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**Mathieu–type Series for the  $\aleph$ –function Occurring in Fokker–Planck Equation**

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**Abstract.** Closed form expressions are obtained for a family of convergent Mathieu type  $\mathbf{a}$ –series and its alternating variant, whose terms contain an  $\aleph$ –function, which naturally occurs in certain problems associated with driftless Fokker–Planck equation with power law diffusion [25]. The  $\aleph$ –function is a generalization of the familiar  $H$ –function and the  $I$ –function. The results derived are of general character and provide an elegant generalization for the closed form expressions the Mathieu–type series associated with the  $H$ –function by Pogány [8], for Fox–Wright functions by Pogány and Srivastava [13] and for generalized hypergeometric  ${}_pF_q$  and Meijer’s  $G$ –function by Pogány and Tomovski [16], and others. For the  $\overline{H}$ –function [7, p. 216] the results are obtained very recently by Pogány and Saxena [11].

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## 1. Introduction and Preliminaries

In order to unify and extend the results for the convergent Mathieu–type  $\mathbf{a}$ –series and its alternating variants whose terms contain the familiar transcendental functions, such as Gauss hypergeometric function  ${}_2F_1$ , generalized hypergeometric function  ${}_pF_q$ , the Fox–Wright

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function, the Meijer’s  $G$ -function and Fox’s  $H$ -function published in a series of papers by Pogány [8, 9, 10], Pogány *et al.* [11, 12, 13, 14, 15, 16] and Srivastava and Tomovski [23].

Inequalities and integral representations for Mathieu-type series are discussed by Cerone and Lenard [1], Pogány and Tomovski [17], Srivastava and Tomovski [23] and others. The results obtained by the authors in this serve as the key formulæ for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature.

In the study of fractional driftless Fokker–Planck equations with power law diffusion coefficients, there arises naturally a special function, which is a special case of the  $\aleph$ , that is Aleph–function. The idea to introduce Aleph–function belongs to Südland *et al.* [24], however the notation and complete definition is presented here in the following manner in terms of the Mellin–Barnes type integrals [also see 25]:

$$\begin{aligned} \aleph[z] &= \aleph_{p_i, q_i, \tau_i; r}^{m, n}[z] = \aleph_{p_i, q_i, \tau_i; r}^{m, n} \left[ z \left| \begin{matrix} (a_j, A_j)_{1, n}, \dots, [\tau_j(a_j, A_j)]_{n+1, p_i} \\ (b_j, B_j)_{1, m}, \dots, [\tau_j(b_j, B_j)]_{m+1, q_i} \end{matrix} \right. \right] \\ &:= \frac{1}{2\pi\omega} \int_{\mathcal{L}} \Omega_{p_i, q_i, \tau_i; r}^{m, n}(s) z^{-s} ds \end{aligned} \tag{1}$$

for all  $z \neq 0$ , where  $\omega = \sqrt{-1}$  and

$$\Omega_{p_i, q_i, \tau_i; r}^{m, n}(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \cdot \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\sum_{i=1}^r \tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} s) \cdot \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - B_{ji} s)}, \tag{2}$$

The integration path  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ ,  $\gamma \in \mathbb{R}$  extends from  $\gamma - i\infty$  to  $\gamma + i\infty$ , and is such that the poles, assumed to be simple, of  $\Gamma(1 - a_j - A_j s)$ ,  $j = \overline{1, n}$  do not coincide with the poles of  $\Gamma(b_j + B_j s)$ ,  $j = \overline{1, m}$ . The parameters  $p_i, q_i$  are non-negative integers satisfying  $0 \leq n \leq p_i$ ,  $1 \leq m \leq q_i$ ,  $\tau_i > 0$  for  $i = \overline{1, r}$ . The parameters  $A_j, B_j, A_{ji}, B_{ji} > 0$  and  $a_j, b_j, A_{ji}, b_{ji} \in \mathbb{C}$ . The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1) are given below:

$$\varphi_\ell > 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_\ell \quad \ell = \overline{1, r}; \tag{3}$$

$$\varphi_\ell \geq 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_\ell \quad \text{and} \quad \Re\{\zeta_\ell\} + 1 < 0, \tag{4}$$

where

$$\varphi_\ell = \sum_{j=1}^n A_j + \sum_{j=1}^m B_j - \tau_\ell \left( \sum_{j=n+1}^{p_\ell} A_{j\ell} + \sum_{j=m+1}^{q_\ell} B_{j\ell} \right) \tag{5}$$

$$\zeta_\ell = \sum_{j=1}^m b_j - \sum_{j=1}^n a_j + \tau_\ell \left( \sum_{j=m+1}^{q_\ell} b_{j\ell} - \sum_{j=n+1}^{p_\ell} a_{j\ell} \right) + \frac{1}{2}(p_\ell - q_\ell) \quad \ell = \overline{1, r}. \tag{6}$$

