(k) = -2e^{-2}.$$

These two equations give $t = -1.333333333$ and $w = 0.6666666668$.
As a result the following series solution can be formed by applying the inverse transformation equation in Eq. (2) up to $N = 20$.

$$
\begin{aligned}
y(x) =& 1.0 - 2.0x + 2.0x^2 - 1.333333333x^3 + 0.6666666668x^4 - 0.2666666667x^5 \\
&+ 0.08888888889x^6 - 0.02539682540x^7 + 0.006349206348x^8 \\
&- 0.001410934745x^9 + 0.0002821869489x^{10} - 0.00005130671797x^{11} \\
&+ 0.000008551119662x^{12} - 0.000001315556871x^{13} + 0.0000001879366960x^{14} \\
&- 0.00000002505822612x^{15} + 0.000000003132278265x^{16} \\
&- 0.0000000003685033252x^{17} + 4.094481391 \times 10^{-11}x^{18} \\
&- 4.309980415 \times 10^{-12}x^{19} + 4.309980412 \times 10^{-13}x^{20}.
\end{aligned}
$$

Example 2

Then, for Example 2 we consider fractional in form $\frac{1}{3}$ order for fifth order boundary value problems. For instance:

$$y^{(5)}(x) = e^x \sqrt[3]{(y(x))} \qquad 0 < x < 1 \tag{11}$$

subject to the boundary conditions

$$y(0) = 1, \quad y'(0) = -\frac{3}{2}, \quad y'(0) = \frac{9}{4}, \quad y(1) = e^{-\frac{3}{2}}, \quad y'(1) = (-3/2)e^{-\frac{3}{2}}. \tag{12}$$

Applying Eq. (12) to Eq. (1) at $x = 0$, the following transformed boundary conditions can be found

$$Y(0) = 1, \quad Y(1) = \frac{-3}{2}, \quad Y(2) = \frac{9}{8}. \tag{13}$$

Next, by using differential transform properties in theorem (13) to Eq. (11) then the transformed equation is given by

$$
\begin{aligned}
Y(k+5) &= \frac{k!}{(k+5)!}\left[\left(\frac{1}{(-\frac{2}{3})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=0,2,4,\dots} \qquad \textit{and} \\
Y(k+5) &= \frac{k!}{(k+5)!}\left[\left(\frac{1}{(\frac{2}{3})^5}\right)\left(\frac{(-1)^k}{k!}\right)k!Y(k)\right]_{k=1,3,5,\dots}
\end{aligned} \tag{14}
$$

which is based on Eq. (1), $t = \dfrac{y'''(0)}{3!} = Y(3)$ and $w = \dfrac{y^{(4)}(0)}{4!} = Y(4)$. On using the transformed boundary conditions in Eq. (13) and transformed equation in Eq. (14), we can get the solution for $Y(k)$, $k \geq 5$ easily. The values of t and w can be evaluated by using boundary conditions in Eq. (12) at $x = 1$ for $N = 20$ by solving two equations such as:

$$\sum_{k=0}^{20} Y(k) = e^{-\frac{3}{2}} \quad \text{and} \quad \sum_{k=0}^{20} kY(k) = (-3/2)e^{-\frac{3}{2}}.$$

These two equations give values for $t = -0.5625000003$ and $w = 0.2109375003$.
As a result the following series solution can be formed by applying the inverse transformation equation in Eq. (2) up to $N = 20$.

$$\begin{aligned} y(x) =& 1.0 - 1.500000000x + 1.125000000x^2 - 0.5625000003x^3 \\ &+ 0.2109375003x^4 - 0.06328125000x^5 + 0.01582031250x^6 \\ &- 0.003390066964x^7 + 0.0006356375561x^8 - 0.0001059395928x^9 \\ &+ 0.00001589093890x^{10} - 0.000002166946213x^{11} \\ &+ 0.0000002708682766x^{12} - 0.00000003125403193x^{13} \\ &+ 0.000000003348646282x^{14} - 0.0000000003348646277x^{15} \\ &+ 3.139355885 \times 10^{-11}x^{16} - 2.770019898 \times 10^{-12}x^{17} \\ &+ 2.308349916 \times 10^{-13}x^{18} - 1.822381515 \times 10^{-14}x^{19} \\ &+ 1.366786134 \times 10^{-15}x^{20}. \end{aligned}$$

