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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Approximation by Complex Potentials Generated by the Euler's Beta Function

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Abstract. In this paper we find the exact orders of approximation of analytic functions by the complex versions of several potentials generated by the Euler's Beta function and by some complex singular integrals.

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1. Introduction

Starting from the Flett real potential defined for any $f \in L^p(\mathbb{R})$ by [see Flett 1]

$$F^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} Q_t(f)(x) dt,$$

where $Q_t(f)(x) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2+t^2} du$ is the classical Poisson-Cauchy real singular integral, in the recent paper [3] we studied the approximation properties for $\alpha \searrow 0$, of its complex version defined by

$$F_U^{\alpha}(f)(z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} Q_t(f)(z) dt,$$

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where $Q_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^{\infty}\frac{f(ze^{-iu})}{u^2+t^2}du$. Also, in the same paper [3], the approximation properties of following types of complex potentials generated by the Gamma function and some other singular integrals were studied :

$$F_U^{\alpha}(f)(z) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}U_t(f)(z)dt,$$

with

$$U_t(f)(z) = P_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-|u|/t}\,du,$$

$$U_t(f)(z) = R_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty}\frac{f(ze^{-iu})}{(u^2+t^2)^2}\,du,$$

$$U_t(f)(z) = W_t^*(f)(z) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/t}\,du,$$

representing the complex versions of the Picard, generalized Poisson-Cauchy and Gauss-Weierstrass singular integrals, respectively.

The goal of the present paper is to find the exact orders of approximation by the complex potentials generated by the Euler's Beta function, that is of the form

$$G_U^{\alpha,\beta}(f)(z) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}U_t(f)(z)dt,$$

for $Q_t(f)(z)$ and for all the $U_t(f)(z)$ defined above.

2. Main Result

For $R > 0$ let us denote $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$. The main result is the following.

Theorem 1. *Let us suppose that $0 < \alpha \le \beta \le 1$, $\alpha + \beta \ge 1$ and that $f : \mathbb{D}_R \to \mathbb{C}$, with $R > 1$, is analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) For $U_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^{\infty}\frac{f(ze^{-iu})}{u^2+t^2}du$ we have that $G_U^{\alpha,\beta}(f)(z)$ is analytic in $\mathbb{D}_R$ and we can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k b_k(\alpha,\beta)\cdot z^k, z \in \mathbb{D}_R,$$

where

$$b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-kt}dt.$$

Also, if f is not constant for $q = 0$, and not a polynomial of degree $\le q-1$ for $q \in \mathbb{N}$, then for all $1 \le r < r_1 < R$, $q \in \mathbb{N}\cup\{0\}$, $\alpha \in (0,\beta]$ we have

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where $\|f\|_r = \sup\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend only on* f*,* q*,* r*,* r_1*,* β*.*

(ii) For $U_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-|u|/t}\,du$ *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{1+t^2k^2}dt$*.*

Also, if f *is not constant for* $q = 0$*, and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$*, then for all* $1 \leq r < r_1 < R$*,* $q \in \mathbb{N} \cup \{0\}$*,* $\alpha \in (0,\beta]$ *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f*,* q*,* r*,* r_1 *and* β*.*

(iii) For $U_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{(u^2+t^2)^2}du$ *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt$*.*

Also, if f *is not constant for* $q = 0$*, and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$*, then for all* $1 \leq r < r_1 < R$*,* $q \in \mathbb{N} \cup \{0\}$*,* $\alpha \in (0,\beta]$ *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f*,* q*,* r*,* r_1 *and* β*.*

(iv) For $U_t(f)(z) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/t}\,du$ *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-(k^2/4)t}dt$*.*

Also, if f *is not constant for* $q = 0$*, and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$*, then for all* $1 \leq r < r_1 < R$*,* $q \in \mathbb{N} \cup \{0\}$*,* $\alpha \in (0,\beta]$ *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f*,* q*,* r*,* r_1 *and* β*.*

Proof.

(i) By Gal [2, p. 213, Theorem 3.2.5, (i)], $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z)=\sum_{k=0}^{\infty}a_k e^{-kt}z^k, \text{ for all } |z|<R \text{ and } t\geq 0.$$