**Remark 1.** If the sum in the denominator of (2) can be simplified in terms of a polynomial in  $s$ , the factors of this polynomial can be expressed by a fraction of Euler’s Gamma function leading to an  $H$ -function instead, see [25, p. 325].

**Remark 2.** It is observed that there is no historical name given to (1), compared to [24]. The Mellin transform of this function is the coefficient of  $z^{-s}$  in the integrand of (1). There are no references containing tables of  $\aleph$ -functions in the literature.

For  $\tau_1 = \tau_2 = \dots = \tau_r = 1$ , in (1) the definition of following  $I$ -function [21] is recovered:

$$I[z] = \aleph_{p_i, q_i, 1; r}^{m, n}[z] = \aleph_{p_i, q_i, 1; r}^{m, n} \left[ z \left| \begin{matrix} (a_j, A_j)_{1, n}, \dots, (a_j, A_j)_{n+1, p_i} \\ (b_j, B_j)_{1, m}, \dots, (b_j, B_j)_{m+1, q_i} \end{matrix} \right. \right] := \frac{1}{2\pi\omega} \int_{\Omega} \Omega_{p_i, q_i, 1; r}^{m, n}(s) z^{-s} ds, \tag{7}$$

where  $\Omega_{p_i, q_i, 1; r}^{m, n}(s)$  is defined in (2). The existence conditions for the integral in (7) are the same as given in (3)–(6) with  $\tau_i = 1, i = \overline{1, r}$ .

If we further set  $r = 1$ , then (7) reduces to the familiar  $H$ -function given e.g. in the monograph [7]:

$$H_{p, q}^{m, n}[z] = \aleph_{p_i, q_i, 1; 1}^{m, n}[z] = \aleph_{p_i, q_i, \tau_i; 1}^{m, n} \left[ z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] := \frac{1}{2\pi\omega} \int_{\Omega} \Omega_{p_i, q_i, 1; 1}^{m, n}(s) z^{-s} ds, \tag{8}$$

where the kernel  $\Omega_{p_i, q_i, 1; 1}^{m, n}(s)$  is given in (2), which itself is a generalization of Meijer’s  $G$ -function [2, p. 207] to which it reduces for  $A_1 = \dots = A_p = 1 = B_1 = \dots = B_q$ . A detailed and comprehensive account of the  $H$ -function is available from the monographs written by Mathai and Saxena [6], Srivastava *et al.* [22], Kilbas and Saigo [4] and Mathai *et al.* [7].

In what follows, the Aleph function will be represented by the contracted notations  $\aleph_{p_i, q_i, \tau_i; r}^{m, n}[z]$  or  $\aleph[z]$ .

Now, consider the Mathieu-type  $\mathbf{a}$ -series  $\Theta_{\lambda, \mu}$  and its alternating variant  $\tilde{\Theta}_{\lambda, \mu}$ , defined by

$$\Theta_{\lambda, \mu} \{ \aleph; \mathbf{c}, x \} := \sum_{j=1}^{\infty} \frac{\aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{c_j} \left| \begin{matrix} (\alpha, \beta), (a_j, A_j)_{1, n}, (a_{ji}, A_{ji})_{n+1, p_i} \\ (b_j, B_j)_{1, m}, (b_{ji}, B_{ji})_{m+1, q_i} \end{matrix} \right. \right]}{c_j^\lambda (c_j + x)^\mu}, \tag{9}$$

$$\tilde{\Theta}_{\lambda, \mu} \{ \aleph; \mathbf{c}, x \} := \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{c_j} \left| \begin{matrix} (\alpha, \beta), (a_j, A_j)_{1, n}, (a_{ji}, A_{ji})_{n+1, p_i} \\ (b_j, B_j)_{1, m}, (b_{ji}, B_{ji})_{m+1, q_i} \end{matrix} \right. \right]}{c_j^\lambda (c_j + x)^\mu} \tag{10}$$

where the convention is followed that the positive sequence  $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$  monotonously increases and tends to infinity; equivalently

$$\mathbf{c}: 0 < c_1 < c_2 < \dots < c_n \uparrow \infty. \tag{11}$$

## 2. Integral Representations of $\Theta_{\lambda,\mu}\{\aleph; \mathbf{c}, x\}$ and $\tilde{\Theta}_{\lambda,\mu}\{\aleph; \mathbf{c}, x\}$

