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Fredholmness of Combinations of Two Idempotents

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Abstract. If P and Q are two idempotents on a Hilbert space, in this paper, we prove that Fredholmness of $aP + bQ - cPQ$ is independent of the choice of a, b, c with $ab \neq 0$.

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1. Introduction

Idempotents are important and have wide applications in the theory of linear algebra and operator theorem. It is shown in [17] that every $n \times n$ matrix over a field of characteristic zero is a linear combination of three idempotents and in [16] that every bounded linear operator on a complex infinite Hilbert space is a sum of at most five idempotents. See also [5],[18],[19].

Let X be a Banach space, and P, Q be two idempotent operators on X. Many researchers (see [1]-[15] and the references within) have addressed stability properties of the linear combination $aP + bQ$; it has been proved that some properties such as invertibility, nullity, Fredholmness, closeness of the range and complementarity of the Kernel of linear combinations of P and Q are independent of the choice of coefficients a and b, provided $ab \neq 0$ and $a + b \neq 0$.

A natural question is whether the results above can be extended to more general situations. In this note we consider the Fredholmness of some special combinations $aP + bQ - cQP$ and $aP + bQ - cPQ - dQP$ when P, Q are idempotents. We prove that Fredholmness and index of any combinations $aP + bQ - cQP$ are independent of the choice of a, b, c with $ab \neq 0$. As an application, we obtain that the invertibility of combinations $aP + bQ - cQP$ are equivalent to the invertibility of $P + Q$ for all $a, b, c \in \mathbb{C}$ with $ab \neq 0$, which generalizes the result of [4]. Moreover, counter examples are shown that the combination $aP + bQ - cPQ - dQP$ fails to retain any such properties.

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2. Preliminaries

Let $\mathscr{H}$ be a Hilbert space, and let all bounded linear operators on $\mathscr{H}$ be denoted by $\mathscr{B}(\mathscr{H})$. An operator $P \in \mathscr{B}(\mathscr{H})$ is said to be idempotent if $P^2 = P$. The set $\mathscr{P}$ of all idempotents in $\mathscr{B}(\mathscr{H})$ is invariant under similarity; that is, is $P \in \mathscr{P}$ and $S \in \mathscr{B}(\mathscr{H})$ is an invertible operator, then $S^{-1}PS$ is still an idempotent since $(S^{-1}PS)^2 = S^{-1}PSS^{-1}PS = S^{-1}P^2S = S^{-1}PS$. An idempotent P is called an orthogonal projection if $P^2 = P = P^*$, where P^* is the adjoint of P. Moreover, for an idempotent $P \in \mathscr{P}$, there exists an invertible operator $U \in \mathscr{B}(\mathscr{H})$ such that $U^{-1}PU$ is an orthogonal projection. In fact, if $P \in \mathscr{P}$, then P can be written in the form of

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{R}(P)^{\perp}$, where $\mathscr{R}(M)$ denotes the range of the operator M. In this case, we have

$$\begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where $\widetilde{P} = \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}$ is invertible and $\widetilde{P}^{-1} = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathscr{H}$. If A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A. An operator T is Fredholm if the nullities of T denoted by nul(T) and T^* are finite and the range of T is closed. For a Fredholm operator T, its index, ind$T$, is by definition nul(T)-nul(T^*). It is know that the Fredholmness of T is preserved under compact perturbations and is equivalent to the existence of an operator T' with $TT' - I$ and $T'T - I$ being compact. For details of Fredholmness, see[3], Chapter XI.

For the proof of the main theorem we need the following two lemmas which are well known, so the proofs are omitted.

Lemma 1 ([3]). *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a bounded linear operator on* $\mathscr{H} \oplus \mathscr{K}$. *Then* A *is a positive operator if and only if* $A_{11} \geq 0$, $A_{22} \geq 0$, $A_{12} = A_{21}^*$ *and there exists a contraction* D *from* $\mathscr{K}$ *into* $\mathscr{H}$ *such that*

$$A = \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}} \\ A_{22}^{\frac{1}{2}} D^* A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix}.$$

Lemma 2 ([3]). *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be an operator on* $\mathscr{H} \oplus \mathscr{K}$, *where* A *is Fredholm with* A' *act on* $\mathscr{H}$ *satisfying* $AA' = I + K_1$ *and* $A'A = I + K_2$ *for some compact operators* K_1 *and* K_2. *Then* T *is Fredholm if and only if* $D - CA'B$ *is. In this case,* $\text{ind}T = \text{ind}A + \text{ind}(D - CA'B)$.

