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**SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY**

# Growth and Chebyshev Approximation of Entire Function Solutions of Helmholtz Equation in $R^2$

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**Abstract.** Some bounds on growth parameters of entire function solution of Helmholtz equation in $R^2$ have been studied in terms of Chebyshev polynomial approximation error in sup norm. Our results extend and improve the results studied by McCoy [9].

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**Key Words and Phrases**: Chebyshev approximation, growth parameters, Bergman integral operator, Helmholtz equation, entire function

## 1. Introduction

For classifying the entire analytic functions by their growth in function theory ,the growth parameters order and type may be computed from the Taylor's coefficients or Chebyshev polynomial approximations.McCoy[8,9] studied the growth of entire analytic function solutions of Helmholtz equation in $R^2$by using function theoretic methods (see R.P.Gilbert[2,3] and McCoy[8])and obtained some bounds on growth parameters in terms of Taylor's coefficients and Chebyshev polynomials approximation errors.

The Helmholtz equation be given in the form

$$[\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + F(r^2)]\phi(r,\theta) = 0 \tag{1}$$

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where $(r,\theta)$ are polar coordinates in $R^2$ and $F(r^2)\neq 0$ is a real valued entire functions with analytic continuation as an entire function of $z\in C$.

Each solution of (1) regular at the origin has a local representation via the Bergman operator[1,7] of the first kind

$$\phi(r,\theta)=B(f(z))=\int_{-1}^{+1}E(r^2,t)f(z(1-t^2/2)(1-t^2)^{-1/2}dt, \tag{2}$$

where

$$E(r^2,t)=1+\sum_{n=1}^{\infty}t^{2n}Q^{(2n)}(r^2)$$

is a real valued analytic function for $t\in[-1,+1]$ that is entire for $r\in[0,\infty)$ and is known as Bergman $Efunction$. In a neighborhood of the origin the solution of (1) has an expansion

$$\phi(r,\theta)=\Sigma_{n=0}^{\infty}a_n\phi_n(r,\theta) \tag{3}$$

where

$$\phi_n(r,\theta)=(\frac{re^{i\theta}}{2})^nG_n(r)$$

and

$$G_n(r)=\int_{-1}^{+1}E(r^2,t)(1-t^2)^{(n-1/2)}dt, n=0,1,2,3,\ldots.$$

The B associate of $\phi$ is given as

$$f(z)=\Sigma_{n=0}^{\infty}a_nz^n. \tag{4}$$

It is known from Gilbert and Colton [4] that $\phi(r,\theta)$ is an entire function if and only if, the associate $f(z)$ is an entire function i.e.,

$$\limsup_{n\to\infty}|a_n|^{1/n}=0. \tag{5}$$

The sets of polynomial solutions of Helmholtz equation are defined as

$$\Pi_n=\{P:P(r,\theta)=\Sigma_{\kappa=0}^{n}a_\kappa\phi_\kappa(r,\theta),a_\kappa real\}$$

The best Chebyshev approximation error in Bernstein's sense be given as

$$E_n(\phi)=\inf\parallel\phi-P\parallel_{r_o}:P\in\Pi_n, \tag{6}$$

$$\parallel\phi-P\parallel_{r_0}=M(r_0,\phi-P),P\in\Pi_n$$

where

$$r_0=r_0(K)=\min\{1,\sup\{r:E(r^2,t)>0,t\in[-1,+1]\}\}>0,$$

and the maximum modulus

$$M(r_0, \phi - P) = \max\{|(\phi - P)(z)| : |z| < r_0\}.$$

Mccoy[9] studied the fast growth of entire function solution $\phi(r,\theta)$ in terms of order $\delta$ and type $\tau$ using the concept of index $k$ i.e., $\delta(k-1) = \infty$ and $\delta(k) < \infty$. Due to lack of suitable inverse operator he obtained bounds on the order and type.It has been noticed that his results do not give any precise information about the growth of those functions for which $\delta(k-1) = \infty$ and $\delta(k) = 0$.To overcome this problem, in this paper we pick up a concept of $(p,q)$-order and $(p,q)$-type introduced by Juneja et al.[5,6]. Roughly speeking, this concept is a modification of the classical definition of order and type, obtained by replacing logarithms by iterated logarithms, where the degrees of iteration are determined by $p$ and $q$, $p \geq q \geq 0$. Our approach unifies the above approach studied by those of McCoy[9] and at the same time it is applicable to every entire function, whether of slow or fast growth. Moreover, we make an attempt to characterize $(p,q)$ growth of $\phi(r,\theta)$ and obtained some bounds on $(p,q)$ order and $(p,q)$ type in terms of Chebyshev polynomial approximation errors defined by (6).

