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Boundary-value Problems with Non-Local Initial Condition for Parabolic Equations with Parameter

John Michael Rassias[1], Erkinjon Tulkinovich Karimov [2,*]

[1] *Pedagogical department, Mathematics and Informatics section, National and Capodistrian University of Athens, Athens, Greece*
[2] *Institute of Mathematics and Information Technologies, Uzbek Academy of Sciences, Tashkent, Uzbekistan*

Abstract. In 2002, J.M.Rassias [14] imposed and investigated the bi-parabolic elliptic bi-hyperbolic mixed type partial differential equation of second order. In the present paper some boundary-value problems with non-local initial condition for model and degenerate parabolic equations with parameter were considered. Also uniqueness theorems are proved and non-trivial solutions of certain non-local problems for forward-backward parabolic equation with parameter are investigated at specific values of this parameter by employing the classical "a-b-c" method. Classical references in this field of mixed type partial differential equations are given by: J.M.Rassias [16] and M.M.Smirnov [25]. Other investigations are achieved by G.C.Wen et al. (in period 1990-2007).

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1. Introduction

Degenerate partial differential equations have numerous applications in Aerodynamics and Hydrodynamics. For example, problems for mixed subsonic and supersonic flows were considered by F.I.Frankl [2]. Reviews of interesting results on degenerated elliptic and hyperbolic equations up to 1965, one can find in the book by M.M.Smirnov [24]. Among other

*Corresponding author.

Email addresses: `jrassias@primedu.uoa.gr` (J. Rassias), `erkinjon@gmail.com` (E. Karimov)

research results on this kind of equations were investigated by J.M.Rassias [14, 15, 16, 17, 18, 19, 20, 21], G.C.Wen [26, 27, 28, 7, 29], A.Hasanov [6] and references therein. Also works by M.Gevrey [4], A.Friedman [3], Yu.Gorkov [5] are well-known on construction fundamental solutions for degenerated parabolic equations. Boundary-value problems with initial non-local condition for model parabolic equations were studied by N.N.Shopolov, for example see [23]. Various non-local problems for mixed type equations containing parabolic type equation were studied by many authors, for instance, see works by Kerefov [12], Sabytov [22], Berdyshev [1], Karimov [10, 11]. However, in 2002, J.M.Rassias [14] imposed and investigated the bi-parabolic elliptic bi-hyperbolic mixed type partial differential equation of second order.

In the present paper some boundary-value problems with non-local initial condition for model and degenerate parabolic equations with parameter were considered. Also uniqueness theorems are proved and non-trivial solutions of certain non-local problems for forward-backward parabolic equation with parameter are investigated at specific values of this parameter by employing the classical "a-b-c" method. Classical references in this field of mixed type partial differential equations are given by: J.M.Rassias [16] and M.M.Smirnov [25]. Other investigations are achieved by G.C.Wen et al. (in period 1990-2007).

2. Non-local Problems for Degenerate Parabolic Equations with Parameter.

Let us consider a parabolic equation

$$y^m u_{xx} - x^n u_y - \lambda x^n y^m u = 0, \tag{1}$$

with two lines of degeneration in the domain $\Phi = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$, where $m, n > 0$, $\lambda \in C$.

The problem 1. To find a regular solution of the equation (1) satisfying boundary conditions

$$u(0,y) = 0, \quad u(1,y) = 0, \quad 0 \le y \le 1, \tag{2}$$

and non-local initial condition

$$u(x,0) = \alpha u(x,1),\ 0 \le x \le 1, \tag{3}$$

where α is non-zero real number.

The following statements are true:

Theorem 1. *Let* $\alpha \in [-1,0) \cup (0,1]$ *,* $\mathrm{Re}\lambda \ge 0$. *If there exists a solution of the problem 1, then it is unique.*

Corollary 1. *The problem 1 can have non-trivial solutions only when parameter* λ *lies outside of the sector* $\Delta = \{\lambda :\ \mathrm{Re}\lambda \ge 0\}$. *These non-trivial solutions represented by*

$$u_{pk}(x,y) = C_{pk}\left(\frac{2}{n+2}\right)^{\frac{1}{n+2}} \mu_k^{\frac{1}{2(n+2)}} x^{\frac{1}{2}} I_{\frac{1}{n+2}}\left(\frac{2\sqrt{\mu_k}}{n+2} x^{\frac{n+2}{2}}\right) e^{(-\ln|\alpha| - ip\pi) y^{m+1}}, \tag{4}$$

where C_{pk} are constants, p,k are natural numbers. Eigenvalues defined as

$$\lambda_{pk} = \mu_k + (m+1)\ln|\alpha| + i(m+1)p\pi.$$

Here μ_k are roots of the equation

$$I_{\frac{1}{n+2}}\left(\frac{2\sqrt{\mu}}{n+2}\right) = 0,$$

where $I_s()$ is the first kind modified Bessel function of s-th order.

