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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Some New Modular Equations of Degree Four and Their Explicit Evaluations

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Abstract. In this paper, we derive several new modular equations of degree 4 by using Ramanujan's modular equations. We also establish several general formulas for explicit evaluations of $h_{4,n}$.

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1. Introduction, Definitions and Notations

In Chapter 16 of his second notebook [7], Ramanujan develops the theory of theta-function and his theta-function is defined by

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \qquad (1) \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \end{aligned}$$

Following Ramanujan, we define

$$\varphi(q) \quad := \quad f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \qquad (2)$$

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$$\begin{aligned}
\psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \\
\chi(q) &:= (-q;q^2)_\infty,
\end{aligned}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \ \mid x \mid < 1,$$

where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{ for } n \geq 1.$$

Let

$$Z(r) := Z(r;x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right),$$

where $r = 2,3,4,6$ and $0 < x < 1$.

Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}, \tag{3}$$

where $r = 2,3,4$ or 6. Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (3). The Ramanujan-Weber class invariant is defined as

$$G_{m/n} = (\alpha(1-\alpha))^{-1/24} \text{ and } G_{mn} = (\beta(1-\beta))^{-1/24}, \tag{4}$$

where β is of degree n over α.
In [9], J. Yi introduced two parameterization $h_{k,n}$ as follows

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}. \tag{5}$$

Yi established some properties and several explicit evaluations of $h_{k,n}$ for different real values of n and k. In [6], M. S. Mahadeva Naika and S. Chandan Kumar have established several new modular equations of degree 2. They have also established general formula for explicit evaluation for $h_{2,n}$. For more detail, one can see [8].

In Section 2, we collect some results which are useful to prove our main results. In Section 3, we establish the modular equations of the ratios of the theta-function $\varphi(q)$. In Section 4, we establish several general formulas for explicit evaluations of $h_{4,n}$ using modular equations established in Section 3.

2. Preliminary Results

In this section, we collect some results which are useful to prove our main results.

Lemma 1. *[3, Entry 10 (i), (v) pp.122] For* $0 < x < 1$*,*

$$\varphi(e^{-y}) = \sqrt{z}, \tag{6}$$

$$\varphi(e^{-4y}) = \frac{1}{2}\sqrt{z}\left(1 + (1-x)^{1/4}\right), \tag{7}$$

where $z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ *and* $y = \pi \dfrac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}$.

Lemma 2. *We have*

1. *[3, Eq. (24.21), p.215] If β is of degree 2 over α, then*

$$\beta = \left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2. \tag{8}$$

2. *[3, Entry 5 (ii), p.230] If β is of degree 3 over α, then*

$$\left(\alpha\beta\right)^{1/4} + \left((1-\alpha)(1-\beta)\right)^{1/4} = 1. \tag{9}$$

3. *[3, Eq. (24.22), p.215] If β is of degree 4 over α, then*

$$\beta = \left(\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right)^4. \tag{10}$$

4. *[3, Entry 13(i), p. 280] If β is of degree 5 over α, then*

$$\left(\alpha\beta\right)^{1/2} + \left((1-\alpha)(1-\beta)\right)^{1/2} + 2\left(\alpha\beta(1-\alpha)(1-\beta)\right)^{1/6} = 1. \tag{11}$$

5. *[3, Entry 19(i), p. 314] If β is of degree 7 over α, then*

$$\left(\alpha\beta\right)^{1/8} + \left((1-\alpha)(1-\beta)\right)^{1/8} = 1. \tag{12}$$

6. *[1, Entry 11.3.3, p. 275] If β has degree 8 over α, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt{2}\sqrt[8]{\beta(1-\alpha)}. \tag{13}$$

7. *[3, Entry 3(x),(xi), p. 352] If β has degree 9 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}, \tag{14}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \tag{15}$$

8. *[4, pp. 387–388] Let*

$$U = 1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}, \tag{16}$$

$$V = 64\left(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{17}$$

and

$$W = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{18}$$

If β has degree 13 over α, then

$$\sqrt{U}(U^3+8W) = \sqrt{W}(11U^2+V). \tag{19}$$

9. *[1, Entry 17.3.26, pp. 391–392] If β has degree 17 over α, then*

$$\begin{aligned} m = &\left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\ &- 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1+\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{17}{m} = &\left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} \\ &- 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}\left\{1+\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}. \end{aligned} \tag{21}$$

3. Some New Modular Equations of Degree Four

In this section, we establish several modular equations of degree four using Ramanujan's modular equations.
Set

$$P := \frac{\varphi(q)}{\varphi(q^4)} \text{ and } Q := \frac{\varphi(q^n)}{\varphi(q^{4n})}. \tag{22}$$

Employing (6) and (7) with $q = e^{-y}$, we find that

$$\frac{2}{P} - 1 = (1-\alpha)^{1/4} \text{ and } \frac{2}{Q} - 1 = (1-\beta)^{1/4}, \tag{23}$$

where β is of degree n over α.

Theorem 1. *If* $X = \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^4)\varphi(q^8)}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^8)}{\varphi(q^4)\varphi(q^2)}$, *then*

$$X^2 - 2X^3Y + X^4Y^2 - 4XY + 4Y^2 - 4XY^3 + 4X^2Y^2 - 2X^3Y^3 + 2X^2Y^4 = 0. \tag{24}$$

Proof. Employing (23) with $n = 2$ in (8), we find that

$$\begin{aligned}(Q^2 - 2Q^2P + Q^2P^2 - 4Q + 4 + 4QP - 4P - 2QP^2 + 2P^2)\\ (-Q^2 + 4Q - 4 - 4QP + 4P + 2QP^2 - 2P^2) = 0.\end{aligned} \tag{25}$$

By examining the first factor near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the second factor does not. By the identity theorem first factor vanishes identically. Hence, by using the fact that $X = PQ$ and $Y = P/Q$, we obtain (24).

Theorem 2. *If* $X = \dfrac{\varphi(q)\varphi(q^3)}{\varphi(q^4)\varphi(q^{12})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{12})}{\varphi(q^4)\varphi(q^3)}$, *then*

$$Y^2 + \frac{1}{Y^2} - 12\left(Y + \frac{1}{Y}\right) + 12\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right) = 4\left(X + \frac{4}{X}\right) + 30. \tag{26}$$

Proof. Employing (23) with $n = 3$ in (9), we find that

$$\begin{aligned}&24PQ^2 + 24QP^2 - 12PQ^3 - 30Q^2P^2 - 12QP^3 + 12P^2Q^3\\ &+ 12P^3Q^2 - 4P^3Q^3 - 16PQ + P^4 + Q^4 = 0.\end{aligned} \tag{27}$$

Using the fact that $X = PQ$ and $Y = P/Q$ in (27), we obtain (26).

Theorem 3. *If* $X = \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^4)\varphi(q^{16})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{16})}{\varphi(q^4)\varphi(q^4)}$, *then*

$$X^4Y^2 + 6X^3Y + X^2 + 16Y^2 + 24XY - 4\sqrt{XY}\left(X^3Y + X^2 + 2X + 8Y\right) = 0. \tag{28}$$

Proof. Employing (23) with $n = 4$ in (10), we find that

$$Q^4 + 16 - 32Q + 24Q^2 - 8Q^3 + Q^4P^4 - 4Q^4P^3 + 6Q^4P^2 - 4Q^4P = 0. \tag{29}$$

Using the fact that $X = PQ$ and $Y = P/Q$ in (29), we obtain (28).

