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Some Properties of a Class of p-valent Analytic Functions Associated with Convolution

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Abstract. In this paper, we define a class $\mathfrak{R}_h^g(p,m,\beta)$ associated with convolution of p-valent analytic functions. Some properties in the form of coefficient inequality, growth and distortion bounds, sufficient conditions with the help of various lemmas, integral means inequality for convolution of two functions and a set of class preserving integral operators of functions belonging to this class are studied.
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1. Introduction

Let A_p denotes a class of functions of the form:

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}\ \left(p\in\mathbb{N}=1,2,3\ldots\right), \tag{1}$$

which are analytic and p-valent in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. Let $g,h\in A_p$ be of the form:

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k},\ b_{p+k}\geq 0 \tag{2}$$

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and

$$h(z) = z^p + \sum_{k=1}^{\infty} c_{p+k} z^{p+k}, c_{p+k} \geq 0. \tag{3}$$

A function $f \in A_p$ is said to be p-valently starlike of order α in Δ, if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ \left(z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N}\right).$$

The class of all p-valent starlike functions of order α is denoted by $S_p^*(\alpha)$. On the other hand, a function $f \in A_p$ is said to be p-valently convex of order α in Δ, if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ \left(z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N}\right).$$

The class of all p-valent convex functions of order α is denoted by $K_p(\alpha)$. Furthermore, a function $f \in A_p$ is said to be p-valently close-to-convex of order α in Δ, if it satisfies the inequality

$$\operatorname{Re}\left\{z^{1-p} f'(z)\right\} > \alpha \ \left(z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N}\right).$$

The class of all p-valent close-to-convex functions of order α is denoted by $CK_p(\alpha)$. If $f \in A_p$ satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta}{p}\frac{\pi}{2} \,(z \in \Delta),$$

for some $0 < \beta \leq p$, then f is said to be p-valently strongly starlike function of order β in Δ and this class is denoted by $\overline{S}_p^*(\beta)$. Further, if $f \in A_p$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\beta}{p}\frac{\pi}{2} \,(z \in \Delta),$$

for some $0 < \beta \leq p$, then f is said to be p-valently strongly convex function of order β in Δ and is denote by $\overline{K}_p(\beta)$, the class of all such functions. Also, if $f \in A_p$ satisfies

$$\left|\arg\left\{z^{1-p} f'(z)\right\}\right| < \frac{\beta}{p}\frac{\pi}{2} \,(z \in \Delta),$$

for some $0 < \beta \leq p$, then f is said to be p-valently strongly close-to-convex function of order β in Δ and denote by $\overline{CK}_p(\beta)$ the class of all such functions. A convolution (Hadamard product) of $f \in A_p$ of the form (1) with $g \in A_p$ of the form (2) is defined by:

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z). \tag{4}$$

Various convolution operators have been defined so far, which can be obtained by taking suitable g in (4). For example the convolution in (4) reduces to the operator $\mathbf{W}^p_{q,s}([\alpha_1,A_1])f(z)$ involving a Wright's generalized hypergeometric function

$$ {}_q\Psi_s[z] \equiv {}_q\Psi_s\left(\begin{array}{c}(\alpha_1,A_1),(\alpha_2,A_2),\ldots,(\alpha_q,A_q)\\(\beta_1,B_1),(\beta_2,B_2),\ldots,(\beta_s,B_s)\end{array};z\right) $$

if $g(z)=z^p\frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}{}_q\Psi_s[z]$, where for $\alpha_i\in\mathbb{C}(\frac{\alpha_i}{A_i}\neq 0,-1,-2,\ldots), i=1,2,\ldots,q$, $\beta_i\in\mathbb{C}(\frac{\beta_i}{B_i}\neq 0,-1,-2,\ldots), i=1,2,\ldots,s$ and $A_i>0$, $i=1,2,\ldots,q$, $B_i>0, i=1,2,\ldots,s$ such that $1+\sum_{i=1}^{s}B_i-\sum_{i=1}^{q}A_i\geq 0$,

$$ {}_q\Psi_s[z]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(\alpha_i+A_ik)}{\prod_{i=1}^{s}\Gamma(\beta_i+B_ik)\,k!}z^k,\ z\in\Delta, \tag{5} $$

$(\prod_{i=1}^{s}B_i^{B_i}\geq\prod_{i=1}^{q}A_i^{A_i}$ in case $1+\sum_{i=1}^{s}B_i-\sum_{i=1}^{q}A_i=0$ [15]). The convolution operator $\mathbf{W}^p_{q,s}([\alpha_1,A_1])f(z)$, for which

$$ b_{p+k}=\frac{\prod_{i=1}^{q}\frac{\Gamma(\alpha_i+A_ik)}{\Gamma(\alpha_i)}}{\prod_{i=1}^{s}\frac{\Gamma(\beta_i+B_ik)}{\Gamma(\beta_i)}\,k!}, $$

is studied by Aouf and Dziok [3, 4], Dziok and Raina [8], and Dziok et al. [9] and Sharma [25] in their respective work and taking $A_i=1, i=1,2,\ldots,q$, $B_i=1, i=1,2,\ldots,s$, for $q\leq s+1$, it reduces to Dziok Srivastava operator [10] which involve a generalized hypergeometric function ${}_qF_s[z]$ and is defined by

$$ {}_q\mathbf{H}^p_s([\alpha_1])f(z)=z^p\,{}_qF_s[z]*f(z) \tag{6} $$

where

$$ {}_qF_s[z]={}_qF_s\left(\alpha_1,\alpha_2,\ldots\alpha_q;\beta_1,\beta_2,\ldots\beta_s;z\right)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}(\alpha_i)_k}{\prod_{i=1}^{s}(\beta_i)_k\ k!}z^k,\ z\in\Delta, $$

the symbol $(\alpha)_k$ is the familiar Pochhammer symbol defined by

$$ (\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, k\in\mathbb{N}_0. $$

The operator ${}_q\mathbf{H}_s^p\left(\left[\alpha_1\right]\right) f(z)$ includes Hohlov operator [13] which involve Gaussian hypergeometric function ${}_2F_1$ as well as Carlson and Shaffer operator [6] defined by Saitoh and Ruschweyh derivative operator [23] (for detail one may refer to [8, 9]). Also, the convolution (4) reduces to the Salagean operator [24] if

$$b_{p+k} = \left(\frac{p+k}{p}\right)^n, \; n \in \mathbb{N}_0$$

and to a generalized Salagean operator [2], if

$$b_{p+k} = \left(\frac{p+\delta k}{p}\right)^n, \delta > 0, \, n \in \mathbb{N}_0.$$