Example 3

Finally we perform fractional-order in form $\frac{1}{4}$ for sixth-order boundary value problems such as:

$$y^{(6)}(x) = e^x \sqrt[4]{(y(x))} \qquad 0 < x < 1 \tag{15}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y'(0) = \frac{-4}{3}, \quad y''(0) = \frac{16}{9}, \quad y(1) = e^{-\frac{4}{3}}, \quad y'(1) = \left(\frac{4}{3}\right) e^{-\frac{4}{3}} \\ &y''(1) = \left(\frac{16}{9}\right) e^{-\frac{4}{3}}. \end{aligned} \tag{16}$$

Applying Eq. (16) to Eq. (1) at $x = 0$, the following transformed boundary conditions can be found

$$Y(0) = 1, \quad Y(1) = \frac{-4}{3}, \quad Y(2) = \frac{8}{9}. \tag{17}$$

Then, by using differential transform properties in theorem (13) to Eq. (15) then the transformed equation is given by

$$Y(k+6) \quad = \quad \frac{k!}{(k+6)!} \left[\left(\frac{1}{\left(-\frac{3}{4}\right)^6} \right) \left(\frac{(-1)^k}{k!} \right) k! Y(k) \right]_{k=0,2,4,\ldots} \qquad and \tag{18}$$

$$Y(k+6) \quad = \quad \frac{k!}{(k+6)!} \left[\left(\frac{1}{\left(\frac{3}{4}\right)^6} \right) \left(\frac{(-1)^k}{k!} \right) k! Y(k) \right]_{k=1,3,5,\ldots} \tag{19}$$

which is based on Eq.(1), $t = \dfrac{y'''(0)}{3!} = Y(3)$, $w = \dfrac{y^{(4)}(0)}{4!} = Y(4)$ and $z = \dfrac{y^{(5)}(0)}{5!} = Y(5)$. Using the transformed boundary conditions in Eq. (17) and transformed equation in Eq. (18), we can get the solution for $Y(k),\ k \geq 6$ easily. The values of t,w and z can be evaluated by using boundary conditions in Eq. (16) at $x = 1$ for $N = 20$ by solving three equations such as:

$$\sum_{k=0}^{20} Y(k) = e^{-\frac{4}{3}}, \quad \sum_{k=0}^{20} kY(k) = \left(\frac{4}{3}\right) e^{-\frac{4}{3}} \quad \text{and} \sum_{k=0}^{20} k(k-1)Y(k) = \left(\frac{16}{9}\right) e^{-\frac{4}{3}}. \tag{20}$$

These three equations give values for $t = -0.4047692491$, 0.1573589866 and $z = -0.5413042140e-1$. Consequently the following series solution can be formed by applying the inverse transformation equation in Eq. (2) up to $N = 20$.

$$\begin{aligned} y(x) =& 1.0 - 1.333333333\,x + 0.8888888889\,x^2 - 0.4047692491\,x^3 + \\ & 0.1573589866\,x^4 - 0.05413042140\,x^5 + 0.007803688462\,x^6 + \\ & 0.001486416850\,x^7 + 0.0002477361417\,x^8 + 0.00003760348951\,x^9 + \\ & 0.000005847526229\,x^{10} + 0.0000009143223916\,x^{11} + \\ & 0.00000006590644330\,x^{12} - 0.000000006759635211\,x^{13} + \\ & 0.0000000006437747820\,x^{14} - 5.863055289 \times 10^{-11}\,x^{15} + \\ & 5.698335785 \times 10^{-12}\,x^{16} - 5.765261068 \times 10^{-13}\,x^{17} + \\ & 2.770487778 \times 10^{-14}\,x^{18} + 1.944201950 \times 10^{-15}\,x^{19} + \\ & 1.296134633 \times 10^{-16}\,x^{20}. \end{aligned}$$

