**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 3, No. 6, 2010, 1086-1092
ISSN 1307-5543 – www.ejpam.com

SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS
DEDICATED TO PROFESSOR HARI M. SRIVASTAVA,
ON THE OCCASION OF HIS 70TH BIRTHDAY

# Coefficient Inequalities for concave Cesáro Operator of Non-concave Analytic Functions

Maslina Darus*, Rabha W. Ibrahim

*School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor Darul Ehsan, Malaysia*

**Abstract.** In this article, we determined the coefficient inequalities for concave Cesáro operator which applied on non-concave analytic functions $f(z)=\sum_{n=0}^{\infty}a_n(f)z^n,\ a_0=0, a_1=2$ in an open unit disk $U:=\{z:|z|<1\}$. Also we discussed the univalence of this operator by using per-Schwarzian derivative.

**2000 Mathematics Subject Classifications**: 30C45

**Key Words and Phrases**: Meromorphic univalent functions, Concave functions, Convex set, Per-Schwarzian derivative; Cesáro operator

## 1. Introduction and Preliminaries

The Cesáro operator $\mathscr{C}$ acts formally on the power series $f(z)=\sum_{n=0}^{\infty}a_n(f)z^n$ as

$$\mathscr{C}f(z)=\sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{k=0}^{n}a_k(f)\Big)z^n.$$

In the past few years, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions. The history of the Cesáro operator goes back to Hardy, who was amongst the first to show that $\mathscr{C}$ is bounded on $H^2$. The boundedness of this operator on various spaces has attracted a lot of attention. In fact that the Cesáro operator is bounded follows from the work of Siskakis [13]. The boundedness

*Corresponding author.

*Email addresses:* maslina@ukm.my (M. Darus), rabhaibrahim@yahoo.com (R. Ibrahim)

 

of $\mathscr{C}$ on $H^1$ was proved with a different method based on a result of Hardy and Littlewood [14]. With similar techniques, Miao [11] proved that $\mathscr{C}$ is bounded even on $H^p, p \in (0,1)$. The Cesáro operator is unbounded on $H^\infty$ (see [6]), so that it is reasonable to work in larger spaces of analytic functions.

In the theory of univalent functions the most important question is to find the coefficient estimates for functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n(g) z^n \tag{1}$$

that are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. Let Co(p) be the family of functions $g : U \to \overline{C}$ where $p \in (0,1)$ that satisfy the following assumption

**Assumption (A):**

(i) $g$ is meromorphic in $U$ and has a simple pole at the point $p$.

(ii) $g(0) = g'(0) - 1 = 0$.

(iii) $g$ maps $U$ conformally onto a set whose complement with respect to $\overline{C}$ is convex.

The family Co(p) has been investigated recently in [1-4,7,15]. In [10], Livingston introduced a necessary and sufficient condition for a function $f$ to be in Co(p)

$$\Re\{-(1+p^2) + 2pz - \frac{(z-p)(1-pz)g''(z)}{g'(z)}\} > 0, \ \forall z \in U.$$

Later Avkhadiev and Wirths (see [4]) proved that for each $g \in Co(p)$ with the expansion in (1) the inequality

$$|b_n(g) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)}| \le |\frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}|$$

is valid. Equality is attained if and only if

$$g(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\theta})z^2}{(1-\frac{z}{p})(1-zp)}. \tag{2}$$

Recently, Bhowmik and Pommerenke (see [5]) obtained certain coefficient estimates for functions have the Laurent expansion

$$g(z) = \sum_{n=-1}^{\infty} B_n(g)(z-p)^n, \ z \in triangle$$

where $\triangle := \{z \in C : |z-p| < 1-p\}$ and $p \in (0,1)$,

$$|B_{n-2} - \frac{(1-p^2 B_{n-1})}{p}| \le \frac{p}{(1-p^4)(1-p)^{n-1}}[1 - (\frac{1-p^4}{p^4})|B_{-1} + \frac{p^2}{1-p^4}|^2, \ n \ge 3$$

and of the form (2).