Further, the convolution (4) reduces to an integral operator involving generalized fractional integral operator $I_{0,z}^{\lambda,\mu,\nu}$, if

$$b_{p+k} = \frac{(p+1)_k\,(p-\mu+\nu+1)_k}{(p-\mu+1)_k\,(p+\lambda+\nu+1)_k}$$

and hence

$$(f*g)(z) = z^{\mu}\frac{\Gamma(p-\mu+1)\,\Gamma(p+\lambda+\nu+1)}{\Gamma(p+1)\,\Gamma(p-\mu+\nu+1)} I_{0,z}^{\lambda,\mu,\nu} f$$

where

$$I_{0,z}^{\lambda,\mu,\nu} z^{\rho} = \frac{\Gamma(\rho+1)\,\Gamma(\rho-\mu+\nu+1)}{\Gamma(\rho-\mu+1)\,\Gamma(\rho+\lambda+\nu+1)} z^{\rho-\mu},$$

$\left(0 \le \lambda < 1, \rho > \max\{0, \mu-\nu\} - 1\right)$. Again, this convolution (4) reduces to the derivative operator involving generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\nu}$, if

$$b_{p+k} = \frac{(p+1)_k\,(p-\mu+\nu+1)_k}{(p-\mu+1)_k\,(p-\lambda+\nu+1)_k}$$

and hence,

$$(f*g)(z) = z^{\mu}\frac{\Gamma(p-\mu+1)\,\Gamma(p-\lambda+\nu+1)}{\Gamma(p+1)\,\Gamma(p-\mu+\nu+1)} J_{0,z}^{\lambda,\mu,\nu} f,$$

where

$$J_{0,z}^{\lambda,\mu,\nu} z^{\rho} = \frac{\Gamma(\rho+1)\,\Gamma(\rho-\mu+\nu+1)}{\Gamma(\rho-\mu+1)\,\Gamma(\rho-\lambda+\nu+1)} z^{\rho-\mu}.$$

The generalized fractional calculus operators $I_{0,z}^{\lambda,\mu,\nu}$ and $J_{0,z}^{\lambda,\mu,\nu}$ defined above are studied in [5], [20, 26]. These generalized fractional calculus operators reduce to fractional calculus operators if we take $\mu = -\lambda$ and $\mu = \lambda$ respectively. Let T_p denotes the subclass of A_p consisting of functions of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad a_{p+k} \ge 0. \tag{7}$$

Motivated with the several work specially the work of Prajapat et al. [21], we consider $\mathfrak{R}_h^g(p, m, \beta)$ class defined as follows:

Definition 1. *A function $f \in T_p$ is said to be a member of the class $\mathfrak{R}_h^g(p,m,\beta)$ if and only if for any $g,\ h \in A_p$ with non-negative coefficients,*

$$\left|\frac{z(f*g)^{m+1}(z)}{(f*h)^m(z)}-(p-m)\right|<\beta,$$

$z \in \Delta, p \in \mathbb{N}, p > m, 0 < \beta \le p, m \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, *where $(f*g)^r(z)$ denotes the r^{th} derivative of $(f*g)$ and is given by*

$$(f*g)^r(z)=\frac{p!}{(p-r)!}z^{p-r}+\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-r)!}a_{p+k}b_{p+k}z^{p+k-r},\ r\in\mathbb{N}_0. \tag{8}$$

Obviously the class $\mathfrak{R}_h^g(p,m,\beta)$ contains the class $S_h^g(p,m,\beta)$, which is defined as follows:

Definition 2. *A function $f(z) \in T_p$ is said to be a member of the class $S_h^g(p,m,\beta)$ if and only if for any $g,\ h \in A_p$ with non-negative coefficients,*

$$Re\left\{\frac{z(f*g)^{m+1}(z)}{(f*h)^m(z)}+m\right\}>p-\beta,$$

$z \in \Delta, p \in \mathbb{N}, p > m, 0 < \beta \le p, m \in \mathbb{N}_0$.

Taking $m = 0$ and 1 respectively and $h(z) = g(z) = \frac{z^p}{1-z}$, the class $\mathfrak{R}_h^g(p,m,\beta)$ coincides with the classes $\overline{S}_p^*(\beta)$ and $\overline{K}_p(\beta)$ respectively and the class $S_h^g(p,m,\beta)$ coincides with the class $S_p^*(p-\beta)$ and $K_p(p-\beta)$ respectively. Also, taking $g(z) = \frac{z^p}{1-z}$, $h(z) = z^p$ and $m = 0$, the class $\mathfrak{R}_h^g(p,m,\beta)$ reduces to the class $\overline{CK}_p(\beta)$ and the class $S_h^g(p,m,\beta)$ reduces to the the class $CK_p(p-\beta)$.

If $h = g$, we denote $\mathfrak{R}_h^g(p,m,\beta) \equiv \mathfrak{R}_g(p,m,\beta)$. Class $\mathfrak{R}_g(1,0,\beta)$ for $g(z) = \frac{z}{1-z}$, coincides with the class studied by Chen et al. [7] as a particular case. In addition, the class $\mathfrak{R}_g(p,0,p(1-\alpha))$ reduces to the class studied by Ali et al. [1]. Taking, for $n + p > 0$, $h(z) = g(z) = \frac{z^p}{(1-z)^{n+p}}$ and $g(z) = \frac{z^p}{(1-z)^{n+p}}, h(z) = z^p$ respectively, the class $\mathfrak{R}_h^g(p,m,\beta)$ reduces to the classes, which were investigated by Raina and Srivastava [22] and these classes coincide with the classes, studied by Güney and Breaz [12] if $n + p = 1$ and are the generalization of the classes investigated by Murugusundaramoorthi and Srivastava [18]. Further, taking $g \in A_1$ so that $b_{k+1} = (1+k)^n$, $n \in \mathbb{N}_0$, the class $\mathfrak{R}_g(1,0,1-\alpha)$ would reduce to the class studied in [1]. Moreover, a class similar to $\mathfrak{R}_g(p,m,\beta)$ is studied by Prajapat et al. [21].