The Laplace transform of the  $\aleph$ -function can be established in the following form

$$\int_0^\infty x^{\lambda-1} e^{-sx} \aleph_{p_i, q_i, \tau_i; r}^{m, n} [\eta x^\rho] dx = s^{-\lambda} \aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{\eta}{s^\rho} \middle| \begin{matrix} (1-\lambda, \rho), (a_j, A_j)_{1, n}, [\tau_i(a_{ji}, A_{ji})]_{n+1, p_i} \\ (b_j, B_j)_{1, m}, [\tau_i(b_{ji}, B_{ji})]_{m+1, q_i} \end{matrix} \right], \tag{12}$$

where  $\lambda, s, \eta \in \mathbb{C}; \Re\{s\} > 0, \rho > 0, \tau_i > 0, i = \overline{1, r}$ , and

$$\Re\{\lambda\} + \rho \min_{1 \leq j \leq m} \frac{\Re\{b_j\}}{B_j} > 0, \quad |\arg(\eta)| < \frac{\pi}{2} \min_{1 \leq \ell \leq r} (\zeta_\ell); \tag{13}$$

the parameter  $\zeta_\ell$  is defined in (6).

The formula (12) can be easily established with the help of the definition (2) of  $\aleph$ -function and using gamma function formula

$$\Gamma(\mu) \zeta^{-\mu} = \int_0^\infty x^{\mu-1} e^{-\zeta x} dx \quad \min(\Re\{\mu\}, \Re\{\zeta\}) > 0.$$

**Theorem 1.** Let  $\lambda > 0, \mu > 0, x > 0, \alpha = 1 - \lambda, \beta = \rho$  and let the sequence  $\mathbf{c}$  satisfies (11). Then there hold the following results:

$$\Theta_{\lambda,\mu}\{\aleph; \mathbf{c}, x\} = \mathfrak{J}_{\mathbf{c}}^\aleph(\lambda + 1, \mu) + \mu \mathfrak{J}_{\mathbf{c}}^\aleph(\lambda, \mu + 1) \tag{14}$$

$$\tilde{\Theta}_{\lambda,\mu}\{\aleph; \mathbf{c}, x\} = \tilde{\mathfrak{J}}_{\mathbf{c}}^\aleph(\lambda + 1, \mu) + \mu \tilde{\mathfrak{J}}_{\mathbf{c}}^\aleph(\lambda, \mu + 1), \tag{15}$$

where

$$\mathfrak{J}_{\mathbf{c}}^\aleph(u, v) := \int_{c_1}^\infty \frac{[c^{-1}(t)]}{t^u(t+x)^v} \aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{t}; u \right] dt, \tag{16}$$

$$\tilde{\mathfrak{J}}_{\mathbf{c}}^\aleph(u, v) := \int_{c_1}^\infty \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(t)]\right)}{t^u(t+x)^v} \aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{t}; u \right] dt, \tag{17}$$

and

$$\aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{t}; u \right] := \aleph_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{t} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{1, n}, [\tau_i(a_{ji}, A_{ji})]_{n+1, p_i} \\ (b_j, B_j)_{1, m}, [\tau_i(b_{ji}, B_{ji})]_{m+1, q_i} \end{matrix} \right].$$

where  $c: \mathbb{R}_+ \mapsto \mathbb{R}_+$  is an increasing function such that  $c(x)|_{x \in \mathbb{N}} = \mathbf{c}, c^{-1}(x)$  is the inverse of  $c(x)$ ,  $[c^{-1}(x)]$  stands for the integer part of the quantity  $c^{-1}(x)$ .