Since $|\sum_{k=0}^{\infty}a_k e^{-kt}z^k|\leq\sum_{k=0}^{\infty}|a_k|\cdot|z|^k<\infty$, this implies that for fixed $|z|<R$, the series in t, $\sum_{k=0}^{\infty}a_k e^{-kt}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z)=\sum_{k=0}^{\infty}a_k b_k(\alpha,\beta)z^k,$$

where

$$b_k(\alpha,\beta)=\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-kt}dt.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z)-f(z)=\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z)-f(z)]dt,$$

which together with the estimate $|U_t(f)(z)-f(z)|\leq C_r(f)t$ in Gal [2, p. 213, Theorem 3.2.5, (iii)], implies

$$|G_U^{\alpha,\beta}(f)(z)-f(z)|\leq\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z)-f(z)|dt$$

$$\leq C_r(f)\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha}(1-t)^{\beta-1}dt=C_r(f)\cdot\frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)}$$

$$=C_r(f)\cdot\frac{\alpha}{\alpha+\beta}\leq C_r(f)\cdot\alpha,$$

for all $|z|\leq r$, where $C_r(f)>0$ is independent of z (and α, β) but depends on f and r. Here we used the well known formula $\frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)}=\frac{\alpha}{\alpha+\beta}$.

Now, let $q\in\mathbb{N}\cup\{0\}$ and $1\leq r<r_1<R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z|\leq r$ and $v\in\gamma$ we have $|v-z|\geq r_1-r$, by using the Cauchy's formula, for all $|z|\leq r$ and $0<\alpha\leq\beta\leq 1$, $\alpha+\beta\geq 1$, we get

$$|[G_U^{\alpha,\beta}(f)]^{(q)}(z)-f^{(q)}(z)|=\frac{q!}{2\pi}\left|\int_\gamma\frac{G_U^{\alpha,\beta}(f)(z)-f(z)}{(v-z)^{q+1}}dv\right|$$

$$\leq C_{r_1}(f)\alpha\cdot\frac{q}{2\pi}\cdot\frac{2\pi r_1}{(r_1-r)^{q+1}}=C^*\alpha,$$

with C^* depending only on f, q, r and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 218-219 in the book Gal [2], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1-e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ an then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt\right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z) - (G_U^{\alpha,\beta}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$
$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt \right]$$
$$\leq \|f^{(q)} - (G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q=0$. In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-pt}]dt \right),$$

we clearly get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-t}]dt.$$

But denoting $g(t)=e^{-t}$, by the mean value theorem there exists $\xi \in (0,1)$ such that $1-e^{-t} = g(0)-g(t) = te^{-\xi} \geq \frac{t}{e}$, which immediately implies

$$V_{\alpha,\beta} \geq \frac{1}{e\cdot Beta(\alpha,\beta)} \int_0^1 t^{\alpha}(1-t)^{\beta-1}dt = \frac{Beta(\alpha+1,\beta)}{e\cdot Beta(\alpha,\beta)}$$
$$= \frac{1}{e}\cdot\frac{\alpha}{\alpha+\beta} \geq \frac{1}{e}\cdot\frac{\alpha}{2\beta} \geq \frac{\alpha}{2e}.$$

Therefore,

$$\frac{1}{2e}\cdot r^p \cdot |a_p| \leq \frac{\|f - G_U^{\alpha,\beta}(f)\|_r}{\alpha},$$

for all $p\geq 1$ and $0<\alpha\leq\beta\leq 1, \alpha+\beta\geq 1$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,\beta]$ with $\lim_{k\to\infty}\alpha_k = 0$ and such that $\lim_{k\to\infty} \frac{\|G_U^{\alpha,\beta}(f)-f\|_r}{\alpha_k} = 0$, then $a_p = 0$ for all $p\geq 1$, that is f is constant on $\overline{\mathbb{D}}_r$.

Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,\beta]} \frac{\|G_U^{\alpha,\beta}(f)-f\|_r}{\alpha} > 0$, which implies that there exists a constant $C_r(f)>0$ such that $\frac{\|G_U^{\alpha,\beta}(f)-f\|_r}{\alpha} \geq C_r(f)$, that is

$$\|G_U^{\alpha,\beta}(f)-f\|_r \geq C_r(f)\alpha, \text{ for all } 0<\alpha\leq\beta\leq 1, \alpha+\beta\geq 1.$$

Now, consider $q\geq 1$ and denote

$$V_{q,\alpha,\beta} = \inf_{p\geq 0} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt \right).$$

Evidently that we have $V_{q,\alpha,\beta} \geq V_{\alpha,\beta} \geq \alpha\cdot\frac{1}{2e}$.

Reasoning as in the case of $q=0$, we obtain

$$\frac{\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r}{\alpha}\ge |a_{q+p}|\frac{(q+p)!}{p!}\cdot\frac{1}{2e}\cdot r^p,$$

for all $p\ge 0$ and $0<\alpha\le\beta\le 1, \alpha+\beta\ge 1$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,\beta]$ with $\lim_{k\to\infty}\alpha_k=0$ and such that $\lim_{k\to\infty}\frac{\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r}{\alpha_k}=0$, then $a_{q+p}=0$ for all $p\ge 0$, that is f is a polynomial of degree $\le q-1$ on $\overline{\mathbb{D}}_r$.