In this paper, we study coefficient inequality, growth and distortion bounds, sufficient conditions with the help of various lemmas, integral means inequality for convolution of two functions and a set of class preserving integral operators for functions belonging to the class $\mathfrak{R}_h^g(p,m,\beta)$.

2. Coefficient Inequality, Growth and Distortion Bounds for the class $\mathfrak{R}_h^g(p,m,\beta)$

A necessary and sufficient coefficient condition for a function $f \in T_p$ to be in the class $\mathfrak{R}_h^g(p,m,\beta)$ is derived in the form of following Theorem:

Theorem 1. *Let the function f be of the form* (7) *and g, $h \in A_p$ of the form* (2) *and* (3) *respectively with $(p+k-m)\,b_{p+k} > (p-m-\beta)\,c_{p+k}$. Then f is in the class $\mathfrak{R}_h^g(p,m,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{(p+k)!\left[(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right]}{(p+k-m)!}a_{p+k}\leq\frac{\beta p!}{(p-m)!}, \tag{9}$$

$p\in\mathbb{N}, p>m, 0<\beta\leq p$. The result is sharp for the function f given by

$$f_k(z)=z^p-\frac{\beta p!\,(p+k-m)!}{(p+k)!\,(p-m)!\left[(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right]}z^{p+k}\ (k\geq 1). \tag{10}$$

Proof. We assume that the inequality (9) holds true, then we have to show that

$$\left|\frac{z\,(f*g)^{m+1}(z)}{(f*h)^m(z)}-(p-m)\right|-\beta<0$$

or,

$$\left|z\,(f*g)^{m+1}(z)-(p-m)\,(f*h)^m(z)\right|-\beta\left|(f*h)^m(z)\right|<0.$$

Using series expansion of $(f*g)^{m+1}$ and $(f*g)^m$ from (8), we have

$$\begin{aligned}
&\left|-\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}}{(p+k-m)!}\left\{(p+k-m)\,b_{p+k}-(p-m)\,c_{p+k}\right\}z^{p+k-m}\right|\\
&-\beta\left|\frac{p!z^{p-m}}{(p-m)!}-\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}c_{p+k}}{(p+k-m)!}z^{p+k-m}\right|\\
\leq\ &\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}}{(p+k-m)!}\left\{(p+k-m)\,b_{p+k}-(p-m)\,c_{p+k}\right\}-\beta\left\{\frac{p!}{(p-m)!}-\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}c_{p+k}}{(p+k-m)!}\right\}\\
=\ &\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}}{(p+k-m)!}\left\{(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right\}-\frac{\beta p!}{(p-m)!}\\
\leq\ &0,\ \text{if (9) holds.}
\end{aligned}$$

Hence, $f\in\mathfrak{R}_h^g(p,m,\beta)$. To prove the converse, we suppose that $f\in\mathfrak{R}_h^g(p,m,\beta)$, that is

$$\left|\frac{z\,(f*g)^{m+1}(z)}{(f*h)^m(z)}-(p-m)\right|<\beta, \tag{11}$$

$z \in \Delta, p \in \mathbb{N}, p > m, 0 < \beta \leq p, m \in \mathbb{N}_0$. Since $|\mathrm{Re}(z)| \leq |z|$ for any z. Choosing z to be real and letting $z \to 1^-$ through real values, (11) yields

$$\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}}{(p+k-m)!}\left\{(p+k-m)\,b_{p+k}-(p-m)\,c_{p+k}\right\}-\beta\left\{\frac{p!}{(p-m)!}-\sum_{k=1}^{\infty}\frac{(p+k)!a_{p+k}c_{p+k}}{(p+k-m)!}\right\}\leq 0$$

or,

$$\sum_{k=1}^{\infty}\frac{(p+k)!\left[(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right]}{(p+k-m)!}a_{p+k}\leq\frac{\beta p!}{(p-m)!}$$

which leads us immediately to the desired inequality (9). Sharpness follows if we take extremal function given by (10).

Corollary 1. *If $f \in \mathfrak{R}_h^g(p,m,\beta)$, then*

$$a_{p+k}\leq\frac{\beta p!\,(p+k-m)!}{(p+k)!\,(p-m)!\left[(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right]}, k\geq 1. \tag{12}$$

The equality in (12) is attained for the function f_k given by (10).

Corollary 2. *Let $f \in \mathfrak{R}_h^g(p,m,\beta)$ and $d_{p+k} := (p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}$ be such that $d_{p+k} \geq d_{p+1}, \forall\ k \geq 1$, then*

$$\sum_{k=1}^{\infty}a_{p+k}\leq\frac{\beta\,(p-m+1)}{(p+1)\,d_{p+1}}. \tag{13}$$

Corollary 3. *Let $f \in \mathfrak{R}_h^g(p,m,\beta)$ and $d_{p+k} := (p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}$ be such that $d_{p+k} \geq d_{p+1}, \forall\ k \geq 1$, then*

$$\sum_{k=1}^{\infty}(p+k)\,a_{p+k}\leq\frac{\beta\,(p-m+1)}{d_{p+1}}.$$

Corollary 4. *Let the function f be of the form (7) and $g,\ h \in A_p$ of the form* (2) *and* (3) *respectively with $(p+k-m)\,b_{p+k} > (p-m-\beta)\,c_{p+k}$,if*

$$\sum_{k=1}^{\infty}\frac{(p+k)!\left[(p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k}\right]}{(p+k-m)!}a_{p+k}\leq\frac{\beta p!}{(p-m)!},$$

$p \in \mathbb{N}, p > m, 0 < \beta \leq p$ holds, then $f \in S_h^g(p,m,\beta)$.