We will omit the proof, because further we consider similar problem in three-dimensional domain in a full detail.

Let Ω be a simple-connected bounded domain in R^3 with boundaries S_i $(i = \overline{1,6})$. Here

$$\begin{array}{l} S_1 = \{(x,y,t) : t = 0, 0 < x < 1, 0 < y < 1\}, \\ S_2 = \{(x,y,t) : x = 1, 0 < y < 1, 0 < t < 1\}, \\ S_3 = \{(x,y,t) : y = 0, 0 < x < 1, 0 < t < 1\}, \\ S_4 = \{(x,y,t) : x = 0, 0 < y < 1, 0 < t < 1\}, \\ S_5 = \{(x,y,t) : y = 1, 0 < x < 1, 0 < t < 1\}, \\ S_6 = \{(x,y,t) : t = 1, 0 < x < 1, 0 < y < 1\}. \end{array}$$

We consider the following degenerate parabolic equation

$$x^n y^m u_t = y^m u_{xx} + x^n u_{yy} - \lambda x^n y^m u \tag{5}$$

in the domain Ω. Here $m > 0$, $n > 0$, $\lambda = \lambda_1 + i\lambda_2,\ \lambda_1, \lambda_2 \in R$.
The problem 2. To find a function $u(x,y,t)$ satisfying the following conditions:

1. $u(x,y,t) \in C\left(\overline{\Omega}\right) \cap C^{2,2,1}_{x,y,t}(\Omega)$;
2. $u(x,y,t)$ satisfies the equation 5 in Ω;
3. $u(x,y,t)$ satisfies boundary conditions

$$u(x,y,t)\Big|_{S_2\cup S_3\cup S_4\cup S_5} = 0\ ; \tag{6}$$

4. and non-local initial condition

$$u(x,y,0) = \alpha u(x,y,1). \tag{7}$$

Here $\alpha = \alpha_1 + i\alpha_2,\ \alpha_1, \alpha_2$ are real numbers, moreover $\alpha_1^2 + \alpha_2^2 \neq 0$.

Theorem 2. *If $\alpha_1^2 + \alpha_2^2 < 1,\ \lambda_1 \geq 0$ and exists a solution of the problem 2, then it is unique.*

Proof. Let us suppose that the problem 2 has two u_1, u_2 solutions. Denoting $u = u_1 - u_2$ we claim that $u \equiv 0$ in Ω.

First we multiply equation (5) to the function $\overline{u}(x, y, t)$, which is complex conjugate function of $u(x, y, t)$. Then integrate it along the domain Ω_ε with boundaries

$$\begin{aligned}
S_{1\varepsilon} &= \{(x,y,t) : t=\varepsilon,\, \varepsilon<x<1-\varepsilon,\, \varepsilon<y<1-\varepsilon\},\\
S_{2\varepsilon} &= \{(x,y,t) : x=1-\varepsilon,\, \varepsilon<y<1-\varepsilon,\, \varepsilon<t<1-\varepsilon\},\\
S_{3\varepsilon} &= \{(x,y,t) : y=\varepsilon,\, \varepsilon<x<1-\varepsilon,\, \varepsilon<t<1-\varepsilon\},\\
S_{4\varepsilon} &= \{(x,y,t) : x=\varepsilon,\, \varepsilon<y<1-\varepsilon,\, \varepsilon<t<1-\varepsilon\},\\
S_{5\varepsilon} &= \{(x,y,t) : y=1-\varepsilon,\, \varepsilon<x<1-\varepsilon,\, \varepsilon<t<1-\varepsilon\},\\
S_{6\varepsilon} &= \{(x,y,t) : t=1-\varepsilon,\, \varepsilon<x<1-\varepsilon,\, \varepsilon<y<1-\varepsilon\}.
\end{aligned}$$