Therefore, because by hypothesis f is not a polynomial of degree $\le q-1$, we obtain $\inf_{\alpha\in(0,\beta]}\frac{\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r}{\alpha}>0$, which implies that there exists a constant $C_{r,q}(f)>0$ such that $\frac{\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r}{\alpha}\ge C_{r,q}(f)$, for all $\alpha\in(0,\beta]$, that is

$$\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r\ge C_{r,q}(f)\alpha, \text{ for all } \alpha\in(0,\beta].$$

(ii) By Gal [2, p. 206, Theorem 3.2.1, (i)], $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z)=\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k, \text{ for all } |z|<R \text{ and } t\ge 0.$$

Since $|\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k|\le\sum_{k=0}^{\infty}|a_k|\cdot|z|^k<\infty$, this implies that for fixed $|z|<R$, the series in t, $\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z)=\sum_{k=0}^{\infty}a_kz^k\frac{1}{Beta(\alpha,\beta)}\int_0^1\frac{t^{\alpha-1}(1-t)^{\beta-1}}{1+t^2k^2}dt.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z)-f(z)=\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z)-f(z)]dt,$$

which together with the estimate $|U_t(f)(z)-f(z)|\le C_r(f)t^2$ in Gal [2, p. 207, Theorem 3.2.1, (iv)], implies

$$|G_U^{\alpha,\beta}(f)(z)-f(z)|\le\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z)-f(z)|dt$$

$$\le C_r(f)\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha+1}(1-t)^{\beta-1}dt=C_r(f)\cdot\frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)}$$

$$=C_r(f)\frac{\alpha+1}{\alpha+\beta+1}\cdot\frac{\alpha}{\alpha+\beta}\le C_r(f)\frac{\alpha(\alpha+1)}{2}\le C_r(f)\alpha,$$

for all $|z| \le r$, where $C_r(f) > 0$ is independent of z (and α, β) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \le r < r_1 < R$. By using the Cauchy's formula and reasoning as in the proof of the above point (i), we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \le C^*\alpha,$$

with C^* depending only on f, q, r and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.1, at pages 209-210 in the book Gal [2], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p \cdot \frac{t^2(q+p)^2}{1+t^2(q+p)^2}.$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ an then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\} t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[f^{(q)}(z) - (G_U^{\alpha,\beta}(f))^{(q)}(z) \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2} \right] dt \right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2} \right] dt \right] \leq \|f^{(q)} - (G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality we immediately obtain

$$|a_p|r^p \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2p^2}{1+t^2p^2} \right] dt \right)$$

$$\leq \|f - G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2p^2}{1+t^2p^2} \right] dt \right),$$

we clearly get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2}{1+t^2} \right] dt$$

$$= \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1 - \frac{1}{1+t^2} \right] dt.$$

But we have $1 - \frac{1}{1+t^2} \geq \frac{t^2}{4}$, for all $t \in [0,1]$. Indeed, denoting $g(t) = 1 - \frac{1}{1+t^2} - \frac{t^2}{4}$, we get $g(0) = 0$ and $g'(t) = \frac{2t}{(1+t^2)^2} - \frac{2t}{4} = 2t\left(\frac{1}{(1+t^2)^2} - \frac{1}{4}\right) \geq 0$, for all $t \in [0,1]$. It follows that $g(t)$ is nondecreasing on $[0,1]$ and therefore $g(t) \geq 0$ for all $t \in [0,1]$.

In conclusion,

$$V_{\alpha,\beta} \geq \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \frac{t^2}{4} dt$$

$$= \frac{1}{4} \cdot \frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)} = \frac{1}{4} \cdot \frac{\alpha+1}{\alpha+\beta+1} \cdot \frac{\alpha}{\alpha+\beta}$$

$$\geq \frac{1}{4} \cdot \frac{\alpha(\alpha+1)}{2} \geq \frac{\alpha}{8}.$$

Now, by following for $q \geq 0$ similar reasonings with those in the above point (i), we get the desired equivalence in the statement.

(iii) By Gal [2, p. 213, Theorem 3.2.5, (i)], $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt}(1+kt)z^k| \leq 2\sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt,$$

where denoting $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt$, we obtain

$$G_U^{\alpha,\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z) - f(z) = \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t^2$ in Gal [2, p. 213-214, Theorem 3.2.5, (iv)], implies

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)|dt$$

$$\leq C_r(f)\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha+1}(1-t)^{\beta-1}dt = C_r(f) \cdot \frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)} \leq C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r. We used here the estimate from the above point (ii).