Theorem 2. *Let $f \in T_p$ of the form (7) be in the class $\mathfrak{R}_h^g(p,m,\beta)$ and g, h be of the form (2), (3) respectively with $d_{p+k} := (p+k-m)\,b_{p+k}-(p-m-\beta)\,c_{p+k} \geq d_{p+1}, \forall\ k \geq 1$, then*

$$|z^p|-\frac{\beta\,(p-m+1)}{(p+1)\,d_{p+1}}\left|z^{p+1}\right|\leq\left|f(z)\right|\leq|z^p|+\frac{\beta\,(p-m+1)}{(p+1)\,d_{p+1}}\left|z^{p+1}\right| \tag{14}$$

and

$$\left|pz^{p-1}\right| - \frac{\beta\left(p-m+1\right)}{d_{p+1}}\left|z^{p}\right| \leq \left|f^{'}\left(z\right)\right| \leq \left|pz^{p-1}\right| + \frac{\beta\left(p-m+1\right)}{d_{p+1}}\left|z^{p}\right|. \tag{15}$$

Also let $g\left(1\right)$ *be finite and* $\zeta := \max\ b_{p+k}\left(k \geq 1\right)$*, then*

$$\left|z^{p}\right| - \frac{\beta\zeta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right| \leq \left|\left(f*g\right)\left(z\right)\right| \leq \left|z^{p}\right| + \frac{\beta\zeta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right|. \tag{16}$$

The bounds are sharp and extremal function may given by

$$f\left(z\right) = z^{p} - \frac{\beta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}z^{p+1}. \tag{17}$$

Proof. Taking absolute value of $f\left(z\right)$ given in (7) and using Corollary 2, we get

$$\left|f\left(z\right)\right| \leq \left|z^{p}\right| + \sum_{k=1}^{\infty} a_{p+k}\left|z^{p+k}\right| \leq \left|z^{p}\right| + \frac{\beta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right|$$

and

$$\left|f\left(z\right)\right| \geq \left|z^{p}\right| - \sum_{k=1}^{\infty} a_{p+k}\left|z^{p+k}\right| \geq \left|z^{p}\right| - \frac{\beta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right|,$$

which prove assertion (14). Again, taking absolute value of $f'(z)$ and using Corollary 3, we get

$$\left|f^{'}\left(z\right)\right| \leq \left|pz^{p-1}\right| + \sum_{k=1}^{\infty}\left(p+k\right)a_{p+k}\left|z^{p+k-1}\right| \leq \left|pz^{p-1}\right| + \frac{\beta\left(p-m+1\right)}{d_{p+1}}\left|z^{p}\right|$$

and

$$\left|f^{'}\left(z\right)\right| \geq \left|pz^{p-1}\right| - \sum_{k=1}^{\infty}\left(p+k\right)a_{p+k}\left|z^{p+k-1}\right| \geq \left|pz^{p-1}\right| - \frac{\beta\left(p-m+1\right)}{d_{p+1}}\left|z^{p}\right|,$$

which prove assertion (15).

Further, taking absolute value of $f*g$, where f and g are of the form (7) and (2) respectively. If $\zeta := \max\ b_{p+k}$, then using corollary (2), we get

$$\left|\left(f*g\right)\left(z\right)\right| \leq \left|z^{p}\right| + \sum_{k=1}^{\infty} a_{p+k}b_{p+k}\left|z^{p+k}\right| \leq \left|z^{p}\right| + \frac{\beta\zeta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right|$$

and

$$\left|\left(f*g\right)\left(z\right)\right| \geq \left|z^{p}\right| - \sum_{k=1}^{\infty} a_{p+k}b_{p+k}\left|z^{p+k}\right| \geq \left|z^{p}\right| - \frac{\beta\zeta\left(p-m+1\right)}{\left(p+1\right)d_{p+1}}\left|z^{p+1}\right|,$$

which prove (16). The bounds in (14), (15) and (16) are sharp, with extremal function given by (10).

3. Sufficient Conditions for Classes $\Re_h^g(p,m,\beta)$ and $S_h^g(p,m,\beta)$

In this section, we obtain sufficient conditions for the classes $\Re_h^g(p,m,\beta)$ and $S_h^g(p,m,\beta)$ with the use of following Lemmas:

Lemma 1. *[14] Let $w(z)$ be analytic in Δ and such that $w(0)=0$. Then if $|w(z)|$ attains its maximum value on circle $|z|=r<1$ at a point $z_0\in\Delta$, we have*

$$z_0 w'(z_0) = kw(z_0),$$

where $k\geq 1$ is a real number.

Lemma 2. *[17] Let $\phi(u,v)$ be a complex valued function:*

$$\phi : D\to\mathbb{C},\ \left(D\subset\mathbb{C}\times\mathbb{C};\ \mathbb{C}\text{is the complex plane}\right),$$

and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) *$\phi(u,v)$ is continuous in D;*

(ii) $(1,0)\in D$ *and Re* $(\phi(1,0))>0$*;*

(iii) *Re* $(\phi(iu_2,v_1))\leq 0$ *for all* $(iu_2,v_1)\in D$ *and such that* $v_1\leq -\left(1+u_2^2\right)/2$.

Let $p(z)=1+p_1z+p_2z^2+\cdots$ be regular in Δ such that $\left(p(z),\ zp'(z)\right)\in D$ for all $z\in\Delta$. If Re $\left(\phi\left(p(z),zp'(z)\right)\right)>0$ $(z\in\Delta)$, then Re $(p(z))>0$ $(z\in\Delta)$.