Then taking real part of the obtained equality and considering

$$\mathrm{Re}\left(y^m\overline{u}u_{xx}\right) = \mathrm{Re}\left(y^m\overline{u}u_x\right)_x - y^m\left|u_x\right|^2, \mathrm{Re}\left(x^n\overline{u}u_{yy}\right) = \mathrm{Re}\left(x^n\overline{u}u_y\right)_y - x^n\left|u_y\right|^2,$$

$$\mathrm{Re}\left(x^n y^m\overline{u}u_t\right) = \left(\frac{1}{2}x^n y^m|u|^2\right)_t,$$

after using Green's formula we pass to the limit at $\varepsilon \to 0$. Then we get

$$\begin{aligned}
&\int\limits_{\partial\Omega}\int \mathrm{Re}\left[y^m\overline{u}u_x\cos(\nu,x) + x^n\overline{u}u_y\cos(\nu,y) - \tfrac{1}{2}x^n y^m|u|^2\cos(\nu,t)\right]d\tau\\
&= \int\int\limits_{\Omega}\int\left(y^m\left|u_x\right|^2 + x^n\left|u_y\right|^2 + \lambda_1 x^n y^m|u|\right)d\sigma
\end{aligned}$$

where ν is outer normal. Considering $\mathrm{Re}\left[\overline{u}u_x\right] = \mathrm{Re}\left[u\overline{u}_x\right]$, $\mathrm{Re}\left[\overline{u}u_y\right] = \mathrm{Re}\left[u\overline{u}_y\right]$ we obtain

$$\begin{aligned}
\mathrm{Re}\int\int\limits_{S_1}\frac{1}{2}x^n y^m|u|^2 d\tau_1 \quad &+ \quad \int\int\limits_{S_2} y^m\mathrm{Re}\left[u\overline{u}_x\right]d\tau_2 - \int\int\limits_{S_3} x^n\mathrm{Re}\left[u\overline{u}_y\right]d\tau_3 - \int\int\limits_{S_4} y^m\mathrm{Re}\left[u\overline{u}_x\right]d\tau_4\\
&+ \quad \int\int\limits_{S_5} x^n\mathrm{Re}\left[u\overline{u}_y\right]d\tau_5 - \mathrm{Re}\int\int\limits_{S_6}\frac{1}{2}x^n y^m|u|^2 d\tau_6 \qquad (8)\\
&= \quad \int\int\limits_{\Omega}\int\left(y^m\left|u_x\right|^2 + x^n\left|u_y\right|^2 + \lambda_1 x^n y^m|u|\right)d\sigma.
\end{aligned}$$

From (8) and by using conditions (6), (8), we find

$$\begin{aligned}
&\frac{1}{2}\left[1-\left(\alpha_1^2+\alpha_2^2\right)\right]\int_0^1\int_0^1 x^n y^m\left|u(x,y,1)\right|dxdy\\
&+\int\int\limits_{\Omega}\int\left(y^m\left|u_x\right|^2 + x^n\left|u_y\right|^2 + \lambda_1 x^n y^m|u|\right)d\sigma = 0. \qquad (9)
\end{aligned}$$

Setting $\alpha_1^2+\alpha_2^2<1$, $\lambda_1\geq 0$, from (9) we have $u\left(x,y,t\right)\equiv 0$ in $\overline{\Omega}$.

We find below non-trivial solutions of the problem 2 at some values of parameter λ for which the uniqueness condition $\mathrm{Re}\lambda=\lambda_1\geq 0$ is not fulfilled.

We search the solution of Problem 2 as follows

$$u\left(x,y,t\right)=X\left(x\right)\cdot Y\left(y\right)\cdot T\left(t\right). \tag{10}$$

After some evaluations we obtain the following eigenvalue problems:

$$\begin{cases} X''\left(x\right)+\mu_1 x^n X\left(x\right)=0 \\ X\left(0\right)=0,\ X\left(1\right)=0; \end{cases} \tag{11}$$

$$\begin{cases} Y''\left(y\right)+\mu_2 y^m Y\left(y\right)=0 \\ Y\left(0\right)=0,\ Y\left(1\right)=0; \end{cases} \tag{12}$$

$$\begin{cases} T'\left(t\right)+\left(\lambda+\mu\right)T\left(t\right)=0 \\ T\left(0\right)=\alpha T\left(1\right). \end{cases} \tag{13}$$

Here $\mu=\mu_1+\mu_2$ is a Fourier constant.