3. Main results

Theorem 1. *Let P and Q in $\mathscr{B}(\mathscr{H})$ be two idempotents, then the Fredholmness of $aP+bQ-cPQ$ is independent of the choice of a, b, c with $ab \neq 0$ and $ind(aP+bQ-cPQ) = ind(P+Q)$.*

Proof. Let P and Q be two idempotents. By the discussion above, since $aP + bQ - cPQ$ is Fredholm if and only if $aS^{-1}PS + bS^{-1}QS - c(S^{-1}PS)(S^{-1}PS)$ is Fredholm, to consider the Fredholmness of $aP + bQ - cPQ$, without loss of generality, we can assume that one of P and Q is an orthogonal projection. For example, assume that Q is an orthogonal projection. Of course, Q is a positive operator. In this case, by Lemma 1, P and Q have the following operator matrix forms:

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} Q_1 & Q_1^{\frac{1}{2}} D Q_2^{\frac{1}{2}} \\ Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}} & Q_2 \end{pmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{R}(P)^{\perp}$, where Q_1 and Q_2 are positive operators on $\mathscr{R}(P)$ and $\mathscr{R}(P)^{\perp}$, respectively, and D is a contraction operator from $\mathscr{R}(P)^{\perp}$ into $\mathscr{R}(P)$. Furthermore, Q_1 and Q_2 have the following operator matrix forms:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{11} \end{pmatrix}, \; Q_2 = \begin{pmatrix} Q_{22} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respect to the space decomposition

$$\mathscr{R}(P) = \mathscr{N}(Q_1) \oplus \mathscr{N}(I - Q_1) \oplus (\mathscr{R}(P) \ominus (\mathscr{N}(Q_1) \oplus \mathscr{N}(I - Q_1)))$$

and the space decomposition

$$\mathscr{R}(P)^{\perp} = (\mathscr{R}(P)^{\perp} \ominus \mathscr{N}(I - Q_2)) \oplus \mathscr{N}(I - Q_2) \oplus \mathscr{N}(Q_2),$$

respectively. Then denote $\mathscr{H}_0 = \mathscr{N}(Q_1)$, $\mathscr{H}_1 = \mathscr{N}(I-Q_1)$, $\mathscr{H}_2 = \mathscr{R}(P) \ominus (\mathscr{N}(Q_1) \oplus \mathscr{N}(I-Q_1))$, $\mathscr{H}_3 = \mathscr{R}(P)^{\perp} \ominus \mathscr{N}(I - Q_2)$ and $\mathscr{H}_4 = \mathscr{N}(I - Q_2)$, $\mathscr{H}_5 = \mathscr{N}(Q_2)$, therefore P and Q have the following matrix representations:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} & P_{13} \\ 0 & I & 0 & P_{21} & P_{22} & P_{23} \\ 0 & 0 & I & P_{31} & P_{32} & P_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathscr{H} = \oplus_{i=0}^{5}\mathscr{H}_i$ for some contraction D_1 from $\mathscr{H}_3$ to $\mathscr{H}_2$. If we let

$$Q_0 = \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix},$$

then Q being an orthogonal projection implies that Q_0 is also an orthogonal projection on $\mathscr{H}_2 \oplus \mathscr{H}_3$. That is, $Q_0 = Q_0^2$. We obtain

$$\begin{cases} Q_{11} = Q_{11}^2 + Q_{11}^{\frac{1}{2}}D_1Q_{22}D_1^*Q_{11}^{\frac{1}{2}}, \\ Q_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{1}{2}} = Q_{11}^{\frac{3}{2}}D_1Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{3}{2}}, \\ Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}} = Q_{22}^{\frac{3}{2}}D_1^*Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{3}{2}}, \\ Q_{22} = Q_{22}^2 + Q_{22}^{\frac{1}{2}}D_1^*Q_{11}D_1Q_{22}^{\frac{1}{2}}. \end{cases}$$

It can be derived by using the injectivity of Q_{11}, $I - Q_{11}$, Q_{22} and $I - Q_{22}$ that

$$\begin{cases} D_1D_1^* = I, \\ D_1^*D_1 = I, \\ Q_{22} = D_1^*(I - Q_{11})D_1. \end{cases} \tag{1}$$