Theorem 4. *If* $X = \dfrac{\varphi(q)\varphi(q^5)}{\varphi(q^4)\varphi(q^{20})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^4)\varphi(q^5)}$, *then*

$$\begin{aligned}
&Y^3 + \frac{1}{Y^3} - 70\left[Y^2 + \frac{1}{Y^2}\right] - 785\left[Y + \frac{1}{Y}\right] + 160\left[\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right] \\
&\times\left[\sqrt{X} + \frac{2}{\sqrt{X}}\right] + 80\left[\sqrt{Y} + \frac{1}{\sqrt{Y}}\right]\left[\sqrt{X^3} + \frac{8}{\sqrt{X^3}} + 10\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\right] \\
&= 80\left[X + \frac{4}{X}\right]\left[5 + 2\left(Y + \frac{1}{Y}\right)\right] + 16\left[X^2 + \frac{16}{X^2}\right] + 1620.
\end{aligned} \tag{30}$$

Proof. Employing (23) with $n = 5$ in (11), we find that

$$\begin{aligned}
&16384 + 147456PQ + 1024aP^2Q^2 - 128aP^5Q^6 - 128aP^6Q^5 + 32aP^6Q^6 \\
&- 2048aP^2Q^3 + 1536aP^2Q^4 - 2048aP^3Q^2 + 4096aP^3Q^3 - 3072aP^3Q^4 \\
&+ 1536aP^4Q^2 - 3072aP^4Q^3 + 2720aP^4Q^4 + 1024aP^3Q^5 - 96aP^3Q^6 \\
&- 1184aP^4Q^5 + 1024aP^5Q^3 - 1184aP^5Q^4 + 176aP^4Q^6 - 96aP^6Q^3 \\
&+ 176aP^6Q^4 - 512aP^2Q^5 + 48aP^2Q^6 - 512aP^5Q^2 + 228864P^2Q^2 + 64Q^6 \\
&+ 48aP^6Q^2 + 61440P^2 + 61440Q^2 - 184320PQ^2 - 184320P^2Q - 960P^5Q^4 \\
&+ 122880PQ^3 - 43776Q^4P - 150528Q^3P^2 + 122880P^3Q - 150528P^3Q^2 \\
&+ 52416P^2Q^4 + 96256P^3Q^3 - 43776P^4Q + 52416P^4Q^2 + 6912Q^5P \\
&- 7872P^2Q^5 - 31872P^3Q^4 - 31872P^4Q^3 - 192Q^6P + 192Q^6P^2 + 4224Q^5P^3 \\
&+ 9600Q^4P^4 + 6912P^5Q - 7872P^5Q^2 + 4224P^5Q^3 - 64P^3Q^6 - 960P^4Q^5 \\
&- 192P^6Q + 192P^6Q^2 - 64P^6Q^3 + 672aP^5Q^5 - 49152P - 49152Q + 64P^6 \\
&- 40960Q^3 + 14592Q^4 - 40960P^3 + 14592P^4 - 2304Q^5 - 2304P^5 = 0,
\end{aligned} \tag{31}$$

where $a = (\alpha\beta)^{1/2}$. Isolating the terms having a on one side of the equation and squaring both sides, we deduce that

$$\begin{aligned}
&(-256PQ - 1600P^2Q^2 + 640PQ^2 + 640P^2Q - 640PQ^3 + 320Q^4P + 1600Q^3P^2 \\
&- 640P^3Q + 1600P^3Q^2 - 785P^2Q^4 - 1620P^3Q^3 + 320P^4Q - 785P^4Q^2 - 70Q^5P \\
&+ 160P^2Q^5 + 800P^3Q^4 + 800P^4Q^3 - 160Q^5P^3 - 400Q^4P^4 - 70P^5Q + 160P^5Q^2 \\
&- 160P^5Q^3 + 80P^4Q^5 + 80P^5Q^4 + Q^6 + P^6 - 16P^5Q^5)(256PQ + 1600P^2Q^2 \\
&- 640PQ^2 - 640P^2Q + P^6Q^6 + 640PQ^3 - 320Q^4P - 1600Q^3P^2 + 640P^3Q \\
&- 1600P^3Q^2 + 815P^2Q^4 + 1620P^3Q^3 - 320P^4Q + 815P^4Q^2 + 70Q^5P - 190P^2Q^5 \\
&- 860P^3Q^4 - 860P^4Q^3 - 6Q^6P + 15Q^6P^2 + 220Q^5P^3 + 490Q^4P^4 + 70P^5Q \\
&- 190P^5Q^2 + 220P^5Q^3 - 20P^3Q^6 - 140P^4Q^5 - 140P^5Q^4 - 6P^6Q + 15P^6Q^2 \\
&- 20P^6Q^3 + Q^6 + P^6 + 15P^4Q^6 + 46P^5Q^5 - 6P^5Q^6 + 15P^6Q^4 - 6P^6Q^5) = 0.
\end{aligned} \tag{32}$$

By examining the first factor near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the second factor does not. By the identity theorem first factor vanishes identically. Hence, by using the fact that $X = PQ$ and $Y = P/Q$, we obtain (30).

Theorem 5. *If* $X = \dfrac{\varphi(q)\varphi(q^7)}{\varphi(q^4)\varphi(q^{28})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{28})}{\varphi(q^4)\varphi(q^7)}$, *then*

$$\begin{aligned}
&Y^4 + \frac{1}{Y^4} - 280\left(Y^3 + \frac{1}{Y^3}\right) - 28\left(Y^2 + \frac{1}{Y^2}\right)\left[349 + 78\left(X + \frac{4}{X}\right)\right] \\
&- 56\left(Y + \frac{1}{Y}\right)\left[1079 + 310\left(X + \frac{4}{X}\right) + 24\left(X^2 + \frac{16}{X^2}\right)\right] \\
&+ 112\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right)\left[515\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right) + 90\left(\sqrt{X^3} + \frac{8}{\sqrt{X^3}}\right)\right. \\
&\left.+4\left(\sqrt{X^5} + \frac{32}{\sqrt{X^5}}\right)\right] + 56\left(\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right)\left[40\left(\sqrt{X^3} + \frac{8}{\sqrt{X^3}}\right)\right. \\
&\left.+313\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\right] + 1176\left(\sqrt{Y^5} + \frac{1}{\sqrt{Y^5}}\right)\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right) \\
&= 16\left[4\left(X^3 + \frac{64}{X^3}\right) + 203\left(X^2 + \frac{16}{X^2}\right) + 2023\left(X + \frac{4}{X}\right)\right] + 106330.
\end{aligned} \tag{33}$$

Proof. Employing (23) with $n = 7$ in (12), we find that

$$-4 + 2Q + 2P - 2PQb + PQc = 0, \tag{34}$$

where $b = (\alpha\beta)^{1/8}$ and $c = (\alpha\beta)^{1/4}$. Isolating the terms having b on one side of the equation and squaring both sides, we deduce that

$$4P^2Q^2c - 16 + 16Q + 16P + 8PQc - 4Q^2 - 8PQ - 4PQ^2c - 4P^2 - 4P^2Qc - P^2Q^2a = 0. \tag{35}$$

Isolating the terms having c on one side of the equation and squaring both sides, we deduce that

$$\begin{aligned}
&8aP^2Q^4 - 32aP^2Q^3 + 32P^2Q^2a + 16aP^4Q^4 + 80aP^3Q^3 - 32aP^3Q^2 \\
&- 256 + 8aP^4Q^2 - 32aP^4Q^3 - 32aP^3Q^4 + 512P + 512Q - 768PQ \\
&- 96P^2Q^2 + 384P^2Q + 384PQ^2 - 64P^3Q - 16P^4 - 384P^2 - 384Q^2 \\
&- 64Q^3P + 128P^3 + 128Q^3 - 16Q^4 - P^4Q^4a^2 = 0.
\end{aligned} \tag{36}$$

Isolating the terms having a on one side of the equation and squaring both sides and then using the fact that $X = PQ$ and $Y = P/Q$, we obtain (33).

Theorem 6. *If* $P = \dfrac{\varphi(q)}{\varphi(q^4)}$ *and* $Q = \dfrac{\varphi(q^8)}{\varphi(q^{32})}$*, then*

$$\begin{aligned}
&\left[P^4+\frac{1}{P^4}\right]\left[-Q^4+16Q^3-112Q^2+448Q-1120+\frac{1792}{Q}-\frac{1792}{Q^2}+\frac{1024}{Q^3}-\frac{256}{Q^4}\right]\\
&+\left[P^3+\frac{1}{P^3}\right]\left[8Q^4+\frac{2048}{Q^4}-\frac{8192}{Q^3}+\frac{14336}{Q^2}-\frac{14336}{Q}\right]+\left[P^2+\frac{1}{P^2}\right]\Big[-28Q^4\\
&-\frac{7168}{Q^4}+\frac{28672}{Q^3}-\frac{50176}{Q^2}+\frac{50176}{Q}\Big]+\left[P+\frac{1}{P}\right]\Big[56Q^4+\frac{14336}{Q^4}-\frac{57344}{Q^3}+\frac{100352}{Q^2}\\
&-\frac{100352}{Q}\Big]+P^3\left[8960+896Q^2-3584Q-128Q^3\right]+30848Q+\frac{125400}{Q}-78144\\
&+\frac{1}{P^3}\left[7936-1536Q-640Q^2+384Q^3\right]+\frac{1}{P^2}\left[-320Q^3-832Q^2+9472Q-29824\right]\\
&+P^2\left[448Q^3-3136Q^2+12544Q-31360\right]+P\left[62720+6272Q^2-896Q^3-25088Q\right]\\
&+\frac{1}{P}\left[61696-23040Q+4736Q^2-384Q^3\right]-70Q^4-\frac{17920}{Q^4}+992Q^3+\frac{71680}{Q^3}\\
&-7456Q^2-\frac{125400}{Q^2}=0.
\end{aligned} \tag{37}$$

Proof. Proof of the identity (37) is similar to the proof of the identity (28) given above except that in place of result (10), result (13) is used.