Solving eigenvalue problems (11), (12) we find

$$\mu_{1k}=\left(\frac{n+2}{2}\widetilde{\mu_{1k}}\right)^2,\ \ \mu_{2p}=\left(\frac{m+2}{2}\widetilde{\mu_{2p}}\right)^2, \tag{14}$$

$$X_k\left(x\right)=A_k\left(\frac{2}{n+2}\right)^{\frac{1}{n+2}}\mu_{1k}^{\frac{1}{2(n+2)}}x^{\frac{1}{2}}J_{\frac{1}{n+2}}\left(\frac{2\sqrt{\mu_{1k}}}{n+2}x^{\frac{n+2}{2}}\right), \tag{15}$$

$$Y_p\left(y\right)=B_p\left(\frac{2}{m+2}\right)^{\frac{1}{m+2}}\mu_{2p}^{\frac{1}{2(m+2)}}y^{\frac{1}{2}}J_{\frac{1}{m+2}}\left(\frac{2\sqrt{\mu_{2p}}}{m+2}x^{\frac{m+2}{2}}\right), \tag{16}$$

where $k,p=1,2,\ldots$, $\widetilde{\mu_{1k}}$ and $\widetilde{\mu_{2p}}$ are roots of equations $J_{\frac{1}{n+2}}\left(x\right)=0$ and $J_{\frac{1}{m+2}}\left(y\right)=0$, respectively.

The eigenvalue problem (13) has non-trivial solution only when $\begin{cases} \alpha_1=e^{\lambda_1+\mu_{kp}}\cos\lambda_2 \\ \alpha_2=e^{\lambda_1+\mu_{kp}}\sin\lambda_2. \end{cases}$

Here $\lambda=\lambda_1+i\lambda_2$, $\alpha=\alpha_1+i\alpha_2$, $\mu_{kp}=\mu_{1k}+\mu_{2p}$. After elementary calculations, we get

$$\lambda_1=-\mu_{kp}+\ln\sqrt{\alpha_1^2+\alpha_2^2},\ \ \lambda_2=\arctan\frac{\alpha_2}{\alpha_1}+s\pi,\ \ s\in Z^+ \tag{17}$$

Corresponding eigenfunctions have the form

$$T_{kp}\left(t\right)=C_{kp}e^{\left[\mu_{kp}-\ln\sqrt{\alpha_1^2+\alpha_2^2}-i\left(\arctan\frac{\alpha_2}{\alpha_1}+s\pi\right)\right]t}. \tag{18}$$

Considering (10), (15), (16) and (18) we can write non-trivial solutions of the problem 2 in the following form:

$$\begin{aligned} u_{kp}\left(x,y,t\right) &= D_{kp}\left(\frac{2}{n+2}\right)^{\frac{1}{n+2}}\left(\frac{2}{m+2}\right)^{\frac{1}{m+2}}\mu_{1k}^{\frac{1}{2(n+2)}}\mu_{2p}^{\frac{1}{2(m+2)}}\sqrt{xy}J_{\frac{1}{n+2}}\left(\frac{2\sqrt{\mu_{1k}}}{n+2}x^{\frac{n+2}{2}}\right) \\ &\times J_{\frac{1}{m+2}}\left(\frac{2\sqrt{\mu_{2p}}}{m+2}y^{\frac{m+2}{2}}\right)e^{\left[\mu_{kp}-\ln\sqrt{\alpha_1^2+\alpha_2^2}-i\left(\arctan\frac{\alpha_2}{\alpha_1}+s\pi\right)\right]t}, \end{aligned}$$

where $D_{kp}=A_k\cdot B_p\cdot C_{kp}$ are constants.

Remark 1. *One can easily see that* $\lambda_1<0$ *in (17), which contradicts to condition* $\mathrm{Re}\lambda=\lambda_1\geq 0$ *of the theorem 2.*

Remark 2. *The following problems can be studied by similar way. Instead of condition (6) we put conditions as follows:*

Problem's name	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
S_2	u_x	u	u	u_x	u	u	u_x	u
S_3	u_y	u	u_y	u	u_y	u	u	u
S_4	u	u_x	u	u_x	u	u_x	u	u
S_5	u	u_y	u_y	u	u	u	u	u_y

3. Non-local Problem for "Forward-backward" Parabolic Equation with Parameter.

In this section we prove the uniqueness of solution of a non-local problem for "forward-backward" parabolic equation with parameter. We have to note work by C.D.Pagani and G.Talenti [13], where boundary-value problems for equation

$$sgn(x)u_y-u_{xx}+ku=f(x,y)$$

were investigated. Existence theorems are proved, with an integral equations technique with the developing of Wiener-Hopf integral equations of the first kind with solutions belonging to Sobolev spaces.

In the domain $D=D_1\cup D_2\cup I_0$, $D_1=\left\{\left(x,y\right):-1\leq x\leq 0,\ 0\leq y\leq 1\right\}$, $I_0=\left\{\left(x,y\right):\ x=0,\ 0\leq y\leq 1\right\}$, $D_2=\left\{\left(x,y\right):0\leq x\leq 1,\ 0\leq y\leq 1\right\}$ let us consider equation

$$Lu=\lambda u, \tag{19}$$

where $\lambda\in R$, $Lu=u_{xx}-sign\left(x\right)u_y$.