## 2. Notations

1. $\log^{[m]} x = \exp^{[-m]} x = \log(\log^{[m-1]} x) = \exp(\exp^{[-m-1]} x)$, $m = 0, \pm 1, \pm 2, \ldots$ provided that $0 < \log^{[m-1]} x < \infty$ with $\log^{[0]} x = \exp^{[0]} x = x$.

2.
$$\Omega(L(p,q)) = \begin{cases} \Omega(L(p,q)) = L(p,q), & \text{if } p > 2; \\ 1 + L(p,q), & \text{if } p = q = 2; \\ max(1, L(p,q)), & \text{if } 3 \leq p = q < \infty; \\ \infty, & \text{if } p = q = \infty. \end{cases}$$

   where $0 \leq \Omega(p,q) \leq \infty$.

## 3. $(p,q)$-Growth of Solutions and Chebyshev Polynomial Approximation

In this section we shall prove our main results.

**Theorem 1.** *Let $\phi(r,\theta)$ be an entire function solution of the Helmholtz equation with expansion*

$$\phi(r,\theta) = \sum_{n=0}^{\infty} a_n \phi_n(r,\theta).$$

*Let $\phi$ and B associate f be entire functions of (p,q)-order $\delta(p,q,\phi)$ and $\delta(p,q,f)$ for a pair of integers (p,q), p≥2, $q \geq 1$. Then the following bounds are valid*

*(i)* $\delta(p,q,\phi) \geq \delta(p,q,f)$

*(ii)* $\delta(p,q,E) \leq \delta(p,q,f)$

*where*

$$\delta(p,q,\phi)=\Omega(L(p,q,\phi)),\delta(p,q,f)=\Omega(L(p,q,f)),\delta(p,q,E)=\Omega(L(p,q,E))$$

*and*

$$L(p,q,\phi)=\limsup_{n\to\infty}\frac{\log^{[p-1]}n}{\log^{[q]}[E_n(\phi)/\mu_n(G_n)]^{-1/n}},$$
$$L(p,q,f)=\limsup_{n\to\infty}\frac{\log^{[p-1]}n}{\log^{[q]}|a_n|^{-1/n}},$$
$$L(p,q,E)=\limsup_{n\to\infty}\frac{\log^{[p-1]}n}{\log^{[q]}[E_n(\phi)]^{-1/n}},$$
$$\mu(G_n)=\int_0^{r_0}G_n(r^2)r^{n+1}dr>0, n=0,1,2,3\ldots.$$

*Proof.*

(i) Using the orthogonality argument in equation (4). we get the identity

$$a_n(\frac{r}{2})^n G_n(r^2)=\frac{1}{2\pi}\int_0^{2\pi}[\phi(r,\theta)-P(r,\theta)]e^{-in\phi}d\phi$$

for $P\in\pi_{n-1}$, n=1,2,3.... Integration above equation over the disk we obtain

$$a_n\mu_n(G_n)/2^n=\frac{1}{2\pi}\int_0^{r_0}\int_0^{2\pi}[\phi(r,\theta)-P(r,\theta)]e^{-in\phi}rdrd\phi$$

or

$$|a_n|\mu_n(G_n)/2^n\le\frac{1}{4\pi}r_0^2E_n(\phi) \tag{7}$$

leads to

$$\limsup_{n\to\infty}\frac{\log^{[p-1]}n}{\log^{[q]}|a_n|^{-1/n}}\le\limsup_{n\to\infty}\frac{\log^{[p-1]}n}{\log^{[q]}[E_n(\phi)/\mu_n(G_n)]^{-1/n}}.$$

Using the $(p,q)$-order coefficient formula for the associate [5, Thm.1,pp.61] we get

$$\delta(p,q,f)\le\delta(p,q,\phi).$$

(ii) Let us consider

$$\begin{aligned}|\phi(r,\theta)| &\le \int_{-1}^{+1}|E(r^2,t)||f(z\frac{1-t^2}{2})-p(z\frac{1-t^2}{2})|(1-t^2)^{\frac{-1}{2}}dt\\ &\le K(r_0)||f-p||_{2/r_0}\end{aligned}$$

where

$$K(r_0) = \max\{K(r^2) : 0 \le r \le r_0\},$$
$$K(r^2) = \max\{E(r^2, t) : t \in [-1, +1]\},$$
$$f(z) - p(z) = \sum_{k=n}^{\infty} a_k (r_0/2)^k z^k.$$