Theorem 7. *If* $X = \dfrac{\varphi(q)\varphi(q^9)}{\varphi(q^4)\varphi(q^{36})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{36})}{\varphi(q^4)\varphi(q^9)}$*, then*

$$\begin{aligned}
&Y^6+\frac{1}{Y^6}-908\left[Y^5+\frac{1}{Y^5}\right]-83582\left[Y^4+\frac{1}{Y^4}\right]-1369692\left[Y^3+\frac{1}{Y^3}\right]\\
&-3\left[Y^2+\frac{1}{Y^2}\right]\left\{2657883+96832\left[X^2+\frac{16}{X^2}\right]\right\}-24\left[Y+\frac{1}{Y}\right]\\
&\times\left\{892353+289628\left[X+\frac{4}{X}\right]\right\}+17323008\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right]\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]\\
&+1831776\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right]\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]+21504\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right]\\
&\times\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right]-29469924=128\Big\{2\left[X^4+\frac{256}{X^4}\right]\\
&+420\left[X^3+\frac{64}{X^3}\right]+9987\left[X^2+\frac{16}{X^2}\right]+75426\left[X+\frac{4}{X}\right]\Big\}.
\end{aligned} \tag{38}$$

Proof. Employing (23) with $n = 9$ in (14) and (15), we find that

$$2Q^2P-4Q^2P^2c-2aQ^2P+4cP^2Q^2b+8cQ^2P-4cPQ^2b+4bP^2Q^2-4Q^2$$

$$\begin{aligned}&4QP+8bQ^2-4cQ^2-12bPQ^2+8cPbQ+2QP^2a+4aQP-4bP^2Q\\&-2QP^2-8cPQ-12cP^2bQ+8bPQ+8QP^2c-4aP^2-4cP^2+8cP^2b=0.\end{aligned}\tag{39}$$

Isolating the terms having b on one side of the equation and squaring both sides, we deduce that

$$\begin{aligned}&72aQ^4P^2-80Q^3P^4ca+192Q^3P^3ca+144Q^2P^4ca-48Q^4P^3ca+96Q^2P^2ac\\&-144Q^3P^2ac+96acP^3Q-240acP^3Q^2+48acQ^4P^2+16Q^4P^4ac-112QP^4ac\\&-16aQ^4Pc+32cQ^3aP-16Q^2P^2+32Q^3P-32Q^3P^2+16Q^2P^3+16Q^4P^4c\\&-4Q^4P^2+16Q^4P+240aP^3Q^3+16a^2Q^3P^2+8a^2Q^3P^3+192Q^3P^3c+72Q^2P^4a\\&-64Q^3P^4a-4a^2Q^4P^2+32acP^4-32a^2Q^2P^3-16aQ^4P-16QP^4a-4Q^2P^4a^2\\&+16QP^4a^2+32a^2QP^3-16a^2Q^2P^2+96Q^2P^2c-240Q^3P^2c-80cQ^4P^3-4Q^2P^4\\&-240Q^3P^2a+96Q^3Pa+96Q^2P^2a+96Q^3Pc-16QP^4c+96aP^3Q-112cQ^4P\\&-144Q^2P^3c-16a^2P^4-16aP^4+32cQ^4-16aQ^4+48Q^2P^4c+144cQ^4P^2-16Q^4\\&-64aQ^4P^3+16Q^4P^4a+32QP^3c-48Q^3P^4c-240aP^3Q^2+8Q^3P^3=0.\end{aligned}\tag{40}$$

By eliminating c and a in the similar manner and then using the fact that $X=PQ$ and $Y=P/Q$, we obtain (37).

Theorem 8. *If* $X=\dfrac{\varphi(q)\varphi(q^{13})}{\varphi(q^4)\varphi(q^{52})}$ *and* $Y=\dfrac{\varphi(q)\varphi(q^{52})}{\varphi(q^4)\varphi(q^{13})}$, *then*

$$\begin{aligned}&Y^7+\frac{1}{Y^7}-6734\left[Y^6+\frac{1}{Y^6}\right]-13\left(Y^5+\frac{1}{Y^5}\right)\left[173721+49184\left(X+\frac{4}{X}\right)\right]\\&-52\left(Y^4+\frac{1}{Y^4}\right)\left[1803735+627156\left(X+\frac{4}{X}\right)+85300\left(X^2+\frac{16}{X^2}\right)\right]\\&-13\left(Y^3+\frac{1}{Y^3}\right)\left[93381075+35045664\left(X+\frac{4}{X}\right)+6898048\left(X^2+\frac{16}{X^2}\right)\right.\\&\left.+522240\left(X^3+\frac{64}{X^3}\right)\right]-26\left(Y^2+\frac{1}{Y^2}\right)\left[255338797+99428448\left(X+\frac{4}{X}\right)\right.\\&\left.+22686560\left(X^2+\frac{16}{X^2}\right)+2651904\left(X^3+\frac{64}{X^3}\right)+112384\left(X^4+\frac{256}{X^4}\right)\right]\\&-13\left(Y+\frac{1}{Y}\right)\left[1355929177+537574080\left(X+\frac{4}{X}\right)+131058304\left(X^2+\frac{16}{X^2}\right)\right.\\&\left.+18059264\left(X^3+\frac{64}{X^3}\right)+1173504\left(X^4+\frac{256}{X^4}\right)+24576\left(X^5+\frac{1024}{X^5}\right)\right]\\&+416\left(\sqrt{Y}+\frac{1}{\sqrt{Y}}\right)\left[35973930\left(\sqrt{X}+\frac{2}{\sqrt{X}}\right)+11349767\left(\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right)\right.\\&+2161984\left(\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right)+222176\left(\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right)+9984\left(\sqrt{X^9}+\frac{512}{\sqrt{X^9}}\right)\end{aligned}$$

$$+128\left(\sqrt{X^{11}}+\frac{2048}{\sqrt{X^{11}}}\right)\Bigg]+832\left(\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right)\Bigg[9431769\left(\sqrt{X}+\frac{2}{\sqrt{X}}\right)$$
$$+2904505\left(\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right)+519072\left(\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right)+45976\left(\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right)$$
$$+1408\left(\sqrt{X^9}+\frac{512}{\sqrt{X^9}}\right)\Bigg]+416\left(\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right)\Bigg[4977453\left(\sqrt{X}+\frac{2}{\sqrt{X}}\right)$$
$$+1447562\left(\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right)+219840\left(\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right)+12528\left(\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right)\Bigg]$$
$$+208\left(\sqrt{Y^7}+\frac{1}{\sqrt{Y^7}}\right)\Bigg[1197554\left(\sqrt{X}+\frac{2}{\sqrt{X}}\right)+308857\left(\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right)$$
$$+31104\left(\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right)\Bigg]+208\left(\sqrt{Y^9}+\frac{1}{\sqrt{Y^9}}\right)\Bigg[53242\left(\sqrt{X}+\frac{2}{\sqrt{X}}\right)$$
$$+10113\left(\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right)\Bigg]+106912\left[\sqrt{Y^{11}}+\frac{1}{\sqrt{Y^{11}}}\right]\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]=24251297512$$
$$+9667573344\left[X+\frac{4}{X}\right]+2403244896\left[X^2+\frac{16}{X^2}\right]+346604544\left[X^3+\frac{64}{X^3}\right]$$
$$+24986624\left[X^4+\frac{256}{X^4}\right]+692224\left[X^5+\frac{1024}{X^5}\right]+4096\left[X^6+\frac{4096}{X^6}\right]. \tag{41}$$