The problem 3. To find a regular solution of the equation (19) from the class of functions $u\left(x,y\right)\in C\left(\overline{D}\right)\cap C^1\left(D\cup I_1\cup I_2\right)$, satisfying non-local conditions

$$k_1u_x(-1,y)+k_2u(-1,y)=k_3u_x(1,y),\ 0\leq y\leq 1, \tag{20}$$

$$k_4 u_x(1,y) + k_5 u(1,y) = k_6 u_x(-1,y),\ \ 0 \le y \le 1; \tag{21}$$

$$u(x,0) = \alpha u(x,1),\ \ -1 \le x \le 1. \tag{22}$$

Here $k_i\ \left(i = \overline{1,6}\right)$, α is given non-zero constant, $I_1 = \left\{(x,y) : x = -1, 0 \le y \le 1\right\}$, $I_2 = \left\{(x,y) : x = 1, 0 \le y \le 1\right\}$.

Note, non-local conditions (20), (21) were used for the first time by N.I.Ionkin and E.I.Moiseev [9, 8].

Theorem 3. *If*

$$|\alpha| = 1,\ \lambda > 0,\ k_3 k_5 = k_2 k_6,\ k_1 k_2 < 0,\ k_4 k_5 > 0 \tag{23}$$

and exists a solution of the problem 3, then it is unique.

Proof. We multiply equation (19) to the function $u(x,y)$ and integrate along the domains D_1 and D_2. Using Green's formula and condition (22), we get

$$\int\limits_0^1 u(-0,y)\, u_x(-0,y)\, dy = \int\limits_{-1}^0 \frac{\alpha^2 - 1}{2} u^2(x,1)\, dx$$

$$+ \int\limits_0^1 u(-1,y)\, u_x(-1,y)\, dy + \int\int\limits_{D_1} \left(u_x^2 + \lambda u^2\right) dx dy,$$

$$\int\limits_0^1 u(+0,y)\, u_x(+0,y)\, dy = \int\limits_0^1 \frac{\alpha^2 - 1}{2} u^2(x,1)\, dx$$

$$+ \int\limits_0^1 u(1,y)\, u_x(1,y)\, dy - \int\int\limits_{D_2} \left(u_x^2 + \lambda u^2\right) dx dy.$$

From conditions (20), (21), we find

$$u(1,y)\, u_x(1,y) = \frac{k_6}{k_5} u_x(-1,y)\, u_x(1,y) - \frac{k_4}{k_5} u_x^2(1,y),$$

$$u(-1,y)\, u_x(-1,y) = \frac{k_3}{k_2} u_x(-1,y)\, u_x(1,y) - \frac{k_1}{k_2} u_x^2(-1,y).$$

Taking above identities into account, we establish

$$\int\limits_{-1}^0 \tfrac{\alpha^2-1}{2} u^2(x,1)\, dx + \int\limits_0^1 \tfrac{1-\alpha^2}{2} u^2(x,1)\, dx + \int\limits_0^1 \left[\tfrac{k_4}{k_5} u_x^2(1,y) - \tfrac{k_1}{k_2} u_x^2(-1,y)\right] dy +$$

$$+ \int\limits_0^1 \left[\tfrac{k_3}{k_2} - \tfrac{k_6}{k_5}\right] u(-1,y)\, u_x(1,y)\, dy + \int\int\limits_{D_1} \left(u_x^2 + \lambda u^2\right) dx dy + \int\int\limits_{D_2} \left(u_x^2 + \lambda u^2\right) dx dy.$$

Considering condition (23), we get $u(x,y) \equiv 0$ in D and the proof of Theorem 3 is complete.

Remark 3. *By similar method one can prove the uniqueness of solution of boundary-value problem with non-local initial condition for equation*

$$0=\begin{cases} y^m u_{xx}+(-x)^n u_y-\lambda(-x)^n y^m u=0, & x<0 \\ y^m u_{xx}-x^n u_y-\lambda x^n y^m u=0, & x>0. \end{cases}$$

Open question. A question is still open, on the unique solvability of boundary value problems for the following equation:

$$0=\begin{cases} y^{m_1}(-x)^{n_2}u_{xx}+(-x)^{n_1}y^{m_2}u_y-\lambda_1 u=0, & x<0 \\ y^{m_1}x^{n_2}u_{xx}-x^{n_1}y^{m_2}u_y-\lambda_2 u=0, & x>0, \end{cases}$$

where λ_1, λ_2 are given complex numbers and m_i, $n_i=const>0$ $(i=1,2)$.

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