*Proof.* Taking  $\zeta = c_n + x$  in (14), setting  $s = c_j$ ;  $\rho = 1, \eta = x$  and inserting  $\alpha = 1 - \lambda, \beta = 1$  in (12), we find that

$$\begin{aligned} \Theta_{\lambda, \mu} \{ \mathfrak{K}; \mathbf{c}, x \} &= \sum_{j=1}^{\infty} \frac{\mathfrak{K}_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{c_j} \middle| \begin{matrix} (1-\lambda, 1), (a_j, A_j)_{1, n}, [\tau_i(a_{ji}, A_{ji})]_{n+1, p_i} \\ (b_j, B_j)_{1, m}, [\tau_i(b_{ji}, B_{ji})]_{m+1, q_i} \end{matrix} \right]}{c_j^\lambda (c_j + x)^\mu} \\ &= \frac{1}{\Gamma(\mu)} \sum_{j=1}^{\infty} \int_0^\infty s^{\lambda-1} e^{-c_j s} \mathfrak{K}_{p_i, q_i, \tau_i; r}^{m, n} [xs] ds \int_0^\infty t^{\mu-1} e^{-(c_j+x)t} dt \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty \int_0^\infty \left( \sum_{j=1}^{\infty} e^{-c_j(s+t)} \right) s^{\lambda-1} t^{\mu-1} e^{-xt} \mathfrak{K}_{p_i, q_i, \tau_i; r}^{m, n} [xs] ds dt, \end{aligned} \tag{18}$$

where, by convergence reasons  $\mu > 0$  is already assumed. Following the lines of the use of Dirichlet series technique used in earlier papers by Pogány and coworkers [10, 11, 12, 13, 14, 15, 16], by means of (12) we conclude

$$\begin{aligned} \Theta_{\lambda, \mu} \{ \mathfrak{K}; \mathbf{c}, x \} &= \frac{1}{\Gamma(\mu)} \int_0^\infty \int_0^\infty \int_{c_1}^\infty s^\lambda t^{\mu-1} e^{-(y+x)t-y s} \mathfrak{K}_{p_i, q_i, \tau_i; r}^{m, n} [xs] [c^{-1}(y)] ds dt dy \\ &+ \frac{1}{\Gamma(\mu)} \int_0^\infty \int_0^\infty \int_{c_1}^\infty s^{\lambda-1} t^\mu e^{-(y+x)t-y s} \mathfrak{K}_{p_i, q_i, \tau_i; r}^{m, n} [xs] [c^{-1}(y)] ds dt dy := J_s + J_t. \end{aligned}$$

Introducing the auxiliary integral

$$\mathfrak{J}_{\mathbf{c}}^{\mathfrak{K}}(u, v) := \int_{c_1}^\infty \frac{[c^{-1}(t)]}{t^u (t+x)^v} \mathfrak{K}_{p_i+1, q_i, \tau_i; r}^{m, n+1} \left[ \frac{x}{t}; u \right] dt,$$

it readily follows that

$$J_s = \mathfrak{J}_{\mathbf{c}}^{\mathfrak{K}}(\lambda + 1, \mu) \quad \text{and} \quad J_t = \mu \cdot \mathfrak{J}_{\mathbf{c}}^{\mathfrak{K}}(\lambda, \mu + 1).$$

This finishes the proof of (16).

The proof of (17) is similar to that of (16), if we employ the definition of the new alternating inner Dirichlet series  $\tilde{\mathcal{D}}_{\mathbf{c}}(\cdot)$  [14, p. 77, Section 4] given below:

$$\tilde{\mathcal{D}}_{\mathbf{c}}(s+t) = \sum_{j=1}^{\infty} (-1)^{j-1} e^{-c_j(s+t)} = (s+t) \int_{c_1}^\infty e^{-(s+t)x} \sin^2 \left( \frac{\pi}{2} [c^{-1}(x)] \right) dx. \tag{19}$$

The application of (19) completes the proof of (17).

### 3. Special Cases

As Aleph function is the most generalized special function, numerous special cases with potentially useful transcendental functions, such Mittag-Leffler functions, Bessel functions,

Whittaker functions, hypergeometric functions, generalized hypergeometric  ${}_pF_q$  function, Meijer's  $G$ -function, Fox-Wright  $\Psi$  function and Fox  $H$ -function and their special cases can be deduced by making suitable changes in the parameters. But, for the sake of brevity, some interesting special cases of Theorem 1 are given below.