In our investigation, we shall use the analytic functions $f(z)$ in the open disk $U$ take the form

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n, \ (z \in U)$$

such that $a_0 = 0$ and $a_1 = 2$. Applied the Cesáro operator $\mathscr{C}$ on $f$ we obtain the operator

$$\mathscr{C} f(z) = \sum_{n=0}^{\infty} \Big( \frac{1}{n+1} \sum_{k=0}^{n} a_k(f) \Big) z^n, \ (z \in U). \tag{3}$$

It is clear that $\mathscr{C} f(z)$ is normalized as follows $\mathscr{C} f(0) = 0$ and $\mathscr{C} f(0)' = 1$. Assume that $\mathscr{C} f(z)$ satisfies the assumption (A). Moreover, it satisfies the expansion

$$\mathscr{C} f(z) = \sum_{n=0}^{\infty} A_n (z-p)^n, \ (z \in \triangle). \tag{4}$$

Our aim is to determine some estimates bound of $A_n$ and $a_n(f)$ for $n \geq 2$.
We need to the following result in the sequel.

**Theorem 1** ([14]). *For each $f \in Co(p)$, there exists a function $\omega$ holomorphic in $U$ such that $\omega(U) \subset \overline{U}$ and*

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{(1-\frac{z}{p})(1-zp)}, \ (z \in U). \tag{5}$$

Next we discuss some other properties of the operator (3) such as univalence of this operator by using per-Schwarzian derivative.
Let $h$ be analytic and locally univalent in $U$. The pre-Schwarzian derivative $T_h$ of $h$ is defined by

$$T_h(z) = \frac{h''(z)}{h'(z)}, \ (z \in U) \tag{6}$$

with the norm

$$\|T_h\| = sup_{z \in U} |T_h|(1-|z|^2).$$

It is known that $\|T_h\| < \infty$ if and only if $h$ is uniformly locally univalent. It is also known that $\|T_h\| \leq 6$ for $h \in \mathscr{S}$ the class of starlike functions and that $\|T_h\| \leq 4$ for $h \in \mathscr{K}$ the class of convex functions (see [9]).

## 2. Coefficient Estimates

In this section, we introduce some coefficient estimates for operator (3) and have the expansion (4). Now, we state our first results

**Theorem 2.** *Let $p \in (0,1)$ and $\mathscr{C} f(z) \in Co(p)$ have the expansion (4). Then*

$$|A_0| \leq \frac{p}{1+p^2}. \tag{7}$$

*The inequality is sharp.*

*Proof.* Let $\mathscr{C}f(z) \in Co(p)$. Then by Theorem 1, there exists a function $\omega(z)$ holomorphic in $U$ and $\omega(U) \subset \overline{U}$ satisfying (5). Assume that

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \ z \in \triangle. \tag{8}$$

Using these two expansions (4) and (8), the power series formulation of (5) takes the form

$$\sum_{n=0}^{\infty} A_n (z-p)^n (1-\frac{z}{p})(1-zp) = z - \frac{p}{1+p^2}[1+\sum_{n=0}^{\infty} c_n (z-p)^n] z^2. \tag{9}$$

Comparing the coefficient of $z$ on both sides of (9), yields the assertion (7).

**Corollary 1.** *Let $p \in (0,1)$ and $\mathscr{C}f \in Co(p)$ have the expansion (4). Then*

$$|A_1| \leq \frac{p^2(3+p^2)}{(1-p^4)(1+2p^3)}. \tag{10}$$

*The result is sharp.*

*Proof.* Comparing the coefficient of $z^2$ on both sides of (9), we obtain

$$A_1 = \frac{\frac{p^2}{1-p^2}(1+c_0)+pA_0}{1+2p^3}. \tag{11}$$

Thus in virtue of Theorem 2 and let $|c_0| \leq 1$ we obtain the assertion (10).

In general we have the following result for $n \geq 2$.