Theorem 9. *If* $X=\dfrac{\varphi(q)\varphi(q^{17})}{\varphi(q^4)\varphi(q^{68})}$ *and* $Y=\dfrac{\varphi(q)\varphi(q^{68})}{\varphi(q^4)\varphi(q^{17})}$, *then*

$$Y^9+\frac{1}{Y^9}-36754\left[Y^8+\frac{1}{Y^8}\right]-17\left[Y^7+\frac{1}{Y^7}\right]\left\{1096631+719552\left[X+\frac{4}{X}\right]\right\}$$
$$-272\left[Y^6+\frac{1}{Y^6}\right]\left\{2211163+874270\left[X^2+\frac{16}{X^2}\right]+2693790\left[X+\frac{4}{X}\right]\right\}$$
$$+68\left[Y^5+\frac{1}{Y^5}\right]\left\{79079901-15277312\left[X^3+\frac{64}{X^3}\right]-114408800\left[X^2+\frac{16}{X^2}\right]\right.$$
$$\left.-157553008\left[X+\frac{4}{X}\right]\right\}+136\left[Y^4+\frac{1}{Y^4}\right]\left\{2016688545-10983072\left[X^4+\frac{256}{X^4}\right]\right.$$
$$\left.-159079808\left[X^3+\frac{64}{X^3}\right]-636257912\left[X^2+\frac{16}{X^2}\right]-237316728\left[X+\frac{4}{X}\right]\right\}$$
$$+68\left[Y^3+\frac{1}{X^3}\right]\left\{39729209249+4069258864\left[X+\frac{4}{X}\right]-6641286816\left[X^2+\frac{16}{X^2}\right]\right.$$
$$\left.-2422235008\left[X^3+\frac{64}{X^3}\right]-296408768\left[X^4+\frac{256}{X^4}\right]-11767808\left[X^5+\frac{1024}{X^5}\right]\right\}$$

$$+272\left[Y^2+\frac{1}{Y^2}\right]\left\{43899000931+7993457534\left[X+\frac{4}{X}\right]-4759955170\left[X^2+\frac{16}{X^2}\right]\right.$$
$$-2288693056\left[X^3+\frac{64}{X^3}\right]-369671072\left[X^4+\frac{256}{X^4}\right]-24953344\left[X^5+\frac{1024}{X^5}\right]$$
$$\left.-572928\left[X^6+\frac{4096}{X^6}\right]\right\}+34\left[Y+\frac{1}{Y}\right]\left\{821963657831+177636127584\left[X+\frac{4}{X}\right]\right.$$
$$-67278544896\left[X^2+\frac{16}{X^2}\right]-39038801664\left[X^3+\frac{64}{X^3}\right]-7258679936\left[X^4+\frac{256}{X^4}\right]$$
$$\left.-610099200\left[X^5+\frac{1024}{X^5}\right]-22257664\left[X^6+\frac{4096}{X^6}\right]-262144\left[X^7+\frac{16384}{X^7}\right]\right\}$$
$$-1088\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right]\left\{18819607015\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]+80443210\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]\right.$$
$$-2329762760\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right]-714027520\left[\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right]-92048640\left[\sqrt{X^9}+\frac{512}{\sqrt{X^9}}\right]$$
$$\left.-5523712\left[\sqrt{X^{11}}+\frac{2048}{\sqrt{X^{11}}}\right]-139264\left[\sqrt{X^{13}}+\frac{8192}{\sqrt{X^{13}}}\right]-1024\left[\sqrt{X^{15}}+\frac{32768}{\sqrt{X^{15}}}\right]\right\}$$
$$-544\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right]\left\{21182793502\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]-724326781\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]\right.$$
$$-2970898992\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right]-832722704\left[\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right]-96598528\left[\sqrt{X^9}+\frac{512}{\sqrt{X^9}}\right]$$
$$\left.-4848384\left[\sqrt{X^{11}}+\frac{2048}{\sqrt{X^{11}}}\right]-81920\left[\sqrt{X^{13}}+\frac{8192}{\sqrt{X^{13}}}\right]\right\}-544\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right]$$
$$\times\left\{6400381178\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]-866738129\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]-1160442480\left[\sqrt{X^5}\right.\right.$$
$$\left.+\frac{32}{\sqrt{X^5}}\right]-270503280\left[\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right]-24447488\left[\sqrt{X^9}+\frac{512}{\sqrt{X^9}}\right]-742144\left[\sqrt{X^{11}}\right.$$
$$\left.\left.+\frac{2048}{\sqrt{X^{11}}}\right]\right\}-1088\left[\sqrt{Y^7}+\frac{1}{\sqrt{Y^7}}\right]\left\{450392085\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right]-185484849\left[\sqrt{X^3}\right.\right.$$
$$\left.+\frac{8}{\sqrt{X^3}}\right]-128003816\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right]-21820492\left[\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right]-1134848\left[\sqrt{X^9}\right.$$

$$
\begin{aligned}
&\left.+\frac{512}{\sqrt{X^9}}\right]\Bigg\} - 1088\left[\sqrt{Y^9}+\frac{1}{\sqrt{Y^9}}\right]\Bigg\{14667567\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right] - 34598985\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]\\
&-13824368\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right] - 1300340\left[\sqrt{X^7}+\frac{128}{\sqrt{X^7}}\right]\Bigg\} + 544\left[\sqrt{Y^{11}}+\frac{1}{\sqrt{Y^{11}}}\right]\\
&\times\Bigg\{3837042\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right] + 5300551\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right] + 1065760\left[\sqrt{X^5}+\frac{32}{\sqrt{X^5}}\right]\Bigg\}\\
&+1632\left[\sqrt{Y^{13}}+\frac{1}{\sqrt{Y^{13}}}\right]\Bigg\{78306\left[\sqrt{X}+\frac{2}{\sqrt{X}}\right] + 41561\left[\sqrt{X^3}+\frac{8}{\sqrt{X^3}}\right]\Bigg\}\\
&+1160896\left[\sqrt{Y^{15}}+\frac{1}{\sqrt{Y^{15}}}\right]\left\{\sqrt{X}+\frac{2}{\sqrt{X}}\right\} + 36888130319124 = 128\Bigg\{512\left[X^8+\frac{65536}{X^8}\right]\\
&+147968\left[X^7+\frac{16384}{X^7}\right] + 9450368\left[X^6+\frac{4096}{X^6}\right] + 230775680\left[X^5+\frac{1024}{X^5}\right]\\
&+2576886724\left[X^4+\frac{256}{X^4}\right] + 13262885376\left[X^3+\frac{64}{X^3}\right]\\
&+21325126145\left[X^2+\frac{16}{X^2}\right] - 65099344519\left[X+\frac{4}{X}\right]\Bigg\}.
\end{aligned}
\tag{42}
$$

Proofs of the identities (41) and (42) are similar to the proof of the identity (38) given above except that in place of result (12), result (19) is used for proving (41); result (20) and (21) is used for proving (42).

4. General Formulas for Explicit Evaluations of $h_{k,n}$

We shall employ modular equations in Section 3 to establish several general formulas for explicit evaluations of $h_{k,n}$.

Theorem 10. *For any positive real number n, we have*

$$
\begin{aligned}
&h_{4,4n}^2 - 2\sqrt{2}h_{4,n}h_{4,4n}^2 - 2\sqrt{2}h_{4,4n} + 2 + 4h_{4,n}h_{4,4n} - 2\sqrt{2}h_{4,n}\\
&- 2\sqrt{2}h_{4,n}^2h_{4,4n} + 2h_{4,n}^2 + 2h_{4,n}^2h_{4,4n}^2 = 0.
\end{aligned}
\tag{43}
$$

Proof. Employing (5) in (24), we obtain (43).

Corollary 1. *We have*

$$h_{4,4} = \left(2-\sqrt{2\sqrt{2}}\right)(\sqrt{2}+1), \tag{44}$$

$$h_{4,1/4} = \frac{(\sqrt{2}+\sqrt[4]{2})}{2}, \tag{45}$$

$$h_{4,2} = 1+\sqrt{2}-\sqrt{\sqrt{2}+1}, \tag{46}$$

$$h_{4,1/2} = \frac{1+\sqrt{\sqrt{2}-1}}{\sqrt{2}}. \tag{47}$$

Proof. Proofs of (44) and (45)
Putting $n=1$ in (43), we find that

$$h^2-4(\sqrt{2}+1)h+4+2\sqrt{2}=0, \tag{48}$$

where $h := h_{4,4}$.
Solving the above equation, we obtain (44) and (45).