**Corollary 1.** [12, Theorem] Let  $\lambda, \mu, x > 0, \alpha = 1 - \lambda, \beta = \rho, \tau_1 = \dots = \tau_r = 1$ , and let the sequence  $\mathbf{c}$  satisfies (11). Then the Aleph function reduces to an  $I$ -function and there holds the following result

$$\Theta_{\lambda, \mu} \left\{ I_{p_i+1, q_i, r}^{m, n+1}; \mathbf{c}, x \right\} = \mathfrak{J}_{\mathbf{c}}^I(\lambda + 1, \mu) + \mu \mathfrak{J}_{\mathbf{c}}^I(\lambda, \mu + 1) \tag{20}$$

$$\tilde{\Theta}_{\lambda, \mu} \left\{ I_{p_i+1, q_i, r}^{m, n+1}; \mathbf{c}, x \right\} = \tilde{\mathfrak{J}}_{\mathbf{c}}^I(\lambda + 1, \mu) + \mu \tilde{\mathfrak{J}}_{\mathbf{c}}^I(\lambda, \mu + 1), \tag{21}$$

where

$$\mathfrak{J}_{\mathbf{c}}^I(u, v) := \int_{c_1}^{\infty} \frac{[c^{-1}(t)]}{t^u(t+x)^v} I_{p_i+1, q_i, r}^{m, n+1} \left[ \frac{x}{t}; u \right] dt, \tag{22}$$

$$\tilde{\mathfrak{J}}_{\mathbf{c}}^I(u, v) := \int_{c_1}^{\infty} \frac{\sin^2 \left( \frac{\pi}{2} [c^{-1}(t)] \right)}{t^u(t+x)^v} I_{p_i+1, q_i, r}^{m, n+1} \left[ \frac{x}{t}; u \right] dt, \tag{23}$$

with

$$I_{p_i+1, q_i, r}^{m, n+1} \left[ \frac{x}{t}; u \right] := \mathfrak{K}_{p_i+1, q_i, 1; r}^{m, n+1} \left[ \frac{x}{t} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{1, n}, (a_{ji}, A_{ji})_{n+1, p_i} \\ (b_j, B_j)_{1, m}, (b_{ji}, B_{ji})_{m+1, q_i} \end{matrix} \right].$$

Here  $c$  remains the same as above in Theorem 1.

When  $r = 1, \tau_1 = 1$ , the Aleph function reduces to Fox's  $H$ -function and Theorem 1 gives rise to the following result given by Pogány [10, Theorem].

**Corollary 2.** Let  $\lambda, \mu, r > 0, \alpha = 1 - \lambda, \beta = \rho = 1$  and let the sequence  $\mathbf{c}$  satisfies the condition given in (11). Then we have

$$\Theta_{\lambda, \mu} \left\{ H_{p+1, q}^{m, n+1}; \mathbf{c}, x \right\} = \mathfrak{J}_{\mathbf{c}}^H(\lambda + 1, \mu) + \mu \mathfrak{J}_{\mathbf{c}}^H(\lambda, \mu + 1) \tag{24}$$

$$\tilde{\Theta}_{\lambda, \mu} \left\{ H_{p+1, q}^{m, n+1}; \mathbf{c}, x \right\} = \tilde{\mathfrak{J}}_{\mathbf{c}}^H(\lambda + 1, \mu) + \mu \tilde{\mathfrak{J}}_{\mathbf{c}}^H(\lambda, \mu + 1), \tag{25}$$

where

$$\mathfrak{J}_{\mathbf{c}}^H(u, v) := \int_{c_1}^{\infty} \frac{[c^{-1}(t)]}{t^u(t+x)^v} H_{p+1, q}^{m, n+1} \left[ \frac{x}{t} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{1, n}, (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, m}, (b_j, B_j)_{m+1, q} \end{matrix} \right] dt \tag{26}$$

$$\tilde{\mathfrak{J}}_{\mathbf{c}}^H(u, v) := \int_{c_1}^{\infty} \frac{\sin^2 \left( \frac{\pi}{2} [c^{-1}(t)] \right)}{t^u(t+x)^v} H_{p+1, q}^{m, n+1} \left[ \frac{x}{t} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{1, n}, (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, m}, (b_j, B_j)_{m+1, q} \end{matrix} \right] dt. \tag{27}$$