**Theorem 3.** *Let $p \in (0,1)$ and $\mathscr{C}f(z) \in Co(p)$ have the expansion (4). Then*

$$|A_n| \leq \frac{p}{(1-p)^n(1+p^2)^2}, \ n \geq 2. \tag{12}$$

*The inequality is sharp.*

*Proof.* Let $p \in (0,1)$ and $\mathscr{C}f(z) \in Co(p)$. Then by compering the coefficient of $(z-p)^n$ on both sides of (9), we obtain

$$A_n = \frac{p}{1+p^2} c_n. \tag{13}$$

But since

$$|c_n| \leq \frac{1-|c_0|^2}{(1-p)^n(1+p)}$$

(see [14]) then yields the assertion (12).

Consequently, the next result present sharp coefficient estimates for all $n \geq 2$ if $\mathscr{C}f \in Co(p)$ of the form (3) and has the expansion (4).

**Theorem 4.** *Let $p \in (0,1)$ and $\mathscr{C}f \in Co(p)$ of the form (3) and have the expansion (4). Then*

$$|\sum_{k=0}^{n} a_n(f)| \leq \frac{p(n+1)}{(1-p)^n(1+p^2)^2}, \; n \geq 2. \tag{14}$$

*The inequality is sharp.*

*Proof.* Equating the right sides of (3) and (4) and applying Theorem 3.

**Corollary 2.** *Let $p \in (0,1)$ and $\mathscr{C}f \in Co(p)$ have the expansion (4). Then*

$$|a_n(f)| \leq \sum_{k=2}^{n} \frac{(k+1)p}{(1-p)^k(1+p^2)^2} + (n-1), \; n \geq 2. \tag{15}$$

*The result is sharp.*

*Proof.* By applying Theorem 4.

## 3. Norm Estimates of the per-Schwarzian Derivative

In this section we determined the norm estimates of the per-Schwarzian derivative for the operator (3).

**Theorem 5.** *Let $p \in (0,1)$ and $\mathscr{C}f \in Co(p)$ of the form (3). Then for $z \to 0$ the per-Schwarzian derivative of $\mathscr{C}f$ satisfies the inequality*

$$\|T_{\mathscr{C}f}\| \leq \frac{(2p+1)^2}{p}. \tag{16}$$

*The result is sharp.*

*Proof.* Let $p \in (0,1)$ and $\mathscr{C}f \in Co(p)$ then in view of Theorem 1, $\mathscr{C}f$ takes the form (5). Differentiating both sides of (5) we obtain

$$\mathscr{C}f'(z) = \frac{H(z)(1-W'(z)) - (z - W(z))H'(z)}{H^2(z)}$$

where $H(z) := (1 - \frac{z}{p})(1 - zp)$ and $W(z) := \frac{p}{1+p^2}(1 + \omega(z))z^2$, or equivalent to

$$\ln \mathscr{C}f'(z) = \ln[H(z)(1-W'(z)) - (z - W(z))H'(z)] - 2\ln H(z).$$

Take the derivative for the above equality we receive

$$\frac{\mathscr{C}f''(z)}{\mathscr{C}f'(z)} = \frac{Q'(z)}{Q(z)} - \frac{2H'(z)}{H(z)}$$

where $Q(z) := [H(z)(1-W'(z)) - (z - W(z))H'(z)]$. Now for $z \to 0$ we obtain the assertion (16).

**Corollary 3.** *Let* $p \in (0,1)$ *and* $\mathscr{C}f \in Co(p)$. *Then* $\mathscr{C}f$ *is uniformly locally univalent when* $z \to 0$.

*Proof.* By applying Theorem 5, we get $\|T_{\mathscr{C}f}\| < \infty$ hence $\mathscr{C}f$ is uniformly locally univalent.