Proof. Proofs of (46) and (47)
Putting $n=1/2$ in (43), we find that

$$(h^2-2(1+\sqrt{2})h+2+\sqrt{2})(h^2-2(\sqrt{2}-1)h+2-\sqrt{2})=0, \tag{49}$$

where $h := h_{4,2}$.
Since the roots of the second factor are imaginary, we deduce that

$$h^2-2(1+\sqrt{2})h+2+\sqrt{2}=0. \tag{50}$$

On solving the above equation, we obtain (46) and (47).

Theorem 11. *For any positive real number n, we have*

$$\begin{aligned} N^2+\frac{1}{N^2}-12\left(N+\frac{1}{N}\right)+12\sqrt{2}\left(\sqrt{M}+\frac{1}{\sqrt{M}}\right)\left(\sqrt{N}+\frac{1}{\sqrt{N}}\right) \\ =8\left(M+\frac{1}{M}\right)+30, \end{aligned} \tag{51}$$

where $M=h_{4,n}h_{4,9n}$ *and* $N=\dfrac{h_{4,n}}{h_{4,9n}}$.

Proof. From the equations (5) and (26), we obtain (51).

Corollary 2. *We have*

$$h_{4,9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}-\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}, \tag{52}$$

$$h_{4,1/9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}+\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}, \tag{53}$$

$$h_{4,3} = \left(\sqrt{2}-1\right)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right), \tag{54}$$

$$h_{4,1/3} = \left(\sqrt{2}+1\right)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right). \tag{55}$$

Proof. [Proofs of (52) and (53)] Putting $n=1$ in (51), we find that

$$x^2-(20-12\sqrt{2})x-32+24\sqrt{2}=0, \tag{56}$$

where $x = h_{4,9}+\dfrac{1}{h_{4,9}}$.
On solving the above equation and $x>1$, we deduce that

$$x = 10-6\sqrt{2}+2\sqrt{51-36\sqrt{2}}. \tag{57}$$

Again solving the above equation, we obtain (52) and (53).

Proof. [Proofs of (54) and (55)] Putting $n=1/3$ in (51), we find that

$$(x^2+2\sqrt{2}x-10)(x-\sqrt{2})^2=0, \tag{58}$$

where $x = h_{4,3}+\dfrac{1}{h_{4,3}}$.
Since $x>1$, we find that

$$x^2+2\sqrt{2}x-10=0. \tag{59}$$

Solving the above equation, we obtain (54) and (55).

Theorem 12. *For any positive real number n, we have*

$$\begin{aligned} &N^3+\frac{1}{N^3}-70\left[N^2+\frac{1}{N^2}\right]-785\left[N+\frac{1}{N}\right]+160\sqrt{2}\left\{\left[\sqrt{N^3}+\frac{1}{\sqrt{N^3}}\right]\right.\\ &\left.\times\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]+\left[\sqrt{N}+\frac{1}{\sqrt{N}}\right]\left[\sqrt{M^3}+\frac{1}{\sqrt{M^3}}+5\left(\sqrt{M}+\frac{1}{\sqrt{M}}\right)\right]\right\}\\ &=160\left[M+\frac{1}{M}\right]\left[5+2\left(N+\frac{1}{N}\right)\right]+64\left[M^2+\frac{1}{M^2}\right]+1620, \end{aligned} \tag{60}$$

where $M = h_{4,n}h_{4,25n}$ *and* $N = \dfrac{h_{4,n}}{h_{4,25n}}$.

Proof. Employing (5) in (30), we obtain (60).

Corollary 3. *We have*

$$h_{4,25} = (\sqrt{2}-1)^4\left(18+8\sqrt{2}+3\sqrt{5}-2\sqrt{27\sqrt{5}+12\sqrt{10}-30\sqrt{2}-20}\right), \tag{61}$$

$$h_{4,1/25} = (\sqrt{2}-1)^4\left(18+8\sqrt{2}+3\sqrt{5}+2\sqrt{27\sqrt{5}+12\sqrt{10}-30\sqrt{2}-20}\right), \tag{62}$$

$$h_{4,5} = \sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}} - \sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}}, \tag{63}$$

$$h_{4,1/5} = \sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}} + \sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}}. \tag{64}$$

Proof. [Proof of (61) and (62)] Putting $n=1$ in (60), we find that

$$(h^4-456h^3+320\sqrt{2}h^3-674h^2+480\sqrt{2}h^2-456h+320\sqrt{2}h+1)(h+1)^2=0, \tag{65}$$

where $h := h_{4,25}$.
Since $h+1 \neq 0$, we find that

$$h^4-456h^3+320\sqrt{2}h^3-674h^2+480\sqrt{2}h^2-456h+320\sqrt{2}h+1=0. \tag{66}$$

The above equation reduces to

$$x^2-2-\left(456-320\sqrt{2}\right)x-674+480\sqrt{2}=0, \tag{67}$$

where $x = h_{4,25}+\dfrac{1}{h_{4,25}}$.
On solving the above equation and $x>2$, we deduce that

$$h_{4,25}+\frac{1}{h_{4,25}} = 228-160\sqrt{2}+102\sqrt{5}-72\sqrt{10}. \tag{68}$$

Solving the above equation, we obtain (61) and (62).

Proof. [Proof of (63) and (64)] Putting $n=1/5$ in (60), we find that

$$\begin{aligned}&(h^4-6h^3+2h^3\sqrt{2}-10h^2+10h^2\sqrt{2}-10h+6\sqrt{2}h-3+2\sqrt{2})\\&(h^4+6h^3+2h^3\sqrt{2}-10h^2-10h^2\sqrt{2}+10h+6\sqrt{2}h-3-2\sqrt{2})\\&(-h-1+\sqrt{2})^2(-h+1+\sqrt{2})^2=0,\end{aligned} \tag{69}$$

where $h := h_{4,5}$.
The first factor of the equation (69) vanishes for the specific value of $q = e^{-\pi\sqrt{5/4}}$ but the other factors does not vanish. Hence,

$$h^4-6h^3+2h^3\sqrt{2}-10h^2+10h^2\sqrt{2}-10h+6\sqrt{2}h-3+2\sqrt{2}=0. \tag{70}$$

The above equation reduces to

$$x^4 - 12x^3 + 12x^2 - 16x - 16 = 0, \tag{71}$$

where $x = h_{4,5} - \dfrac{1}{h_{4,5}}$.
The above equation (71) can be written as

$$x^2 - (6 + 2\sqrt{5})x - 2 - 2\sqrt{5} = 0. \tag{72}$$

Solving the above equation and $x < 1$, we find that

$$x = 3 + \sqrt{5} - 2\sqrt{4 + 2\sqrt{5}}. \tag{73}$$

Again solving the above equation, we obtain (63) and (64).

Theorem 13. *For any positive real number n, we have*

$$\begin{aligned}
&N^4 + \frac{1}{N^4} - 280\left(N^3 + \frac{1}{N^3}\right) - 28\left(N^2 + \frac{1}{N^2}\right)\left[349 + 156\left(M + \frac{1}{M}\right)\right] \\
&- 56\left[N + \frac{1}{N}\right]\left[1079 + 620\left(M + \frac{1}{M}\right) + 96\left(M^2 + \frac{1}{M^2}\right)\right] - 106330 \\
&+ 112\left[\sqrt{N} + \frac{1}{\sqrt{N}}\right]\left[515\sqrt{2}\left(\sqrt{M} + \frac{1}{\sqrt{M}}\right) + 180\sqrt{2}\left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\right. \\
&\left. +16\sqrt{2}\left(\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right)\right] + 56\left[\sqrt{N^3} + \frac{1}{\sqrt{N^3}}\right]\left[80\sqrt{2}\left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\right. \\
&\left. +313\sqrt{2}\left(\sqrt{M} + \frac{1}{\sqrt{M}}\right)\right] + 1176\sqrt{2}\left[\sqrt{N^5} + \frac{1}{\sqrt{N^5}}\right]\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right] \\
&= 32\left[16\left(M^3 + \frac{1}{M^3}\right) + 406\left(M^2 + \frac{16}{M^2}\right) + 2023\left(M + \frac{4}{M}\right)\right],
\end{aligned} \tag{74}$$

where $M = h_{4,n}h_{4,49n}$ *and* $N = \dfrac{h_{4,n}}{h_{4,49n}}$.