Lemma 3. *[19] Let a function $p(z)$ be analytic in Δ, $p(0)=1$, and $p(z)\neq 0$ $(z\in\Delta)$. If there exists a point $z_0\in\Delta$ such that*

$$\left|\arg\ p(z)\right| < \frac{\pi}{2}\beta \text{ for } |z|<|z_0|$$

and

$$\left|\arg\ p(z_0)\right| = \frac{\pi}{2}\beta$$

with $0<\beta\leq 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il\beta$$

where

$$l\geq 1 \text{ when } \arg\ p(z_0) = \frac{\pi}{2}\beta$$

and

$$l\leq -1 \text{ when } \arg\ p(z_0) = -\frac{\pi}{2}\beta.$$

Theorem 3. *Let the function $f \in A_p$, if for $g,\ h \in A_p,\ p \in \mathbb{N}, p > m, 0 < \beta \leq p$,*

$$\left| 1 + \frac{z\left(f * g\right)^{m+2}(z)}{\left(f * g\right)^{m+1}(z)} - \frac{z\left(f * h\right)^{m+1}(z)}{\left(f * h\right)^{m}(z)} \right| < \frac{\beta}{(p-m)+\beta}, \tag{18}$$

holds, then $f \in \mathfrak{R}_h^g\left(p, m, \beta\right)$.

Proof. Let $w(z)$ be defined by

$$\frac{z\left(f * g\right)^{m+1}(z)}{\left(f * h\right)^{m}(z)} = (p-m) + \beta w(z).$$

Clearly $w(z)$ is analytic in Δ and $w(0) = 0$. Differentiating logarithmically, we obtain

$$1 + \frac{z\left(f * g\right)^{m+2}(z)}{\left(f * g\right)^{m+1}(z)} - \frac{z\left(f * h\right)^{m+1}(z)}{\left(f * h\right)^{m}(z)} = \frac{z\beta w^{'}(z)}{\left[(p-m) + \beta w(z)\right]}.$$

Suppose that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z|<|z_0|} |w(z)| = \left|w\left(z_0\right)\right| = 1 \quad \left(w(z_0) \neq 1\right).$$

Then using Jack's Lemma 1, we get $z_0 w^{'}\left(z_0\right) = kw\left(z_0\right)$ $(k \geq 1)$. Therefore, letting $w\left(z_0\right) = e^{i\theta}$ $(\theta \neq 0)$,

$$\begin{aligned} \left| 1 + \frac{z_0\left(f * g\right)^{m+2}\left(z_0\right)}{\left(f * g\right)^{m+1}\left(z_0\right)} - \frac{z_0\left(f * h\right)^{m+1}\left(z_0\right)}{\left(f * h\right)^{m}\left(z_0\right)} \right| &= \left| \frac{z_0 \beta w^{'}\left(z_0\right)}{(p-m) + \beta w\left(z_0\right)} \right| \\ &= \frac{\beta k}{\left\{(p-m)^2 + \beta^2 + 2\beta\left(p-m\right)\cos\ \theta\right\}^{\frac{1}{2}}} \\ &\geq \frac{\beta}{p-m+\beta}, \end{aligned}$$

which contradicts the condition (18), we have $|w(z)| < 1$ for all $z_0 \in \Delta$, consequently, we conclude that $f \in \mathfrak{R}_h^g\left(p, m, \beta\right)$.

Taking $h = g$, we get following inclusion result with the help of Jack's Lemma.

Theorem 4. *For $p > m$, $\mathfrak{R}_g\left(p, m+1, \beta\right) \subset \mathfrak{R}_g\left(p, m, \alpha\right)$, where*

$$0 < \alpha \leq \frac{-\left(p-m-\beta+1\right) \pm \sqrt[2]{\left(p-m-\beta+1\right)^2 + 4\beta\left(p-m\right)}}{2} \leq p-m. \tag{19}$$

Proof. Let $f \in \Re_g\left(p, m+1, \beta\right)$. Then

$$\left|\frac{z\left(f*g\right)^{m+2}(z)}{\left(f*g\right)^{m+1}(z)} - (p-m-1)\right| < \beta \tag{20}$$

and let $w(z)$ be defined by

$$\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*g\right)^{m}(z)} - (p-m) = \alpha w(z). \tag{21}$$

Clearly $w(z)$ is analytic in Δ and $w(0) = 0$. Differentiating logarithmically, we obtain

$$\begin{aligned}
\frac{z\left(f*g\right)^{m+2}(z)}{\left(f*g\right)^{m+1}(z)} &= (p-m-1) + \alpha w(z) + \frac{\alpha z w'(z)}{(p-m)+\alpha w(z)} \\
\frac{z\left(f*g\right)^{m+2}(z)}{\left(f*g\right)^{m+1}(z)} - (p-m-1) &= \alpha w(z)\left[1 + \frac{\alpha z w'(z)}{\alpha w(z)}\frac{1}{(p-m)+\alpha w(z)}\right].
\end{aligned}$$

Now, suppose that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z|<|z_0|} |w(z)| = \left|w(z_0)\right| = 1 \quad \left(w(z_0) \neq 1\right).$$

Using Jack's Lemma 1, we have $z_0 w'(z_0) = k w(z_0)$ $(k \geq 1)$. Therefore, letting $w(z_0) = e^{i\theta}$ $(\theta \neq 0)$,

$$\begin{aligned}
\left|\frac{z_0\left(f*g\right)^{m+2}(z_0)}{\left(f*g\right)^{m+1}(z_0)} - (p-m-1)\right| &= \alpha\left|w(z_0)\right|\left|1 + \frac{\alpha z_0 w'(z_0)}{\alpha w(z_0)}\frac{1}{(p-m)+\alpha w(z_0)}\right| \\
&= \alpha\left|1 + \frac{k}{(p-m)+\alpha e^{i\theta}}\right| \\
&\geq \alpha\left[1 + \frac{k}{(p-m)}\mathrm{Re}\left\{\frac{1+\frac{\alpha}{(p-m)}\cos\theta - i\frac{\alpha}{(p-m)}\sin\theta}{1+\left(\frac{\alpha}{(p-m)}\right)^2 + \frac{2\alpha}{(p-m)}\cos\theta}\right\}\right] \\
&= \alpha\left[1 + \frac{k}{(p-m)}\left\{\frac{1}{2+\frac{\left(\frac{\alpha}{(p-m)}\right)^2-1}{1+\frac{\alpha}{(p-m)}\cos\theta}}\right\}\right] \\
&\geq \alpha\left[1 + \frac{1}{(p-m)}\left\{\frac{(p-m+\alpha)(p-m)}{2(p-m+\alpha)(p-m)+\alpha^2-(p-m)^2}\right\}\right] \\
&= \alpha\left\{\frac{p-m+\alpha+1}{p-m+\alpha}\right\},
\end{aligned}$$

on using (19), it gives

$$\left|\frac{z\left(f*g\right)^{m+2}(z)}{\left(f*g\right)^{m+1}(z)}-\left(p-m-1\right)\right|\geq\beta$$

which contradicts (20). Hence $|w(z)|<1$ and from (21), it follows that $f\in\Re_g\left(p,m,\alpha\right)$.