The Fox-Wright function  ${}_p\Psi_q$  is defined [7, p. 23] by the power series in the form

$${}_p\Psi_q \left[ \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \middle| z \right] := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + A_j n)}{\prod_{j=1}^q \Gamma(b_j + B_j n)} \frac{z^n}{n!}. \tag{28}$$

with  $a_j, b_j \in \mathbb{C}, A_j, B_j \in \mathbb{R}, A_i \cdot B_j \neq 0 (i = \overline{1, p}, j = \overline{1, q})$  and  $\sum_{j=1}^q B_j - \sum_{j=1}^p A_j > -1$ . Now, employing the identity [7, p. 25]

$${}_p\Psi_q \left[ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -z \right] = H_{p,q+1}^{1,p} \left[ z \middle| \begin{matrix} (1 - a_p, A_p) \\ (0, 1), (1 - b_q, B_q) \end{matrix} \right], \tag{29}$$

pointing out that we can express it via the Aleph function as

$${}_p\Psi_q \left[ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -z \right] = \aleph_{p,q+1,1;1}^{1,p} \left[ z \middle| \begin{matrix} (1 - a_p, A_p) \\ (0, 1), (1 - b_q, B_q) \end{matrix} \right],$$

it is not difficult to deduce the corresponding results for Fox–Wright function.

Let us define

$$\Theta_{\lambda,\mu} \{ {}_{p+1}\Psi_q; \mathbf{c}, x \} := \sum_{j=1}^{\infty} \frac{{}_{p+1}\Psi_q \left[ \begin{matrix} (\alpha, \beta), (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -\frac{x}{c_j} \right]}{c_j^\lambda (c_j + x)^\mu}, \tag{30}$$

and

$$\tilde{\Theta}_{\lambda,\mu} \{ {}_{p+1}\Psi_q; \mathbf{c}, x \} := \sum_{j=1}^{\infty} \frac{(-1)^{j-1} {}_{p+1}\Psi_q \left[ \begin{matrix} (\alpha, \beta), (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -\frac{x}{c_j} \right]}{c_j^\lambda (c_j + x)^\mu}. \tag{31}$$

We then obtain the following

**Corollary 3.** *Let  $\lambda \notin \mathbb{N}, \mu > 0, r > 0, (\alpha, \beta) = (1 - \lambda, 1), (b_q, \beta_q) = (1, 1)$  and let the sequence  $\mathbf{c}$  satisfies (11). Then we have*

$$\Theta_{\lambda,\mu} \{ {}_{p+1}\Psi_q; \mathbf{c}, r \} = \mathfrak{J}_{\mathbf{c}}^\Psi(\lambda + 1, \mu) + \mu \mathfrak{J}_{\mathbf{c}}^\Psi(\lambda, \mu + 1) \tag{32}$$

$$\tilde{\Theta}_{\lambda,\mu} \{ {}_{p+1}\Psi_q; \mathbf{c}, r \} = \tilde{\mathfrak{J}}_{\mathbf{c}}^\Psi(\lambda + 1, \mu) + \mu \tilde{\mathfrak{J}}_{\mathbf{c}}^\Psi(\lambda, \mu + 1), \tag{33}$$

where

$$\mathfrak{J}_{\mathbf{c}}^\Psi(u, v) := \int_{c_1}^{\infty} \frac{[c^{-1}(t)]}{t^u (t + x)^v} \cdot {}_{p+1}\Psi_q \left[ \begin{matrix} (1 - u, 1), (a_p, A_p) \\ (1, 1), (b_{q-1}, B_{q-1}) \end{matrix} \middle| -\frac{x}{t} \right] dt \tag{34}$$

and

$$\tilde{\mathfrak{J}}_{\mathbf{c}}^\Psi(u, v) := \int_{c_1}^{\infty} \frac{\sin^2 \left( \frac{\pi}{2} [c^{-1}(t)] \right)}{t^u (t + x)^v} \cdot {}_{p+1}\Psi_q \left[ \begin{matrix} (1 - u, 1), (a_p, A_p) \\ (1, 1), (b_{q-1}, B_{q-1}) \end{matrix} \middle| -\frac{x}{t} \right] dt. \tag{35}$$

**Remark 3.** *Finally, it is interesting to observe that by virtue of the relation*

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1} \left[ -z \middle| \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right] = \aleph_{1,2,1;1}^{1,1} \left[ -z \middle| \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right],$$

where  $E_{\alpha,\beta}(z)$  is the Mittag–Leffler function [Chapter 18 3] and [p. 80 5], defined by

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(an + \beta)},$$

where  $\alpha, \beta \in \mathbb{C}; \Re\{\alpha\}, \Re\{\beta\} > 0$ , the similar type of results for the Mittag–Leffler function can be deduced from Corollary 2.

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