Note that

$$aP + bQ - cPQ = \tag{2}$$

$$= \begin{pmatrix} U_{11} & 0 & U_{13} & U_{14} & U_{15} & U_{16} \\ 0 & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} \\ 0 & 0 & V_{11} & V_{12} & U_{35} & U_{36} \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & U_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

with respect to the space decomposition $\mathscr{H} = \oplus_{i=0}^{5}\mathscr{H}_i$, where

$$\begin{aligned} &U_{11} = aI, && U_{13} = -cP_{11}Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}}, \\ &U_{14} = aP_{11} - cP_{11}Q_{22}, && U_{15} = aP_{12} - cP_{12}, \\ &U_{16} = aP_{13}, && U_{22} = (a + b - c)I, \\ &U_{23} = -cP_{21}Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}}, && U_{24} = aP_{21} - cP_{21}Q_{22}, \\ &U_{25} = aP_{22} - cP_{22}, && U_{26} = aP_{23}, \\ &U_{35} = aP_{32} - cP_{32}, && U_{36} = aP_{33}, \\ &U_{55} = bI. \end{aligned}$$

and

$$\begin{aligned}
V_{11} &= aI + bQ_{11} - c(Q_{11} + P_{31}Q_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}}) \\
&= aI + bQ_{11} - c(Q_{11} + P_{31}D_1^*Q_{11}^{\frac{1}{2}}(I-Q_{11})^{\frac{1}{2}}), \\
V_{12} &= aP_{31} + bQ_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{1}{2}} - c(Q_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{1}{2}} + P_{31}Q_{22}), \\
&= aP_{31} + bQ_{11}^{\frac{1}{2}}(I-Q_{11})^{\frac{1}{2}}D_1 - c(Q_{11}^{\frac{1}{2}}(I-Q_{11})^{\frac{1}{2}}D_1 \\
&\quad + P_{31}D_1^*(I-Q_{11})^{\frac{1}{2}}D_1), \\
V_{21} &= bQ_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}} = bD_1^*Q_{11}^{\frac{1}{2}}(I-Q_{11})^{\frac{1}{2}}, \\
V_{22} &= bQ_{22} = bD_1^*(I-Q_{11})D_1.
\end{aligned}$$

We claim that $aP+bQ-cPQ$ is Fredholm if and only if $I-Q_{11}$ is invertible and $I-P_{31}D_1^*(I-P_{11})^{-\frac{1}{2}}P_{11}^{\frac{1}{2}}$ is Fredholm. Indeed, if $aP+bQ-cPQ$ is Fredholm, then, letting A be an operator on $\mathscr{H}$ such that

$$K = (aP+bQ-cPQ)A - I$$

is compact, we have, with

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \text{ and } K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{R}(P) \oplus \mathscr{R}(P)^{\perp},$$

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} I+K_1 & K_2 \\ K_3 & I+K_4 \end{pmatrix}.$$

Carrying out the mulitiplication here yields

$$bQ_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}}A_2 + bQ_{22}A_4 = I + K_4$$

or

$$bQ_{22}^{\frac{1}{2}}(D_1^*Q_{11}^{\frac{1}{2}}A_2 + Q_{22}^{\frac{1}{2}}A_4) = I + K_4.$$

This shows that $Q_{22}^{\frac{1}{2}}$ is Fredholm and hence so is Q_{22}. Therefore, Q_{22} is invertible and thus so is $I-Q_{11}$ by (1). The Fredholmness of $aP+bQ-cPQ$ is equivalent to that of

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

by (3), which is in turn equivalent to that of

$$V_{11} - V_{12}V_{22}'V_{21} = aI + bQ_{11} - (aP_{31} + bQ_{11}^{\frac{1}{2}}D_1Q_{22}^{\frac{1}{2}})(bQ_{22})'(bQ_{22}^{\frac{1}{2}}D_1^*Q_{11}^{\frac{1}{2}})$$

by Lemma 2. But this letter operator is equal to

$$aI + bQ_{11} - (aP_{31} + bQ_{11}^{\frac{1}{2}}D_1D_1^*(I-Q_{11})^{\frac{1}{2}}D_1)D_1^*(I-Q_{11})^{-\frac{1}{2}}D_1D_1^*Q_{11}^{\frac{1}{2}},$$

which can be further simplified to

$$a(I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}})$$

by (1). This proves one direction. For the other, if $I - Q_{11}$ is invertible and $I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}}$ is Fredholm then we can reverse the above arguments to show that $aP + bQ - cPQ$ is Fredholm. The equivalence of Fredholmness of $aP + bQ - cPQ$ and $P + Q$ follows easily. Finally, we also have

$$\mathrm{ind}(aP + bQ - cPQ) = \mathrm{ind}(I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}}) = \mathrm{ind}(P + Q),$$

which complete the proof.

As an application, we immediately have the following corollary.

Corollary 1. *Let P, Q be two idempotents in $\mathscr{B}(X)$. Then*

(i) the invertibility of $aP + bQ - cQP$ is independent of the choice of $a, b, c \in \mathbb{C}$ and $ab \neq 0$.

(ii) the invertibility of $aP + bQ - cQP$ is equivalent to the invertibility of $aP + bQ$ for all choice of $a, b, c \in \mathbb{C}$ and $ab \neq 0$.