Proof. Employing (5) in (33), we obtain (74).

Corollary 4. *We have*

$$h_{4,7} = \left(\frac{3 + \sqrt{7}}{\sqrt{2}}\right)\left(2\sqrt{2} - \sqrt{7}\right), \tag{75}$$

$$h_{4,1/7} = \left(\frac{3 - \sqrt{7}}{\sqrt{2}}\right)\left(2\sqrt{2} + \sqrt{7}\right). \tag{76}$$

Proof. [Proof of (75) and (76)] Putting $n = 1/7$ in (74), we find that

$$\begin{aligned}&(h^2-12h+7\sqrt{2}h+1)(h^2+12h+7\sqrt{2}h+1)(-h^2+\sqrt{2}h-1)^2\\&(-h^2+2h+3\sqrt{2}h-1)^2(-h^2-2h+3\sqrt{2}h-1)^2=0,\end{aligned} \tag{77}$$

where $h := h_{4,7}$.
We observe that the first factor of the equation (77) vanishes for specific value of $q = e^{-\pi\sqrt{7/4}}$, but the other factors does not vanish. Hence, we obtain (75) and (76).

Theorem 14. *For any positive real number n, we have*

$$\begin{aligned}&\left[16M^4+\frac{1}{M^4}\right]\left[-N^4+16N^3-56N^2+112\sqrt{2}N-280+\frac{224\sqrt{2}}{N}-\frac{224}{N^2}\right.\\&\left.+\frac{64\sqrt{2}}{N^3}-\frac{16}{N^4}\right]+16\sqrt{2}\left[M^3+\frac{1}{8M^3}\right]\left[4N^4+\frac{64}{N^4}-\frac{512\sqrt{2}}{N^3}+\frac{1792}{N^2}-\frac{1792\sqrt{2}}{N}\right]\\&+8\left[M^2+\frac{1}{4M^2}\right]\left[-28N^4-\frac{323}{N^4}+\frac{1792\sqrt{2}}{N^3}-\frac{6272}{N^2}+\frac{6272\sqrt{2}}{N}\right]+1984\sqrt{2}N^3\\&+32\sqrt{2}\left[M+\frac{1}{2M}\right]\left[7N^4+\frac{112}{N^4}-\frac{448\sqrt{2}}{N^3}+\frac{1568}{N^2}-\frac{1568\sqrt{2}}{N}\right]-280N^4-\frac{4480}{N^4}\\&+512\sqrt{2}M^3\left[35+7N^2-14\sqrt{2}N-\sqrt{2}N^3\right]+\frac{17920\sqrt{2}}{N^3}-14912N^2-\frac{62700}{N^2}\\&+\frac{64\sqrt{2}}{M^3}\left[31-6\sqrt{2}N-5N^2+3\sqrt{2}N^3\right]+1792M^2\left[\sqrt{2}N^3-7N^2+14\sqrt{2}N-35\right]\\&+\frac{64}{M^2}\left[-5\sqrt{2}N^3-13N^2+74\sqrt{2}N-233\right]+30848\sqrt{2}N+\frac{62700\sqrt{2}}{N}-78144\\&+1792\sqrt{2}M\left[35+7N^2-\sqrt{2}N^3-14\sqrt{2}N\right]\\&+\frac{128\sqrt{2}}{M}\left[241-90\sqrt{2}N+37N^2-3\sqrt{2}N^3\right]=0,\end{aligned} \tag{78}$$

where $M = h_{4,n}$ *and* $N = h_{4,64n}$.

Proof. Employing (5) in (37), we obtain (78).

Corollary 5. *We have*

$$h_{4,8} = \sqrt{2}(\sqrt{2}+1)^{3/2}\left[\sqrt{2}+\sqrt{\sqrt{2}+1}-\sqrt{4+\sqrt{2+10\sqrt{2}}}\right] \tag{79}$$

$$h_{4,1/8} = \frac{(\sqrt{2}-1)^{3/2}}{\sqrt{2}\left[\sqrt{2}+\sqrt{\sqrt{2}+1}-\sqrt{4+\sqrt{2+10\sqrt{2}}}\right]} \tag{80}$$

Proof. [Proof of (79) and (80)] Putting $n = 1/8$ in (78), we find that

$$(-h^4+16h^3+12\sqrt{2}h^3-36h^2-24\sqrt{2}h^2+32h+24\sqrt{2}h-12-8\sqrt{2})$$
$$(-h^4-16h^3+12\sqrt{2}h^3-36h^2+24\sqrt{2}h^2-32h+24\sqrt{2}h-12+8\sqrt{2}) \tag{81}$$
$$(h^2+4h+2\sqrt{2}h-2-2\sqrt{2})^2(h^2-4h+2\sqrt{2}h-2+2\sqrt{2})^2=0,$$

where $h := h_{4,8}$.
We observe that the first factor of the equation (81) vanishes for specific value of $q = e^{-\pi\sqrt{2}}$, but the other factors does not vanish. Hence, we obtain (79) and (80).

Theorem 15. *For any positive real number n, we have*

$$\begin{aligned}
&N^6+\frac{1}{N^6}-908\left[N^5+\frac{1}{N^5}\right]-83582\left[N^4+\frac{1}{N^4}\right]-1369692\left[N^3+\frac{1}{N^3}\right]\\
&-3\left[N^2+\frac{1}{N^2}\right]\left\{2657883+387328\left[M^2+\frac{1}{M^2}\right]\right\}-24\left[N+\frac{1}{N}\right]\\
&\times\left\{892353+579256\left[M+\frac{1}{M}\right]\right\}+17323008\sqrt{2}\left[\sqrt{N}+\frac{1}{\sqrt{N}}\right]\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]\\
&+3663552\sqrt{2}\left[\sqrt{N^3}+\frac{1}{\sqrt{N^3}}\right]\left[\sqrt{M^3}+\frac{1}{\sqrt{M^3}}\right]+86016\sqrt{2}\left[\sqrt{N^5}+\frac{1}{\sqrt{N^5}}\right]\\
&\times\left[\sqrt{M^5}+\frac{1}{\sqrt{M^5}}\right]-29469924=128\left\{32\left[M^4+\frac{1}{M^4}\right]\right.\\
&\left.+3360\left[M^3+\frac{1}{M^3}\right]+39948\left[M^2+\frac{1}{M^2}\right]+150852\left[M+\frac{1}{M}\right]\right\},
\end{aligned} \tag{82}$$

where $M = h_{4,n}h_{4,81n}$ *and* $N = \dfrac{h_{4,n}}{h_{4,81n}}$.

Proof. Employing (5) in (38), we obtain (82).

Corollary 6. *We have*

$$h_{4,9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}-\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}, \tag{83}$$

$$h_{4,1/9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}+\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}. \tag{84}$$

Proof. [Proof of (83) and (84)] Putting $n = 1$ in (82), we find that

$$\begin{aligned}
&(h^4-20h^3+12\sqrt{2}h^3-30h^2+24h^2\sqrt{2}-20h+12\sqrt{2}h+1)\\
&(h^4+20h^3+12\sqrt{2}h^3-30h^2-24h^2\sqrt{2}+20h+12\sqrt{2}h+1)\\
&(-h^4-6h^3+6\sqrt{2}h^3-14h^2+8h^2\sqrt{2}-6h+6\sqrt{2}h-1)^2\\
&(-h^4+6h^3+6\sqrt{2}h^3-14h^2-8h^2\sqrt{2}+6h+6\sqrt{2}h-1)^2=0,
\end{aligned} \tag{85}$$

where $h := h_{4,9}$.

We observe that the first factor of the equation (85) vanishes for specific value of $q = e^{-\pi\sqrt{9/4}}$, but the other factors does not vanish. Hence, we obtain (75) and (76).