From this we get for $P \in \prod_{n-1}$

$$E_n(\phi) \le \| \phi - P \|_{r_0} \le K(r_0) \| f - p \|_{2/r_0}$$

we have

$$e_n(f) = \inf\{\| f - p \|_{2/r_0} : p \in \Pi_{n-1}\}$$

and

$$\Pi_n = \{p : p(z) = \sum_{k=0}^{n} a_k (r_0/2)^k z^k, a_k real\}.$$

By using to Reddy's [10] extension of Bernstein theorem, for given $\varepsilon > 0$ there is an $N(\varepsilon) > 0$ such that

$$e_n(f) \le K(r_0)[2^n |a_n| / (r_0 + \varepsilon)^n]$$

for all n $\ge N(\varepsilon)$ or

$$E_n(\phi)^{-1/n} \ge | a_n |^{-1/n} \frac{(r_0 + \varepsilon)}{2} K(r_0)^{-1/n} \tag{8}$$

or

$$\log^{[q]} E_n(\phi)^{-1/n} \ge \log^{[q]} | a_n |^{-1/n} + 0(1)$$

or

$$\limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} E_n(\phi)^{-1/n}} \le \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} | a_n |^{-1/n}}.$$

Again using the $(p, q)$order coefficient formula for the associate [5, Thm.1, pp 61] we obtain

$$\delta(p, q, E) \le \delta(p, q, f).$$

Hence the proof of *(ii)* is completed.

**Theorem 2.** *Let $\phi(r, \theta)$ be an entire function solution of the Helmholtz equation with expansion $\phi(r, \theta)$ and B- associate f(z) have the same index-pair (p,q). Then the (p,q) types satisfy*

*(i)*

$$\left[\frac{T(p, q, \phi)}{M(\phi)}\right]^{\frac{1}{(\delta(p,q,\phi)-A)}} \ge \left[\frac{T(p, q, f)}{M(f)}\right]^{\frac{1}{(\delta(p,q,f)-A)}} \liminf_{n\to\infty} (\log^{[p-2]} n)^{\frac{1}{\delta(p,q,\phi)-A} - \frac{1}{\delta(p,q,f)-A}}.$$

*(ii)*

$$[\frac{T(p,q,f)}{M(f)}]^{\frac{1}{(\delta(p,q,f)-A)}} \geq [\frac{T(p,q,E)}{M(E)}]^{\frac{1}{(\delta(p,q,E)-A)}}.\beta.\liminf_{n\to\infty}(\log^{[p-2]} n)^{\frac{1}{\delta(p,q,f)-A}-\frac{1}{\delta(p,q.E)-A}},$$

*where*

$$T(p,q,\phi) = M(\phi)\upsilon(p,q,\phi),$$
$$T(p,q,E) = M(E)\upsilon(p,q,E),$$
$$T(p,q,f) = M(F)\upsilon(p,q,f)$$

*and*

$$\frac{1}{\upsilon(p,q,\phi)} = \liminf_{n\to\infty}\frac{[\log^{[q-1]}[E_n(\phi)/\mu_n(r_n)]^{-1/n}]^{(\delta(p,q,\phi)-A)}}{\log^{[p-2]} n},$$
$$\frac{1}{\upsilon(p,q,E)} = \liminf_{n\to\infty}\frac{[\log^{[q-1]} E_n(\phi)^{-1/n}]^{(\delta(p,q,E)-A)}}{\log^{[p-2]} n},$$
$$\frac{1}{\upsilon(p,q,f)} = \liminf_{n\to\infty}\frac{[\log^{[q-1]} \mid a_n \mid^{-1/n}]^{(\delta(p,q,f)-A)}}{\log^{[p-2]} n}.$$

*Here* $A = 1$, *if* $p = q$ *and* $A = 0$ *if* $p > q$ *and*

$$M(\phi) = \begin{cases} \frac{(\delta(2,2,\phi)-1)^{\delta(2,2,\phi)-1}}{(\delta(2,2,\phi))^{\delta(2,2,\phi)}}, & \text{if } (p,q) = (2,2),; \\ \frac{1}{e\delta(2,1,\phi)}, & \text{if } (p,q) = (2,1),; \\ 1, & \text{if } 3 \leq p = q < \infty. \end{cases}$$

*M(f) and M(E) are defined similarly.Also* $\beta = (r_0/2)$ *if* $(p,q) = (2,1)$ *and* $\beta$*=1; otherwise.*

*Proof.*

(i) From (7) we have

$$\frac{[|a_n|^{-1/n}]^{\delta(2,1,f)}}{n} \geq \{\frac{\{2[E_n(\phi)/\mu_n(G_n)]^{-1/n}\}^{\delta(2,1,\phi)}}{n}\}^{\frac{\delta(2,1,f)}{\delta(2,1,\phi)}}.n^{\frac{\delta(2,1,f)}{\delta(2,1,\phi)}-1}.$$