Proof.

(i) Let $a_0P + b_0Q - c_0QP$ be invertible for some $a_0, b_0, c_0 \in \mathbb{C}$ with $a_0b_0 \neq 0$. Then $a_0P + b_0Q - c_0QP$ is Fredholm with the nullity and defect equal to zero. By the above Theorem , $aP + bQ - cQP$ is invertible for all $a, b, c \in \mathbb{C}$ with $ab \neq 0$.

(ii) Let $c = 0$, then the (ii) follows from (i).

Remark 1. *Let $c = 0$, we obtain the Theorems of [4] and [7].*

As to the invertibility of $aP + bQ - cPQ$, there is an natural question that does the combination $aP + bQ - cPQ - dQP$ retain the invertibility for any $ab \neq 0$ and $a + b = c + d$. However, there is an counterexample to note that this is impossible. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$, then P, Q are idempotent and the determinant of $aP + bQ - cPQ - dQP$ is 0 when $a = 12, b = -5, c = 10, d = -3$ with $a + b = c + d$, and is -3 when $a = 1, b = 1, c = -1, d = -1$ with $a + b = c + d$. So the invertibility of $aP + bQ - cPQ - dQP$ depending on the choice of scalars a, b, c, d with $a + b = c + d$. Therefore the idea of generalize the invertibility of $aP + bQ - cPQ$ or $aP + bQ - cQP$ to the invertibility of $aP + bQ - cPQ - dQP$ or more generally $aP + bQ - cPQ - dQP - ePQP - fQPQ - \cdots$ can not be achieved.

References

[1] J.K. Baksalary and O.M. Baksalary. Nonsingularity of linear combinations of idempotent matrices. *Linear Algebra and its Applications*, 388: 25-29, 2004.

[2] J.K. Baksalary and O.M. Baksalary. Idempotency of linear combinations of three idempotent matrices, two of which are disjoint. *Linear Algebra and its Applications* 388: 67-78, 2004.

[3] J.B. Conway. *A Course in Functional Analysis*, 2nd ed., Springer, New York, 1990.

[4] H. Du, X. Yao and C. Deng. Invertibility of linear combinations of two idempotents. *Proceedings of American Mathematical Society*, 134: 1451-1457, 2006.

[5] P.A. Fillmore. On sums of projections. *Journal of Functional Analysis*, 4: 146-152, 1969.

[6] J. Gro and G. Trenkler. Nonsingularity of the difference of two oblique projectors. *SIAM J. Matrix Anal. Appl.*, 21: 390-395, 1999.

[7] H.L. Gau and P.Y. Wu. Fredholmness of linear combinations of two idempotents. *Integral Equations and Opertor Theory*, 59: 579-583, 2007.

[8] H.L. Gau, C.J. Wang and N.C. Wong. Invertibility and Fredholmness of linear combinations of quadratic, *k*-potent and nilpotent operators. *Operators and Matrices*, 2: 193-199, 2008.

[9] R. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, New York and Basel, 1988.

[10] J.J. Koliha and V. Rakočević. Invertibility of the sum of idempotents. *Linear and Multilinear Algebra*, 50: 285-292, 2002.

[11] J.J. Koliha and V. Rakočević. Invertibility of the difference of idempotents. *Linear and Multilinear Algebra*, 51: 97-110, 2003.

[12] J.J. Koliha and V. Rakočević. I.Straškraba. The difference and sum of projectors. *Linear Algebra Appl.*, 388: 279-288, 2004.

[13] J.J. Koliha and V. Rakočević. Fredholm properties of the difference of orthogonal projections in a Hilbert space. *Integr. Equ. Oper. Theory*, 52: 125-134, 2005.

[14] J.J. Koliha and V. Rakočević. The nullity and rank of linear combinations of idempotent matrices. *Linear Algebra Appl.*, 418: 11-14, 2006.

[15] J.J. Koliha and V. Rakočevićs. Stability theorems for linear combinations of idempotents. *Integr. Equ. Oper. Theory*, 58: 597-601, 2007.

[16] C. Pearcy and D. Topping. Sums of small numbers of idempotents. *Michigan Journal of Mathematics*, 14(4): 453-465, 1967.

[17] V. Rabanovic. Every matrix is a linear combination of three idempotents. *Linear Algebra and its Applications*, 390: 137-143, 2004.

[18] P.Y. Wu. Sums of idempotent matrices. *Linear Algebra and its Applications*, 142: 43-54, 1990.

[19] P.Y. Wu. Additive combinations of special operations. *Functional Analysis and Operator Theorem*, 30: 337-361, 1994.