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Asymptotic Attractors of Benjamin-Bona-Mahony Equations

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Abstract. In this paper, we consider the long time behavior of solution for the Benjamin-Bona-Mahony equations with periodic boundary conditions. By the method of orthogonal decomposition, we show that the existence of asymptotic attractor which overcome difficulty come from the precision of approximate inertial manifolds. Moreover, the dimensions estimate of the asymptotic attractor is obtained.

Key words: Benjamin-Bona-Mahony equation, Asymptotic attractor, Dimensions estimate, Orthogonal decomposition.

1. Introduction

It is well known that the concept of an inertial manifold plays an important role in the investigation of the long-time behavior of infinite dimensional dynamical systems, see, for example, [6, 8]. Inertial manifold is a finite dimensional invariant manifold in the phase space H of the system which attracts exponentially all orbits. It is constructed as the graph of a mapping from PH to $(I-P)H$, where P is a projection of finite dimension N. However the existence usually holds under a restrictive spectral gap condition. To investigate the case when the spectral gap condition does not hold the concepts of approximate inertial manifolds [7] have been introduced.

But the precision of approximate inertial manifolds is inextricable difficulty at all times. To overcome this difficulty, recently, new concept of **asymptotic attractor** has been introduced [12].

Now let us recall the definition of asymptotic attractor. We consider the solution $u(t)$ of a differential equation

$$u_t + Au = F(u), \tag{1.1}$$

with initial data

$$u(0) = u_0. \tag{1.2}$$

The variable $u(t)$ belongs to a linear space $\mathbf{E}$ called the phase space, and F is a mapping of $\mathbf{E}$ into itself. The semigroup $\left\{S(t)\right\}_{t\geq 0}$ associated to problems (1.1)-(1.2):

$$S(t) : u_0 \in \mathbf{E} \to u(t) \in \mathbf{E}. \tag{1.3}$$

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If B is a bounded absorbing set, then

$$\mathscr{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s, u_0\in B} S(t)u_0}. \tag{1.4}$$

is global attractor for problems (1.1)-(1.2).

Definition 1.1. *[12]. Let $\mathscr{E}$ be a finite-dimensional subspace of the phase space $\mathbf{E}$, and let B be a bounded absorbing set in $\mathbf{E}$. Suppose there exists a number $t^*(B) > 0$ such that for all $u_0 \in B$ and all $t \geq t^*(B)$, there exists a sequence $\{u^k(t)\}_N \subset \mathscr{E}$ such that*

$$\|u^k(t) - S(t)u_0\|_{\mathbf{E}} \to 0, \quad k \to \infty. \tag{1.5}$$

Then the sequence of sets $\mathscr{A}^k$ defined by

$$\mathscr{A}^k = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s, u_0\in B} u^k(t)} \tag{1.6}$$

is called an asymptotic attractor of the problem (1.1)-(1.2).

In this paper, we will show that the existence of the asymptotic attractor for the following Benjamin-Bona-Mahony equations with periodic boundary conditions

$$u_t - \delta u_{xxt} - \mu u_{xx} + uu_x = f(x), \tag{1.7}$$
$$u(x,0) = u_0(x), \tag{1.8}$$

where $u(x,t) = u(x+2\pi,t)$, $x \in \mathbb{R}^1$, $\int_0^{2\pi} u(x,t)dx = 0$ and δ, μ are positive constants. The Benjamin-Bona-Mahony equation was proposed in [3] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions, as well as the decay rates of solutions for this equation were studied by many authors, see, for example, [1, 2, 4]. On the other hand, the long-time behavior for this equation were considered also by many authors, see, for example, [5, 9–11, 13–15].

Here, by the method of orthogonal decomposition, we show that the existence of asymptotic attractor for problems (1.7)-(1.8). Furthermore, the dimensions estimate of the asymptotic attractor is obtained. Throughout this paper, we set Ω=$(0,2\pi)$, $\|u\|^2 = \int_0^{2\pi} |u|^2 dx$ and

$$\dot{H}^1_{per}(\Omega) =: \left\{ u \middle| u \in L^2(\Omega), u_x \in L^2(\Omega); \int_0^{2\pi} u(x,t)dx = 0; \right.$$
$$\left. u(x,t) = u(x+2\pi,t), x \in R^1 \right\}.$$

Applying Faedo-Galerkin method, it is easy to prove that the problems (1.7)-(1.8) exists a unique solution $u(t) \in \dot{H}^1_{per}(\Omega)$ if $u_0(x) \in \dot{H}^1_{per}(\Omega)$ and $f(x) \in L^2(\Omega)$. Moreover, there are $t_0 > 0$ and $\rho_0 > 0$ such that

$$B = \left\{ u(t) \in \dot{H}^1_{per}(\Omega) : \|u(t)\|^2 + \delta\|u_x(t)\|^2 \leq \rho_0^2, \quad t \geq t_0 \right\}$$

is a bounded absorbing set. Now we are in position to state our main result:

Theorem 1.1. *If $u_0(x) \in \dot{H}^1_{per}(\Omega)$ and $f(x) \in L^2(\Omega)$, the semigroup $S(t)$ associated with problems (1.7)-(1.8) possesses an asymptotic attractor $\mathscr{A}^k$ in $\dot{H}^1_{per}(\Omega)$. Moreover, the dimensions of $\mathscr{A}^k$ satisfies*

$$N_{\mathscr{A}^k} = min\left(N \in \mathbb{N} \left| \frac{2(4\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)^2}{\rho_0^2 c_1 \mu (N+1)^2} \leq 1\, , \frac{2\left(\sqrt{2}\rho_0\delta^{-\frac{1}{4}} + c\rho_0\delta^{-\frac{1}{2}}\right)^2}{c_1\mu(N+1)^2} < 1 \right. \right),$$

where $c_1 = min\left(\frac{\mu(N+1)^2}{2}, \frac{\mu}{2\delta}\right)$.

2. Asymptotic Attractor

In this section, we show that the existence of asymptotic attractor for problems (1.7)-(1.8) by the method of orthogonal decomposition. Let $\{\sin kx, \cos kx, k = 1, 2, \cdots\}$ is an orthonormal basis of $\dot{L}^2_{per}([0, 2\pi])$, denote

$$H_N = Span\{\sin kx, \cos kx, k = 1, 2, \cdots, N\}.$$

Let $P_N\colon \dot{L}^2_{per}([0, 2\pi]) \to H_N$, $Q_N = I - P_N$. For any $u(x, t) \in \dot{L}^2_{per}([0, 2\pi])$, we denote

$$p = P_N u, \quad q = Q_N u.$$

By projecting (1.7) on the H_N, we have

$$p_t - \delta p_{xxt} - \mu p_{xx} + P_N(uu_x) = P_N f, \tag{2.1}$$

and

$$q_t - \delta q_{xxt} - \mu q_{xx} + Q_N(uu_x) = Q_N f. \tag{2.2}$$

For any $u_0(x) \in B$, we set $u^k = p + q^k$ satisfies:

$$\begin{cases} q^0_t - \delta q^0_{xxt} - \mu q^0_{xx} + Q_N(pp_x) = Q_N f, \\ q^0(x,t) = q^0(x + 2\pi, t), \\ q^0(x, 0) = Q_N u_0. \end{cases} \tag{2.3}$$

$$\begin{cases} q^k_t - \delta q^k_{xxt} - \mu q^k_{xx} + Q_N(u^{k-1}u^{k-1}_x) = Q_N f, \\ q^k(x,t) = q^k(x + 2\pi, t), \\ q^k(x, 0) = Q^k_N u_0. \end{cases} \tag{2.4}$$

where $Q^k_N = Q_N - Q_{2^{k+1}N}$, $k = 1, 2, \cdots$.

Thus by (2.3)-(2.4), we can get a sequence $\{u^k(t)\}$ for problems (1.7)-(1.8). To prove Theorem 1.2, we only to check the condition (1.5), that is, we only prove the following Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *Assume that $u(x,t)$ is solution for problems (1.7)-(1.8) with $u_0(x) \in B$, and q^k $(k = 0,1,2,\cdots)$ satisfy (2.3)-(2.4), there are $N_0 \in \mathbb{N}$ and $t_1^*(B) > 0$ such that for $N \geq N_0$ we have*

$$\|u^k\|^2 + \delta\|u_x^k\|^2 \leq 2\rho_0^2, \quad t \geq t_1^*(B), k = 0,1,2,\cdots. \tag{2.5}$$

Proof : We only need to prove the following inequality:

$$\|q^k\|^2 + \delta\|q_x^k\|^2 \leq \rho_0^2. \tag{2.6}$$

Here we verify (2.6) by the inductive method. Firstly, multiplying (2.3) by q^0, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|q^0\|^2 + \delta\|q_x^0\|^2) + \mu\|q_x^0\|^2 \\
\leq\ & \|p\|^{\frac{1}{2}}\|p_x\|^{\frac{3}{2}}\|q^0\| + \|f\|\|q^0\| \\
\leq\ & \rho_0^{\frac{1}{2}}\delta^{-\frac{3}{4}}\rho_0^{\frac{3}{2}}\|q^0\| + \|f\|\|q^0\| \\
\leq\ & (\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)\|q^0\| \\
\leq\ & (\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)\frac{1}{N+1}\|q_x^0\| \\
\leq\ & \frac{\mu}{2}\|q_x^0\|^2 + \frac{1}{2\mu(N+1)^2}(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}(\|q^0\|^2 + \delta\|q_x^0\|^2) + \mu\|q_x^0\|^2 \leq \frac{(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{\mu(N+1)^2}.$$