Theorem 5. *Let $f\in A_p$ if*

$$Re\left\{\left[\delta\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}+(1-\delta)z\left\{\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}\right\}^{'}\right]\right\}>\gamma\ \left(z\in\Delta\right),$$

for some $\gamma\left(\gamma<\delta\left(p-m\right)\right)$, $0\leq\delta\leq1$, then $f\in S_h^g\left(p,m,\beta\right)$, where $\beta=\frac{2\left(\delta\left(p-m\right)-\gamma\right)}{1+\delta}\leq p$.

Proof. If $\delta=1$, the result holds. Let $0\leq\delta<1$, define the function $p(z)$ by

$$\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}=\left(p-m-\beta\right)+\beta p(z). \tag{22}$$

Then $p(z)=1+p_1z+p_2z^2+\cdots$ is regular in Δ. It follows from (22) that

$$1+z\frac{\left(f*g\right)^{m+2}(z)}{\left(f*g\right)^{m+1}(z)}-z\frac{\left(f*h\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}=\frac{\beta zp^{'}(z)}{\left(p-m-\beta\right)+\beta p(z)},$$

or,

$$z\frac{\left(f*h\right)^{m}(z)}{\left(f*g\right)^{m+1}(z)}\left\{\frac{\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}\right\}^{'}=\frac{\beta zp^{'}(z)-\left\{\left(p-m-\beta\right)+\beta p(z)\right\}}{\left(p-m-\beta\right)+\beta p(z)},$$

or, equivalently

$$z^2\left\{\frac{\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}\right\}^{'}=\beta zp^{'}(z)-\left\{\left(p-m-\beta\right)+\beta p(z)\right\}.$$

Therefore, we have

$$\begin{aligned}&\text{Re}\left\{\left[\delta\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}+(1-\delta)z\left\{\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}\right\}^{'}\right]-\gamma\right\}\\ =\ &\text{Re}\left\{\left[\frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}+(1-\delta)z^2\left\{\frac{\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)}\right\}^{'}\right]-\gamma\right\}\end{aligned}$$

$$= \quad \mathrm{Re}\left\{\delta\left(p-m-\beta\right)+\beta\delta p\left(z\right)+\left(1-\delta\right)\beta z p^{'}\left(z\right)-\gamma\right\}>0$$

If we define a function $\phi\left(u,v\right)$ by

$$\phi\left(u,v\right)=\delta\left(p-m-\beta\right)+\beta\delta u+\left(1-\delta\right)\beta v-\gamma \tag{23}$$

with $u=u_1+iu_2$ and $v=v_1+iv_2$, then

(i) $\phi\left(u,v\right)$ is continuous in $D\subset\mathbb{C}\times\mathbb{C}$;

(ii) $(1,0)\in D$ and Re $\phi\left(1,0\right)=\delta\left(p-m\right)-\gamma>0$;

(iii) For all $\left(iu_2,v_1\right)\in D$ and such that for $v_1\leq-\left(1+u_2^2\right)/2$, we get

$$\begin{aligned}\mathrm{Re}\left\{\phi\left(iu_2,v_1\right)\right\} &= \delta\left(p-m-\beta\right)+\left(1-\delta\right)\beta v_1-\gamma\\ &\leq \delta\left(p-m-\beta\right)-\left(1-\delta\right)\beta\left(1+u_2^2\right)/2-\gamma\\ &= -\left(1-\delta\right)\beta u_2^2/2\\ &\leq 0.\end{aligned}$$

Therefore, $\phi\left(u,v\right)$ satisfies the conditions of Lemma 2. This show that Re $\left(p\left(z\right)\right)>0$ $(z\in\Delta)$, i.e.

$$\mathrm{Re}\left\{\frac{z\left(f*g\right)^{m+1}\left(z\right)}{\left(f*h\right)^{m}\left(z\right)}\right\}>p-m-\beta\ \ (z\in\Delta)$$

which proves that $f\left(z\right)\in S_h^g\left(p,m,\beta\right)$.

Theorem 6. *Let for $p\in\mathbb{N},p>m,m\in\mathbb{N}_0,0<\beta\leq p$, if*

$$\left|\arg\left\{\frac{1}{\left(p-m\right)}\left\{\frac{z\left(f*g\right)^{m+1}\left(z\right)}{\left(f*h\right)^{m}\left(z\right)}+z\left(\frac{z\left(f*g\right)^{m+1}\left(z\right)}{\left(f*h\right)^{m}\left(z\right)}\right)^{'}\right\}\right\}\right|<\frac{\beta}{p}\frac{\pi}{2}+\tan^{-1}\left(\frac{\beta}{p}\right)\ (z\in\Delta), \tag{24}$$

then $\left|\arg\left\{\frac{z\left(f*g\right)^{m+1}\left(z\right)}{\left(f*h\right)^{m}\left(z\right)}\right\}\right|<\frac{\beta}{p}\frac{\pi}{2}\ (z\in\Delta)$. *In particular, if*

$$\left|\arg\left\{\frac{zf^{'}\left(z\right)}{pf\left(z\right)}\left(2+\frac{zf^{''}\left(z\right)}{f^{'}\left(z\right)}-\frac{zf^{'}\left(z\right)}{f\left(z\right)}\right)\right\}\right|<\frac{\beta}{p}\frac{\pi}{2}+\tan^{-1}\left(\frac{\beta}{p}\right)\ (z\in\Delta),$$

then $f\in\overline{S}_p^*\left(\beta\right)$.