5 Result

There are numerical results for differential transformation method and comparison to exact solution of fifth-order BVPs for fractional order $\frac{1}{2}$ and $\frac{1}{3}$. We also provided sixth-order boundary value problems for fractional-order of degree $\frac{1}{4}$. They are in Table 1, Table 2 and Table 3 respectively.

x	Exact solution	DTM (N=21)	Error
0.0	1	1	0
0.1	0.8187307532	0.8187307532	0
0.2	0.6703200461	0.6703200461	0
0.3	0.5488116361	0.5488116361	0
0.4	0.4493289640	0.4493289640	0
0.5	0.3678794413	0.3678794413	0
0.6	0.3011942120	0.3011942120	0
0.7	0.2465969642	0.2465969642	0
0.8	0.2018965183	0.2018965183	0
0.9	0.1652988883	0.1652988884	0.1×10^{-9}
1.0	0.1353352833	0.1353352834	0.1×10^{-9}

Table 1 : Comparison numerical result for Example 1

x	Exact solution	DTM (N=21)	Error
0.0	1	1	0
0.1	0.8607079765	0.8607079765	0
0.2	0.7408182206	0.7408182206	0
0.3	0.6376281517	0.6376281517	0
0.4	0.5488116361	0.5488116361	0
0.5	0.4723665528	0.4723665528	0
0.6	0.4065696598	0.4065696597	0.1×10^{-9}
0.7	0.3499377491	0.3499377490	0.1×10^{-9}
0.8	0.3011942119	0.3011942118	0.1×10^{-9}
0.9	0.2592402607	0.2592402606	0.1×10^{-9}
1.0	0.2231301601	0.2231301601	0

Table 2 : Comparison numerical result for Example 2

x	Exact solution	DTM (N=21)	Error
0.0	1	1	0
0.1	0.8751733191	0.8751659889	0.0000073302
0.2	0.7659283385	0.7658857067	0.426318×10^{-4}
0.3	0.6703200462	0.6702203323	0.997139×10^{-4}
0.4	0.5866462197	0.5864923238	0.0001538959
0.5	0.5134171191	0.5132373530	0.1797661×10^{-3}
0.6	0.4493289642	0.4491646634	0.0001643008
0.7	0.3932407212	0.3931270548	0.0001136664
0.8	0.3441537873	0.3441018869	0.519004×10^{-4}
0.9	0.3011942119	0.3011846781	0.0000095338
1.0	0.2635971382	0.2635971384	0.2×10^{-9}

Table 3 : Comparison numerical result for Example 3.

From the tables, we can see that Differential Transformation Method has minor error. By the way, for Table 1 it has no error from points 0.0 to 0.8. For points 0.9 and 1.0, the error is $0.1e - 9$. The maximum error for Example 1 is $\sum_{i=0}^{1} x_i = 2.0 \times 10^{-10}$. That means DTM is very accurate with exact solution for fractional order $\frac{1}{2}$ of fifth-order BVP. By comparison to fractional order $\frac{1}{3}$, the maximum error is $\sum_{i=0}^{1} x_i = 4.0 \times 10^{-10}$. From point $x = 0.0$ to $x = 0.5$ it has no error but at points $x = 0.6$ to 0.9, the error is 1×10^{-10} and it declines to 0 for point 1.0 while for sixth-order BVPs of $\frac{1}{4}$ fractional order function, the maximum error is $\sum_{i=0}^{1} x_i = 0.9364059e - 3$. From Table 3, we can see it has no error only for point $x = 0.0$.

6 Conclusion

The results of this research support the idea that DTM has high degree of accuracy in numerical solution. We introduced general form of nth-order BVPs for fractional order function. This research will serve as a base for future studies in nonlinear function especially in fractional order function. Therefore, we proved the DTM method very successful and powerful in numerical solution for the bounded domains. The computations in the examples were computed by using Maple 9.

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