Noting that

$$\begin{aligned}
\mu\|q_x^0\|^2 &= \frac{\mu}{2}\|q_x^0\|^2 + \frac{\mu}{2}\|q_x^0\|^2 \\
&\geq \left(\frac{\mu(N+1)^2}{2}\|q^0\|^2 + \frac{\mu}{2\delta}\delta\|q_x^0\|^2\right) \\
&\geq c_1(\|q^0\|^2 + \delta\|q_x^0\|^2),
\end{aligned}$$

we have

$$\frac{d}{dt}(\|q^0\|^2 + \delta\|q_x^0\|^2) + c_1(\|q^0\|^2 + \delta\|q_x^0\|^2) \leq \frac{(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{\mu(N+1)^2}.$$

By Gronwall's Lemma, we have

$$\begin{aligned}
\|q^0(t)\|^2 + \delta\|q_x^0(t)\|^2 \leq\ & (\|q^0(0)\|^2 + \delta\|q_x^0(0)\|^2)e^{-c_1 t} \\
& + \frac{(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{c_1\mu(N+1)^2}(1 - e^{-c_1 t}).
\end{aligned}$$

There exists a $t_{11}^*(B) > 0$, such that for $\forall t \geq t_{11}^*(B)$, we have

$$\|q^0(t)\|^2 + \delta\|q_x^0(t)\|^2 \leq \frac{2(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{c_1\mu(N+1)^2}.$$

Let N is large enough, such that

$$\frac{2(\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{\rho_0^2 c_1\mu(N+1)^2} \leq 1, \tag{2.7}$$

we have

$$\|q^0(t)\|^2 + \delta\|q_x^0(t)\|^2 \leq \rho_0^2, \quad t \geq t_{11}^*(B). \tag{2.8}$$

Now assume that $\|q^{k-1}\|^2 + \delta\|q_x^{k-1}\|^2 \leq \rho_0^2$ holds, we shall prove that for $\forall k$ (2.6) is holds. Multiplying (2.4) by q^k, we have

$$\frac{1}{2}\frac{d}{dt}(\|q^k\|^2 + \delta\|q_x^k\|^2) + \mu\|q_x^k\|^2 \leq (4\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)\|q^k\|.$$

By using similar argument as above, we can obtain

$$\frac{d}{dt}(\|q^k\|^2 + \delta\|q_x^k\|^2) + c_1(\|q^k\|^2 + \delta\|q_x^k\|^2) \leq \frac{(4\rho_0^2\delta^{-\frac{3}{4}} + \|f\|)^2}{\mu(N+1)^2}.$$

By Gronwall's Lemma, there exists a $t_{12}^*(B) > 0$ such that for $\forall t \geq t_{12}^*(B)$ we have

$$\|q^k(t)\|^2 + \delta\|q_x^k(t)\|^2 \leq \frac{2(4\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)^2}{c_1\mu(N+1)^2}.$$

Let N is large enough, such that

$$\frac{2(4\delta^{-\frac{3}{4}}\rho_0^2 + \|f\|)^2}{\rho_0^2 c_1\mu(N+1)^2} \leq 1, \tag{2.9}$$

we have

$$\|q^k(t)\|^2 + \delta\|q_x^k(t)\|^2 \leq \rho_0^2, \quad t \geq t_{12}^*(B). \tag{2.10}$$

Let $t_1^*(B) = \max(t_{11}^*(B), t_{12}^*(B))$, then (2.6) follows from (2.8) and (2.10). The proof of Lemma 2.1 is completed.

Lemma 2.2. *Under the hypotheses of Lemma 2.1, there are* $N_1 \in \mathbb{N}$ *and* $t_2^*(B) > 0$ *such that for* $N \geq N_1$ *we have*

$$\|q^k - q\|^2 + \delta\|q_x^k - q_x\|^2 \to 0, \quad k \to \infty, t \geq t_2^*(B). \tag{2.11}$$

Proof : Here we verify (2.11) by the inductive method. Firstly, set $w^0 = q^0 - q$, by (2.2) and (2.3) we have

$$w_t^0 - \delta w_{xxt}^0 - \mu w_{xx}^0 + Q_N(pp_x - uu_x) = 0. \tag{2.12}$$

Multiplying (2.12) by w^0 we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|w^0\|^2 + \delta\|w_x^0\|^2) + \mu\|w_x^0\|^2 \\ \le\ & 2\|u\|^{\frac{1}{2}}\|u_x\|^{\frac{3}{2}}\|w^0\| \le 2\rho_0^2\delta^{-\frac{3}{4}}\|w^0\|. \end{aligned}$$