Proof. Let

$$p(z):=\frac{1}{\left(p-m\right)}\left\{\frac{z\left(f*g\right)^{m+1}\left(z\right)}{\left(f*h\right)^{m}\left(z\right)}\right\}.$$

We obtain

$$zp^{'}(z) = \frac{z}{(p-m)} \left\{ \frac{z\left(f*g\right)^{m+1}(z)}{\left(f*h\right)^{m}(z)} \right\}^{'}.$$

Suppose that there exists point $z_0 \in \Delta$ such that

$$\left|\arg p(z)\right| < \frac{\beta}{p}\frac{\pi}{2} \text{ for } |z| < \left|z_0\right|, \ \left|\arg p\left(z_0\right)\right| = \frac{\beta}{p}\frac{\pi}{2}.$$

Then applying Lemma 3, we write that

$$\frac{z_0 p^{'}\left(z_0\right)}{p\left(z_0\right)} = il\frac{\beta}{p}$$

where

$$l \geq 1 \text{ when } \arg p\left(z_0\right) = \frac{\beta}{p}\frac{\pi}{2}$$

and

$$l \leq -1 \text{ when } \arg p\left(z_0\right) = -\frac{\beta}{p}\frac{\pi}{2}.$$

Then it follows that

$$\begin{aligned}
& \arg\left\{\frac{1}{(p-m)}\left(\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)} + z_0\left[\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)}\right]^{'}\right)\right\} \\
= & \arg\left\{p\left(z_0\right) + z_0 p^{'}\left(z_0\right)\right\} \\
= & \arg\left\{p\left(z_0\right)\left(1 + \frac{z_0 p^{'}\left(z_0\right)}{p\left(z_0\right)}\right)\right\} \\
= & \arg p\left(z_0\right) + \arg\left(1 + il\frac{\beta}{p}\right) \\
= & \arg p\left(z_0\right) + \tan^{-1}\left(l\frac{\beta}{p}\right).
\end{aligned}$$

When $\arg p\left(z_0\right) = \frac{\beta}{p}\frac{\pi}{2}$, we have

$$\begin{aligned}
& \arg\left\{\frac{1}{(p-m)}\left\{\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)} + z_0\left(\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)}\right)^{'}\right\}\right\} \qquad (25) \\
= & \frac{\beta}{p}\frac{\pi}{2} + \tan^{-1}\left(l\frac{\beta}{p}\right) \\
\geq & \frac{\beta}{p}\frac{\pi}{2} + \tan^{-1}\left(\frac{\beta}{p}\right).
\end{aligned}$$

Similarly, if $\arg p\left(z_0\right) = -\frac{\beta}{p}\frac{\pi}{2}$, then we obtain that

$$\arg\left\{\frac{1}{(p-m)}\left(\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)} + z_0\left[\frac{z_0\left(f*g\right)^{m+1}\left(z_0\right)}{\left(f*h\right)^{m}\left(z_0\right)}\right]'\right)\right\} \tag{26}$$

$$\begin{aligned} &= -\frac{\beta}{p}\frac{\pi}{2} + \tan^{-1}\left(l\frac{\beta}{p}\right) \\ &\leq -\left(\frac{\beta}{p}\frac{\pi}{2} + \tan^{-1}\left(\frac{\beta}{p}\right)\right). \end{aligned}$$

Thus we see that (25) and (26) contradicts the condition (24). Consequently, we conclude that

$$\left|\arg p(z)\right| < \frac{\beta}{p}\frac{\pi}{2}\ (z \in \Delta).$$

This proves Theorem 6.

4. Integral Means Inequality for the Class $\mathfrak{R}_h^g\left(p, m, \beta\right)$

Definition 3. *[Subordination Principle]. For two functions f_1 and f_2, analytic in Δ, we say that the function $f_1(z)$ is subordinate to $f_2(z)$ in Δ, and write*

$$f_1(z) \prec f_2(z) \quad (z \in \Delta),$$

if there exists a Schwartz function $w(z)$, analytic in Δ with

$$w(0) = 0 \ \text{ and } \ |w(z)| < 1,$$

such that

$$f_1(z) = f_2(w(z)) \quad (z \in \Delta).$$

In particular, if the function f_2 is univalent in Δ, the subordination is equivalent to

$$f_1(0) = f_2(0) \ \text{ and } \ f_1(\Delta) \subset f_2(\Delta).$$

Littlewood [16] proved the following subordination result (See also Duren [11]).

Lemma 4. *[16] If f_1 and f_2 are analytic in Δ with $f_1 \prec f_2$, then for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} \left|f_1(z)\right|^{\tau} d\theta \leq \int_0^{2\pi} \left|f_2(z)\right|^{\tau} d\theta.$$

Theorem 7. *Let $g(z)$, $h(z)$ be of the form (2), (3) respectively and $f \in \mathfrak{R}_h^g(p,m,\beta)$ be of the form (7) and let for some $i \in \mathbb{N}$,*

$$\frac{\varphi_i}{b_{p+i}} = \min_{k\geq 1} \frac{\varphi_k}{b_{p+k}} ,$$

where $\varphi_k := \frac{(p+k)!d_{p+k}}{(p+k-m)!}$ and $d_{p+k} := \left[(p+k-m)\, b_{p+k} - (p-m-\beta)\, c_{p+k}\right] > 0$. Also let for such $i \in \mathbb{N}$, functions f_i and g_i be defined respectively by

$$f_i(z) = z^p - \frac{\beta p!\,(p+i-m)!}{d_{p+i}\,(p+i)!\,(p-m)!} z^{p+i} \ , \ g_i = z^p + b_{p+i} z^{p+i}, \tag{27}$$

If there exists an analytic function w defined by

$$\{w(z)\}^i = \frac{d_{p+i}\,(p+i)!\,(p-m)!}{b_{p+i}\beta p!\,(p+i-m)!} \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k$$

then, for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|(f*g)(z)\right|^{\tau} d\theta \leq \int_0^{2\pi} \left|(f_i * g_i)\right|^{\tau} d\theta \ \ (\tau > 0).$$