Theorem 16. *For any positive real number n, we have*

$$
\begin{aligned}
&N^7 + \frac{1}{N^7} - 6734\left[N^6 + \frac{1}{N^6}\right] - 13\left[N^5 + \frac{1}{N^5}\right]\left[173721 + 98368\left[M + \frac{1}{M}\right]\right] \\
&- 52\left[N^4 + \frac{1}{N^4}\right]\left[1803735 + 1254312\left[M + \frac{1}{M}\right] + 341200\left[M^2 + \frac{1}{M^2}\right]\right] \\
&- 13\left[N^3 + \frac{1}{N^3}\right]\left[93381075 + 70091328\left[M + \frac{1}{M}\right] + 27592192\left[M^2 + \frac{1}{M^2}\right]\right. \\
&\left. +4177920\left[M^3 + \frac{1}{M^3}\right]\right] - 26\left[N^2 + \frac{1}{N^2}\right]\left[255338797 + 198856896\left[M + \frac{1}{M}\right]\right. \\
&\left. +90748240\left[M^2 + \frac{1}{M^2}\right] + 21215232\left[M^3 + \frac{1}{M^3}\right] + 1798144\left[M^4 + \frac{1}{M^4}\right]\right] \\
&- 13\left[N + \frac{1}{N}\right]\left[1355929177 + 1075148160\left[M + \frac{1}{M}\right] + 524233216\left[M^2 + \frac{1}{M^2}\right]\right. \\
&\left. +144474112\left[M^3 + \frac{1}{M^3}\right] + 18776064\left[M^4 + \frac{1}{M^4}\right] + 786432\left[M^5 + \frac{1}{M^5}\right]\right] \\
&+ 416\sqrt{2}\left[\sqrt{N} + \frac{1}{\sqrt{N}}\right]\left[35973930\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right] + 22699534\left[\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right]\right. \\
&+ 8647936\left[\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right] + 1777408\left[\sqrt{M^7} + \frac{1}{\sqrt{M^7}}\right] + 159744\left[\sqrt{M^9} + \frac{1}{\sqrt{M^9}}\right] \\
&\left. +4096\left[\sqrt{M^{11}} + \frac{1}{\sqrt{M^{11}}}\right]\right] + 832\sqrt{2}\left[\sqrt{N^3} + \frac{1}{\sqrt{N^3}}\right]\left[9431769\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right]\right. \\
&+ 5809010\left[\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right] + 2076288\left[\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right] + 367808\left[\sqrt{M^7} + \frac{1}{\sqrt{M^7}}\right] \\
&\left. +22528\left[\sqrt{M^9} + \frac{1}{\sqrt{M^9}}\right]\right] + 416\sqrt{2}\left[\sqrt{N^5} + \frac{1}{\sqrt{N^5}}\right]\left[4977453\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right]\right. \\
&\left. +2895124\left[\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right] + 879360\left[\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right] + 100224\left[\sqrt{M^7} + \frac{1}{\sqrt{M^7}}\right]\right] \\
&+ 208\sqrt{2}\left[\sqrt{N^7} + \frac{1}{\sqrt{N^7}}\right]\left[1197554\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right] + 617714\left[\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+124416\left[\sqrt{M^5}+\frac{1}{\sqrt{M^5}}\right]\Bigg]+208\sqrt{2}\left[\sqrt{N^9}+\frac{1}{\sqrt{N^9}}\right]\left[53242\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]\right.\\
&+20226\left[\sqrt{M^3}+\frac{1}{\sqrt{M^3}}\right]\Bigg]+106912\sqrt{2}\left[\sqrt{N^{11}}+\frac{1}{\sqrt{N^{11}}}\right]\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]\\
&=19335146688\left[M+\frac{1}{M}\right]+9612979584\left[M^2+\frac{1}{M^2}\right]+2772836352\left[M^3+\frac{1}{M^3}\right]\\
&+399785984\left[M^4+\frac{1}{M^4}\right]+22151168\left[M^5+\frac{1}{M^5}\right]+262144\left[M^6+\frac{1}{M^6}\right]\\
&+24251297512,
\end{aligned}
\tag{86}
$$

where $M=h_{4,n}h_{4,169n}$ *and* $N=\dfrac{h_{4,n}}{h_{4,169n}}$.

Proof. Employing (5) in (41), we obtain (86).

Corollary 7. *We have*

$$
h_{4,13}=-\sqrt{\frac{1279+355\sqrt{13}+12\sqrt{a_1}}{2}}+\sqrt{\frac{1281+355\sqrt{13}+12\sqrt{a_1}}{2}}, \tag{87}
$$

$$
h_{4,1/13}=\sqrt{\frac{1279+355\sqrt{13}+12\sqrt{a_1}}{2}}+\sqrt{\frac{1281+355\sqrt{13}+12\sqrt{a_1}}{2}}. \tag{88}
$$

where $a_1=22733+6305\sqrt{13}$.

Proof. [Proof of (87) and (88)] Putting $n=1/13$ in (86), we find that

$$
\begin{aligned}
&(h^4-50h^3+34\sqrt{2}h^3-154h^2+110\sqrt{2}h^2-190h+134\sqrt{2}h-99+70\sqrt{2})\\
&(h^4+50h^3+34\sqrt{2}h^3-154h^2-110\sqrt{2}h^2+190h+134\sqrt{2}h-99-70\sqrt{2})\\
&(-h^4-20h^3+16\sqrt{2}h^3-54h^2+36\sqrt{2}h^2-44h+32\sqrt{2}h-17+12\sqrt{2})^2\\
&(-h^4+20h^3+16\sqrt{2}h^3-54h^2-36\sqrt{2}h^2+44h+32\sqrt{2}h-17-12\sqrt{2})^2\\
&(-h+1+\sqrt{2})^2(-h-1+\sqrt{2})^2=0,
\end{aligned}
\tag{89}
$$

where $h:=h_{4,13}$.
We observe that the first factor of the equation (89) vanishes for specific value of $q=e^{-\pi\sqrt{13/4}}$, but the other factors does not vanish. Hence, we find that

$$
x^4-116x^2-144-100x^3-240x=0, \tag{90}
$$

where $x=h_{4,13}+\dfrac{1}{h_{4,13}}$.
The above equation can be written as

$$
(x^2-50x-14\sqrt{13}x-10\sqrt{13}-34)(x^2-50x+14\sqrt{13}x+10\sqrt{13}-34)=0. \tag{91}
$$

Since the roots of second factor are imaginary, we deduce that

$$x^2 - 50x - 14\sqrt{13}x - 10\sqrt{13} - 34 = 0. \tag{92}$$

On solving the above equation, we obtain (87) and (88).

Theorem 17. *For any positive real number n, we have*

$$\begin{aligned}
&N^9 + \frac{1}{N^9} - 36754\left[N^8 + \frac{1}{N^8}\right] - 17\left[N^7 + \frac{1}{N^7}\right]\left\{1096631 + 1439104\left[M + \frac{1}{M}\right]\right\}\\
&- 272\left[N^6 + \frac{1}{N^6}\right]\left\{2211163 + 3497080\left[M^2 + \frac{1}{M^2}\right] + 5387580\left[M + \frac{1}{M}\right]\right\}\\
&+ 68\left[N^5 + \frac{1}{N^5}\right]\left\{79079901 - 122218496\left[M^3 + \frac{1}{M^3}\right] - 457635200\left[M^2 + \frac{1}{M^2}\right]\right.\\
&\left.-315106016\left[M + \frac{1}{M}\right]\right\} + 136\left[N^4 + \frac{1}{N^4}\right]\left\{2016688545 - 175729152\left[M^4 + \frac{1}{M^4}\right]\right.\\
&\left.-1272638464\left[M^3 + \frac{1}{M^3}\right] - 2545031648\left[M^2 + \frac{1}{M^2}\right] - 474633456\left[M + \frac{1}{M}\right]\right\}\\
&+ 68\left[N^3 + \frac{1}{M^3}\right]\left\{39729209249 - 376569856\left[M^5 + \frac{1}{M^5}\right] + 8138517728\left[M + \frac{1}{M}\right]\right.\\
&\left.-26565147264\left[M^2 + \frac{1}{M^2}\right] - 19377880064\left[M^3 + \frac{1}{M^3}\right] - 4742540288\left[M^4 + \frac{1}{M^4}\right]\right\}\\
&+ 272\left[N^2 + \frac{1}{N^2}\right]\left\{43899000931 - 36667392\left[M^6 + \frac{1}{M^6}\right] + 15986915068\left[M + \frac{1}{M}\right]\right.\\
&-19039820680\left[M^2 + \frac{1}{M^2}\right] - 1839544448\left[M^3 + \frac{1}{M^3}\right] - 5914737152\left[M^4 + \frac{1}{M^4}\right]\\
&\left.-798507008\left[M^5 + \frac{1}{M^5}\right]\right\} + 34\left[N + \frac{1}{N}\right]\left\{821963657831 + 355272255168\left[M + \frac{1}{M}\right]\right.\\
&-269114179584\left[M^2 + \frac{1}{M^2}\right] - 312310413312\left[M^3 + \frac{1}{M^3}\right] - 1424490496\left[M^6 + \frac{1}{M^6}\right]\\
&\left.-1161388787976\left[M^4 + \frac{1}{M^4}\right] - 19523174400\left[M^5 + \frac{1}{M^5}\right] - 33554432\left[M^7 + \frac{1}{M^7}\right]\right\}\\
&- 1088\sqrt{2}\left[\sqrt{N} + \frac{1}{\sqrt{N}}\right]\left\{18819607015\left[\sqrt{M} + \frac{1}{\sqrt{M}}\right] + 160886420\left[\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right]\right.\\
&-9319051040\left[\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right] - 5712220160\left[\sqrt{M^7} + \frac{1}{\sqrt{M^7}}\right] - 1472778240\left[\sqrt{M^9}\right.\\
&\left.+\frac{1}{\sqrt{M^9}}\right] - 176758784\left[\sqrt{M^{11}} + \frac{1}{\sqrt{M^{11}}}\right] - 8912896\left[\sqrt{M^{13}} + \frac{1}{\sqrt{M^{13}}}\right]
\end{aligned}$$