By using similar argument as above, we can obtain

$$\frac{d}{dt}(\|w^0\|^2 + \delta\|w_x^0\|^2) + c_1(\|w^0\|^2 + \delta\|w_x^0\|^2) \le \frac{4\rho_0^4}{\delta^{\frac{3}{2}}\mu(N+1)^2}. \tag{2.13}$$

By Gronwall's Lemma, there exists a $t_{20}^*(B) > 0$, such that

$$\|w^0(t)\|^2 + \delta\|w_x^0(t)\|^2 \le \frac{8\rho_0^4}{c_1\delta^{\frac{3}{2}}\mu(N+1)^2}, \quad t \ge t_{20}^*(B). \tag{2.14}$$

Denote $w^k = q^k - q$, by (2.2) and (2.4), we have

$$w_t^k - \delta w_{xxt}^k - \mu w_{xx}^k + Q_N(u^{k-1}u_x^{k-1} - uu_x) = 0, \tag{2.15}$$

where $k = 1, 2, \cdots$. Here we note that

$$u^{k-1}u_x^{k-1} - uu_x = u^{k-1}w_x^{k-1} + w^{k-1}u_x.$$

Multiplying (2.15) by w^k we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|w^k\|^2 + \delta\|w_x^k\|^2) + \mu\|w_x^k\|^2 \\ \le\ & \sqrt{2}\delta^{-\frac{1}{4}}\rho_0\|w_x^{k-1}\|\|w^k\| + \delta^{-\frac{1}{2}}\rho_0\|w^{k-1}\|_{L^\infty}\|w^k\| \\ \le\ & \left(\sqrt{2}\rho_0\delta^{-\frac{1}{4}} + c\rho_0\delta^{-\frac{1}{2}}\right)\|w_x^{k-1}\|\|w^k\| \\ \le\ & \frac{1}{N+1}\left(\sqrt{2}\rho_0\delta^{-\frac{1}{4}} + c\rho_0\delta^{-\frac{1}{2}}\right)\|w_x^{k-1}\|\|w_x^k\| \\ \le\ & \frac{1}{2\mu(N+1)^2}\left(\sqrt{2}\rho_0\delta^{-\frac{1}{4}} + c\rho_0\delta^{-\frac{1}{2}}\right)^2\|w_x^{k-1}\|^2 + \frac{\mu}{2}\|w_x^k\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{d}{dt}(\|w^k\|^2 + \delta\|w_x^k\|^2) + c_1(\|w^k\|^2 + \delta\|w_x^k\|^2) \\ \le\ & \frac{1}{\mu(N+1)^2}\left(\sqrt{2}\rho_0\delta^{-\frac{1}{4}} + c\rho_0\delta^{-\frac{1}{2}}\right)^2\|w_x^{k-1}\|^2. \end{aligned} \tag{2.16}$$

where $k=1,2,\cdots$. Let $k=1$ in (2.16), we have

$$\begin{aligned} &\frac{d}{dt}(\|w^1\|^2+\delta\|w_x^1\|^2)+c_1(\|w^k\|^2+\delta\|w_x^k\|^2) \\ \leq\ &\frac{1}{\mu(N+1)^2}\Big(\sqrt{2}\rho_0\delta^{-\frac{1}{4}}+c\rho_0\delta^{-\frac{1}{2}}\Big)^2\|w_x^0\|^2. \end{aligned} \tag{2.17}$$

By Gronwall's lemma, there is a $t_{21}^*(B)>0$ such that

$$\begin{aligned} &\|w^1(t)\|^2+\delta\|w_x^1(t)\|^2 \\ \leq\ &\frac{2}{c_1\mu(N+1)^2}\Big(\sqrt{2}\rho_0\delta^{-\frac{1}{4}}+c\rho_0\delta^{-\frac{1}{2}}\Big)^2\|w_x^0(t)\|^2, \quad t\geq t_{21}^*(B). \end{aligned} \tag{2.18}$$

By the inductive method, there is a $t_{2k}^*(B)>0$ such that

$$\begin{aligned} &\|w^k\|^2+\delta\|w_x^k\|^2 \\ \leq\ &\frac{2^k}{c_1^k\mu^k(N+1)^{2k}}\Big(\sqrt{2}\rho_0\delta^{-\frac{1}{4}}+c\rho_0\delta^{-\frac{1}{2}}\Big)^{2k}\|w_x^0(t)\|^2, \quad t\geq t_{2k}^*(B) \end{aligned} \tag{2.19}$$

where $k=1,2,\cdots$. If N is large enough, such that

$$\frac{2\Big(\sqrt{2}\rho_0\delta^{-\frac{1}{4}}+c\rho_0\delta^{-\frac{1}{2}}\Big)^2}{c_1\mu(N+1)^2}<1, \tag{2.20}$$

then (2.11) follows from (2.14) and (2.19). The proof of Lemma 2.2 is completed.

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