Proof. Convolution of f and g is defined as:

$$(f*g)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = z^p \left(1 - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k\right)$$

Similarly, from (27), we obtain

$$\begin{aligned}(f_i * g_i)(z) &= z^p - \frac{b_{p+i}\beta p!\,(p+i-m)!}{d_{p+i}\,(p+i)!\,(p-m)!} z^{p+i} \\ &= z^p \left(1 - \frac{b_{p+i}\beta p!\,(p+i-m)!}{d_{p+i}\,(p+i)!\,(p-m)!} z^i\right).\end{aligned}$$

To prove the theorem, we must show that for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|1 - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k\right|^{\tau} d\theta \leq \int_0^{2\pi} \left|1 - \frac{b_{p+i}\beta p!\,(p+i-m)!}{d_{p+i}\,(p+i)!\,(p-m)!} z^i\right|^{\tau} d\theta.$$

Thus, by applying Lemma 4, it would suffice to show that

$$1 - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k \prec 1 - \frac{b_{p+i}\beta p!\,(p+i-m)!}{d_{p+i}\,(p+i)!\,(p-m)!} z^i. \tag{28}$$

If the subordination (28) holds true, then there exist an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k = 1 - \frac{b_{p+i} \beta p! (p+i-m)!}{d_{p+i} (p+i)! (p-m)!} \{w(z)\}^i .$$

From the hypothesis of the theorem, there exists an analytic function w given by

$$\{w(z)\}^i = \frac{d_{p+i} (p+i)! (p-m)!}{b_{p+i} \beta p! (p+i-m)!} \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^k$$

which readily yields $w(0) = 0$. Thus for such function w, using the hypothesis in the coefficient inequality for the class $\mathfrak{R}_h^g(p, m, \beta)$, we get

$$\begin{aligned} |w(z)|^r &\leq \frac{d_{p+i} (p+i)! (p-m)!}{b_{p+i} \beta p! (p+i-m)!} \sum_{k=1}^{\infty} a_{p+k} b_{p+k} |z|^k \\ &\leq |z| \frac{d_{p+i} (p+i)! (p-m)!}{b_{p+i} \beta p! (p+i-m)!} \sum_{k=1}^{\infty} a_{p+k} b_{p+k} \\ &\leq |z| < 1. \end{aligned}$$

Therefore the subordination (28) holds true, thus the theorem is proved.

5. Class-preserving Integral-Operators for the Class $\mathfrak{R}_h^g(p, m, \beta)$

In this section, we present several integral operators which preserve class $\mathfrak{R}_h^g(p, m, \beta)$. For $f \in \mathfrak{R}_h^g(p, m, \beta)$, we define the integral operators by

$$\begin{aligned} L_1 f(z) &= \frac{(p+c)}{z^c} \int_0^z t^{c-1} f(t)\, dt, c > -p, \\ L_2 f(z) &= \frac{(p+c)^\sigma}{z^c \Gamma(\sigma)} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\sigma-1} f(t)\, dt, c > -p, \sigma \geq 0, \\ L_3 f(z) &= \binom{p+c+\sigma-1}{p+c-1} \frac{\sigma}{z^c} \int_0^z \left(1 - \frac{t}{z}\right)^{\sigma-1} t^{c-1} f(t)\, dt, c > -p, \ \sigma \geq 0. \end{aligned}$$

Theorem 8. *Let $f \in \mathfrak{R}_h^g(p, m, \beta)$, then for $p > m, 0 < \beta \leq p, c > -p$ and $\sigma \geq 0$, $L_j f \in \mathfrak{R}_h^g(p, m, \beta)$, $j = 1, 2, 3$.*

Proof. Let $f \in T_p$ of the form (7) be in the class $\mathfrak{R}_h^g(p,m,\beta)$, then

$$L_1 f(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k} \right) a_{p+k} z^{p+k},$$

$$L_2 f(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k} \right)^{\sigma} a_{p+k} z^{p+k},$$

$$L_3 f(z) = z^p - \sum_{k=1}^{\infty} \frac{(p+c)_k}{(p+c+\sigma)_k} a_{p+k} z^{p+k}.$$

Since $\left(\frac{c+p}{c+p+k} \right) < 1$, for $\sigma \geq 0$, $\left(\frac{c+p}{c+p+k} \right)^{\sigma} \leq 1$ and $\frac{(p+c)_k}{(p+c+\sigma)_k} \leq 1, k \geq 1$, by Theorem 1, we see that

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{(p+k)!\left[(p+k-m)\, b_{p+k} - (p-m-\beta)\, c_{p+k}\right]}{(p+k-m)!} \left(\frac{c+p}{c+p+k} \right) a_{p+k} \\ \leq \quad & \sum_{k=1}^{\infty} \frac{(p+k)!\left[(p+k-m)\, b_{p+k} - (p-m-\beta)\, c_{p+k}\right]}{(p+k-m)!} a_{p+k} \leq \frac{\beta p!}{(p-m)!}. \end{aligned}$$

Hence, by Theorem 1, $L_1 f(z) \in \mathfrak{R}_h^g(p,m,\beta)$. Also

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{(p+k)!\left[(p+k-m)\, b_{p+k} - (p-m-\beta)\, c_{p+k}\right]}{(p+k-m)!} \left(\frac{c+p}{c+p+k} \right)^{\sigma} a_{p+k} \\ \leq \quad & \sum_{k=1}^{\infty} \frac{(p+k)!\left[(p+k-m)\, b_{p+k} - (p-m-\beta)\, c_{p+k}\right]}{(p+k-m)!} a_{p+k} \leq \frac{\beta p!}{(p-m)!}. \end{aligned}$$

Hence, $L_2 f(z) \in \mathfrak{R}_h^g(p,m,\beta)$. Similarly, we obtain that $L_3 f(z) \in \mathfrak{R}_h^g(p,m,\beta)$.

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