Consider the class $\Sigma$ of all analytic functions $F$ satisfy

$$\|F\|_\Sigma = sup_{z\in U}(1-|z|^2)|\frac{F'(z)}{F(z)}| < \infty.$$

Also denoted by $Rg(z) := zg'(z)$. Define the extended Cesáro operator in term of integral operator as follows (see [15])

$$\mathscr{C}_g[f](z) = \int_0^1 f(\xi z)Rg(\xi z)\frac{d\xi}{\xi}. \tag{17}$$

Then we have the following result

**Theorem 6.** *Let* $f$ *and* $Rg$ *in the class* $\Sigma$. *Then* $T_{\mathscr{C}_g[f]}$ *is bounded and uniformly locally univalent.*

*Proof.* Differentiating both sides of (17) we obtain

$$\mathscr{C}_g[f]'(z) = f(z)Rg(z).$$

Or equivalent to

$$\ln \mathscr{C}_g[f]'(z) = \ln f(z) + \ln Rg(z).$$

Take the derivative for both sides of the above equality

$$\frac{\mathscr{C}_g[f]''(z)}{\mathscr{C}_g[f]'(z)} = \frac{f'(z)}{f(z)} + \frac{Rg'(z)}{Rg(z)}.$$

Hence we obtain

$$\begin{aligned}
|T_{\mathscr{C}_g[f]}| &\le |\frac{f'(z)}{f(z)}| + |\frac{Rg'(z)}{Rg(z)}| \\
&\le sup_{z\in U}(1-|z|^2)|\frac{f'(z)}{f(z)}| + sup_{z\in U}(1-|z|^2)|\frac{Rg'(z)}{Rg(z)}| \\
&= \|f\|_\Sigma + \|Rg\|_\Sigma < \infty
\end{aligned}$$

yields that $T_{\mathscr{C}_g[f]}$ is bounded and uniformly locally univalent.

**ACKNOWLEDGEMENTS** The work presented here was supported by UKM-ST-06-FRGS0107-2009.

## References

[1] F. G. Avkhadiev, K. J. Wirths, Convex holes produce lower bounds for coefficients, Complex Variables, 47, 553-563. 2002.

[2] F. G. Avkhadiev, C. Pommerenke, K.J.Wirths, On the sufficient of concave univalent functions, Math. Nachr. 271, 3-9. 2004.

[3] F. G. Avkhadiev, K. J. Wirths, On a conjectur of Livingston, Mathematica(Cluj) 46(69), 19-23. 2004.

[4] F. G. Avkhadiev, K. J. Wirths, A proof of Livingston conjectur, Forum Math. 19, 149-158. 2007

[5] B. Bhowmik, C. Pommerenke, Coefficient inequalities for concave and meromorphically starlike univalent functions. Ann. Polon.Math. 93(2), 177-186. 2008.

[6] N. Danikas, A. G. Siskakis, The Cesáro operator on bounded analytic functions, Analysis, 13(3), 295-299. 1993

[7] M. Darus, R. W. Ibrahim, Coefficient Inequalities for a New Class of Univalent Functions, Lobachevskii Journal of Mathematics, 29(4), 221-229. 2008.

[8] Z. Fang, Z. Zhou, Extended Cesáro operator on Zygmund spaces in the unit ball, arXiv: 0709.1436v [math. FA] 10 Sep 2007, PP1-8.

[9] Y. C. Kim, T. Sugawa, Norm estimates of the per-Schwarzian derivative for certain classes of univalent functions, Proc. of the Edinburg Mathematical Society, 49, 131-143. 2006.

[10] A. E. Livingston, Convex meromorphic mappings, Ann. Polon. Math., 59, 275-291. 1994.

[11] J. Miao, The Cesáro operator is bounded on $H^p$ for $0 < p < 1$, Proc. Amer. Math. Soc., 116 (4), 1077-1079. 1992.

[12] St. Ruscheweyh, Two remarks on bounded analytic functions, Serdica 11, 200-202. 1985.

[13] A. G. Siskakis, The Cesáro operator is bounded on $H^1$, Proc. Amer. Math. Soc., 110(2), 461-462. 1990.

[14] A. G. Siskakis, Composition semigroups and the Cesáro operator on $H^p$, J. London Math. Soc. (2) 36, 153-164. 1987.

[15] K. J. Wirths, On the residuum of concave univalent functions, Serdica Math. J., 32, 209-214. 2006.