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. By using the Cauchy's formula and reasoning as in the proof of the above point (i), we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \leq C^*\alpha,$$

with C^* depending only on f, q, r and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 219-220 in the book Gal [2], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1-(1+(q+p)t)e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ an then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z) - (G_U^{\alpha,\beta}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+(q+p)t)e^{-(q+p)t}\right] dt \right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+(q+p)t)e^{-(q+p)t}\right] dt \right]$$

$$\leq \|f^{(q)} - (G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality we immediately obtain

$$|a_p|r^p \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+pt)e^{-pt}\right] dt \right)$$

$$\leq \|f - G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+pt)e^{-pt}\right] dt \right),$$

we immediately get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+t)e^{-t}\right] dt.$$

But we have $1-(1+t)e^{-t} \geq \frac{t^2}{e}$, for all $t \in [0,1]$. Indeed, denoting $g(t) = 1-(1+t)e^{-t} - \frac{t^2}{e}$, we have $g(0) = 0$ and $g'(t) = te^{-t} - \frac{t}{e} = t\left(\frac{1}{e^t} - \frac{1}{e}\right) \geq 0$ for all $t \in [0,1]$. This implies that $g(t)$ is nondecreasing on $[0,1]$ and therefore $g(t) \geq 0$ for all $t \in [0,1]$.

Therefore,

$$V_{\alpha,\beta} \geq \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \frac{t^2}{2e} dt$$

$$= \frac{Beta(\alpha+2,\beta)}{2e \cdot B(\alpha,\beta)} = \frac{1}{2e} \cdot \frac{\alpha+1}{\alpha+\beta+1} \cdot \frac{\alpha}{\alpha+\beta}$$

$$\geq \frac{1}{2e} \cdot \frac{\alpha(\alpha+1)}{2} \geq \frac{\alpha}{4e}.$$

Now, by following for $q \geq 0$ similar reasonings with those in the above point (i), we get the desired equivalence in the statement.

(iv) By Gal [2, p. 223, Theorem 3.2.8, (i)], $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-k^2 t/4} z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-k^2 t/4} z^k| \leq \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k e^{-k^2 t/4} z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{-(k^2/4)t} dt,$$

where denoting $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{-(k^2/4)t} dt$ we can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z) - f(z) = \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z) - f(z)] dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t$ in Gal [2, p. 224, Theorem 3.2.8, (iv)], implies

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)| dt$$

$$\leq C_r(f) \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha}(1-t)^{\beta-1} dt = C_r(f) \cdot \frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)} \leq C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. By using the Cauchy's formula and reasoning as in the proof of the above point (i), we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \leq C^* \alpha,$$

with C^* depending only on f, q, r and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.8, at pages 227-228 in the book Gal [2], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$ we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1-e^{-(q+p)^2t/4}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ an then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi} d\varphi \right\} t^{\alpha-1})1-t)^{\beta-1} dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-e^{-(q+p)^2t/4}\right] dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[\frac{1}{Beta(\alpha,\beta)} \int_0^{\infty} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1} e^{-t} \left[1-e^{-(q+p)^2t/4}\right] dt \right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)} \int_0^1 [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[f^{(q)}(z) - (G_U^{\alpha,\beta}(f))^{(q)}(z) \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-e^{-(q+p)^2t/4}\right] dt \right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-e^{-(q+p)^2t/4}\right] dt \right]$$

$$\leq \|f^{(q)} - (G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality we immediately obtain

$$|a_p| r^p \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1 - e^{-p^2 t/4} \right] dt \right) \leq \|f - G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1 - e^{-p^2 t/4} \right] dt \right),$$

by simple calculation we get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1 - e^{-t/4} \right] dt.$$

But denoting $g(t) = e^{-t/4}$, by the mean value theorem there exists $\xi \in (0,1)$ such that $1 - e^{-t/4} = g(0) - g(t) = t\frac{e^{-\xi/4}}{4} \geq \frac{t}{4e^{1/4}}$, which immediately implies

$$V_{\alpha,\beta} \geq \frac{1}{4e^{1/4} \cdot Beta(\alpha,\beta)} \int_0^1 t^{\alpha}(1-t)^{\beta-1} dt = \frac{Beta(\alpha+1,\beta)}{4e^{1/4} \cdot Beta(\alpha,\beta)}$$

$$= \frac{1}{4e^{1/4}} \cdot \frac{\alpha}{\alpha+\beta} \geq \frac{1}{4e^{1/4}} \cdot \frac{\alpha}{2\beta} \geq \frac{\alpha}{8e^{1/4}}.$$

Now, by following for $q \geq 0$ similar reasonings with those in the above point (i), we get the desired equivalence in the statement.

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