$$-131072\left[\sqrt{M^{15}}+\frac{1}{\sqrt{M^{15}}}\right]\Bigg\}-544\sqrt{2}\left[\sqrt{N^{3}}+\frac{1}{\sqrt{N^{3}}}\right]\Bigg\{21182793502\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]$$

$$-1448653562\left[\sqrt{M^{3}}+\frac{1}{\sqrt{M^{3}}}\right]-11883595968\left[\sqrt{M^{5}}+\frac{1}{\sqrt{M^{5}}}\right]-6661781632\Bigg[\sqrt{M^{7}}$$

$$+\frac{1}{\sqrt{M^{7}}}\Bigg]-1545576448\left[\sqrt{M^{9}}+\frac{1}{\sqrt{M^{9}}}\right]-155148288\left[\sqrt{M^{11}}+\frac{1}{\sqrt{M^{11}}}\right]$$

$$-5242880\left[\sqrt{M^{13}}+\frac{1}{\sqrt{M^{13}}}\right]\Bigg\}-544\sqrt{2}\left[\sqrt{N^{5}}+\frac{1}{\sqrt{N^{5}}}\right]\Bigg\{6400381178\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]$$

$$-1733476258\left[\sqrt{M^{3}}+\frac{1}{\sqrt{M^{3}}}\right]-4641769920\left[\sqrt{M^{5}}+\frac{1}{\sqrt{M^{5}}}\right]-2164026240\Bigg[\sqrt{M^{7}}$$

$$+\frac{1}{\sqrt{M^{7}}}\Bigg]-391159808\left[\sqrt{M^{9}}+\frac{1}{\sqrt{M^{9}}}\right]-23748608\left[\sqrt{M^{11}}+\frac{1}{\sqrt{M^{11}}}\right]\Bigg\}$$

$$-1088\sqrt{2}\left[\sqrt{N^{7}}+\frac{1}{\sqrt{N^{7}}}\right]\Bigg\{450392085\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]-370969698\left[\sqrt{M^{3}}+\frac{1}{\sqrt{M^{3}}}\right]$$

$$-512015264\left[\sqrt{M^{5}}+\frac{1}{\sqrt{M^{5}}}\right]-174563936\left[\sqrt{M^{7}}+\frac{1}{\sqrt{M^{7}}}\right]-18157568\Bigg[\sqrt{M^{9}}$$

$$+\frac{1}{\sqrt{M^{9}}}\Bigg]\Bigg\}-1088\sqrt{2}\left[\sqrt{N^{9}}+\frac{1}{\sqrt{N^{9}}}\right]\Bigg\{14667567\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]-69197970\Bigg[\sqrt{M^{3}}$$

$$+\frac{1}{\sqrt{M^{3}}}\Bigg]-55297472\left[\sqrt{M^{5}}+\frac{1}{\sqrt{M^{5}}}\right]-10402720\left[\sqrt{M^{7}}+\frac{1}{\sqrt{M^{7}}}\right]\Bigg\}$$

$$+544\sqrt{2}\left[\sqrt{N^{11}}+\frac{1}{\sqrt{N^{11}}}\right]\Bigg\{3837042\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]+10601102\left[\sqrt{M^{3}}+\frac{1}{\sqrt{M^{3}}}\right]$$

$$+4263040\left[\sqrt{M^{5}}+\frac{1}{\sqrt{M^{5}}}\right]\Bigg\}+1632\sqrt{2}\left[\sqrt{N^{13}}+\frac{1}{\sqrt{N^{13}}}\right]\Bigg\{78306\left[\sqrt{M}+\frac{1}{\sqrt{M}}\right]$$

$$
\begin{aligned}
&+83122\left[\sqrt{M^3}+\frac{1}{\sqrt{M^3}}\right]\Bigg\}+1160896\sqrt{2}\left[\sqrt{N^{15}}+\frac{1}{\sqrt{N^{15}}}\right]\left\{\sqrt{M}+\frac{1}{\sqrt{M}}\right\}\\
&=128\Bigg\{131072\left[M^8+\frac{1}{M^8}\right]+18939904\left[M^7+\frac{1}{M^7}\right]+106103083008\left[M^3+\frac{1}{M^3}\right]\\
&+604823552\left[M^6+\frac{1}{M^6}\right]+7384821760\left[M^5+\frac{1}{M^5}\right]+41230187584\left[M^4+\frac{1}{M^4}\right]\\
&+85300504580\left[M^2+\frac{1}{M^2}\right]-130198689038\left[M+\frac{1}{M}\right]\Bigg\}-36888130319124
\end{aligned}
\tag{93}
$$

where $M = h_{4,n}h_{4,289n}$ *and* $N = \dfrac{h_{4,n}}{h_{4,289n}}$.

Proof. Employing (5) in (42), we obtain (93).

Corollary 8. *We have*

$$h_{4,17} = \frac{v-\sqrt{v^2-4}}{2}, \tag{94}$$

$$h_{4,1/17} = \frac{v+\sqrt{v^2-4}}{2}, \tag{95}$$

where

$$v=\left(\sqrt{2}-1\right)^2\left[11+\sqrt{2}-\sqrt{17}\left(\sqrt{2}-1\right)+2\sqrt{\sqrt{17}\left(9-2\sqrt{2}\right)-2\left(13\sqrt{2}-6\right)}\right].$$

Proof. [Proof of (94) and (95)] Putting $n = 1/17$ in (93), we find that

$$
\begin{aligned}
&(h^8-116h^7+76\sqrt{2}h^7-608h^6+456\sqrt{2}h^6-1516h^5+1012h^5\sqrt{2}-1858h^4\\
&+1392h^4\sqrt{2}-1516h^3+1012\sqrt{2}h^3-608h^2+456\sqrt{2}h^2-116h+76\sqrt{2}h+1)\\
&(h^8+116h^7+76\sqrt{2}h^7-608h^6-456\sqrt{2}h^6+1516h^5+1012h^5\sqrt{2}-1858h^4\\
&-1392h^4\sqrt{2}+1516h^3+1012\sqrt{2}h^3-608h^2-456\sqrt{2}h^2+116h+76\sqrt{2}h+1)\\
&(-h^4-50h^3+38\sqrt{2}h^3-62h^2+40\sqrt{2}h^2-50h+38\sqrt{2}h-1)^2(h-1)^2\\
&(-h^4+50h^3+38\sqrt{2}h^3-62h^2-40\sqrt{2}h^2+50h+38\sqrt{2}h-1)^2(h+1)^2=0,
\end{aligned}
\tag{96}
$$

where $h := h_{4,17}$.

We observe that the first factor of the equation (96) vanishes for specific value of $q = e^{-\pi\sqrt{17/4}}$, but the other factors does not vanish hence, we find that

$$x^4-612x^2-640+76\sqrt{2}x^3+784\sqrt{2}x-116x^3-1168x+456\sqrt{2}x^2+480\sqrt{2}=0, \tag{97}$$

where $x = h_{4,17} + \dfrac{1}{h_{4,17}}$.

Solving the above equation, we obtain (94) and (95).

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