Proceeding to limit infimum as $n \to \infty$ and using the $(2,1)$-type coefficient formula for the associate [6,Thm.1,pp.181] we get

$$\frac{1}{e\delta(2,1,f)T(2,1,f)} \geq 2^{\delta(2,1,f)}(\frac{1}{e\delta(2,1,\phi)T(2,1,\phi)})^{\delta(2,1,f)/\delta(2,1,\phi)}\liminf_{n\to\infty} n^{\frac{\delta(2,1,f)}{\delta(2,1,\phi)}-1}$$

or

$$(\delta(2,1,\phi)T(2,1,\phi))^{\frac{1}{\delta(2,1,\phi)}} \geq 2(\delta(2,1,f)T(2,1,f))^{\frac{1}{\delta(2,1,f)}}\liminf_{n\to\infty}\{(n/e)^{\frac{1}{\delta(2,1,\phi)}-\frac{1}{\delta(2,1,f)}}\}. \tag{9}$$

From (7) we can obtain that

$$\frac{[\log|a_n|^{-1/n}]^{\delta(2,2,f)-1}}{n} \geq \{\frac{[\log 2[E_n(\phi)/\mu_n(G_n)]^{-1/n}]^{\delta(2,2,\phi)-1}}{n}\}^{\frac{\delta(2,2,f)-1}{\delta(2,2,\phi)-1}} n^{(\frac{\delta(2,2,f)-1}{\delta(2,2,\phi)-1})-1}.$$

Applying the limit infimum and taking into account the $(2,2)$-type coefficient formula for associate [6,Thm.1,pp.181] we obtain

$$\frac{(\delta(2,2,f)-1)^{\delta(2,2,f)-1}}{\delta(2,2,f)^{\delta(2,2,f)}}.\frac{1}{T(2,2,f)} \geq [\frac{(\delta(2,2,\phi)-1)^{\delta(2,2,\phi)-1}}{\delta(2,2,\phi)^{\delta(2,2,\phi)}}.\frac{1}{T(2,2,\phi)}]^{\frac{\delta(2,2,f)-1}{\delta(2,2,\phi)-1}} .\liminf_{n\to\infty} n^{(\frac{\delta(2,2,f)-1}{\delta(2,2,\phi)-1}-1)}$$

or

$$(T(2,2,\phi))^{\frac{1}{\delta(2,2,\phi)-1}} \geq (\frac{\delta(2,2,\phi)-1}{\delta(2,2,f)-1})\frac{(\delta(2,2,f))^{\delta(2,2,f)/(\delta(2,2,f)-1)}}{(\delta(2,2,\phi))^{\delta(2,2,\phi)/(\delta(2,2,\phi)-1)}}(T(2,2,f))^{\frac{1}{\delta(2,2,f)-1}} .\liminf_{n\to\infty} n^{(\frac{1}{\delta(2,2,\phi)-1}-\frac{1}{\delta(2,2,f)-1})}. \quad (10)$$

Hence for $(p,q)=(2,2)$ the proof is completed. Now for $3 \leq p = q < 0$ we can easily obtain from (7) that

$$\frac{[\log^{[q-1]} \mid a_n \mid^{-1/n}]^{\delta(p,q,f)}}{\log^{[p-2]} n} \geq \{\frac{[\log^{[q-1][E_n(\phi)/\mu(r_n)]^{-1/n}}]^{\delta(p,q,\phi)}}{\log^{[p-2]} n}\}^{\frac{\delta(p,q,f)}{\delta(p,q,\phi)}}.(\log^{[p-2]} n)^{\frac{\delta(p,q,f)}{\delta(p,q,\phi)}-1}.$$

Proceeding to limit infimum as $n \to \infty$ and the $(p,q)$ type coefficient formula for associate [6, Thm.1,pp.181] taking into account we obtain

$$(\frac{1}{T(p,q,f)})^{\frac{1}{\delta(p,q,f)}} \geq (\frac{1}{T(p,q,\phi)})^{\frac{1}{\delta(p,q,\phi)}} \liminf_{n\to\infty}(\log^{[p-2]} n)^{(\frac{1}{\delta(p,q,\phi)}-\frac{1}{\delta(p,q,f)})}$$

or

$$T(p,q,\phi)^{\frac{1}{\delta(p,q,\phi)}} \geq T(p,q,f)^{\frac{1}{\delta(p,q,f)}} \liminf_{n\to\infty}(\log^{[p-2]} n)^{(\frac{1}{\delta(p,q,\phi)}-\frac{1}{\delta(p,q,f)})}. \quad (11)$$

Combining (9),(10) and (11) we get the required result i.e., (i).

(ii) Following the lines of proof of (i) with equation (8) the result (ii) can be prove easily. Hence the proof is left for the reader.

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