EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 3, 2010, 531-540
ISSN 1307-5543 – www.ejpam.com

SPECIAL ISSUE ON
GRANGER ECONOMETRICS AND STATISTICAL MODELING
DEDICATED TO THE MEMORY OF PROF. SIR CLIVE W.J. GRANGER

A New Skew-normal Model for the Application-Oriented Skew-t Model

John T. Chen

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403

Abstract. Among many papers of Professor Clive W. J. Granger, the one that strongly draws my attention is his work [7] using the skew-t model to analyze common factors in conditional distributions for bivariate time series. Different from many existing versions of theory-oriented skew-t models, the skew-t model that Professor Granger and his collaborators used was directly motivated by applications in analyzing economics data. This application-oriented skew-t model has discernible features on enabling model flexibility and keeping the practical standardizing conditions [10]. On the other hand, the skew-t model is in need of a proper statistical justification to solidify its theoretical foundation. In this paper, we initiate a new skew normal family that enhances the skew-t model in [10] and [7].

2000 Mathematics Subject Classifications: 62E15, 62P20

Key Words and Phrases: Skew normal model, Student-t principle, excess kurtosis, asymmetry

1. Introduction

Granger [6], [7] study common factors in conditional distributions for bivariate time series, in which two typical macro-economic variables (income and consumption) are modeled with individual growth varying over a business cycle. In order to test the influence of a business cycle index variable over the conditional copula of income and consumption growth, [7] consider linear models for the conditional mean of two series: the US real per capita disposable income and US real per capita consumption on non-durables. For the business cycle, they use the [12] experimental coincident index, which is a business cycle indicator of the two series. More detail and the data can be found from Jim Stock's web page (`http://`

Email address: `jchen@bgnet.bgsu.edu`

`ksghome.harvard.edu/JStock.Academic.Ksg/xri/0201/xindex.asc`). For the conditional variance, they apply the autoregressive conditional heteroscedasticity (ARCH) model of [4].

Considering excess kurtosis and asymmetry, [7] use the following model with skew parameter λ and degrees of freedom ν to fit the data:

$$f(t;\lambda,\nu)=\begin{cases} bc(1+\frac{1}{\nu-2}(\frac{by+a}{1-\lambda})^2)^{-(\nu+1)/2} & \text{if } t\le -\frac{a}{b} \\ bc(1+\frac{1}{\nu-2}(\frac{by+a}{1+\lambda})^2)^{-(\nu+1)/2} & \text{if } t> -\frac{a}{b} \end{cases} \tag{1}$$

where

$$\begin{aligned} a &= 4\lambda c\frac{\nu-2}{\nu-1} \\ c &= \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)}} \\ b &= \sqrt{1+3\lambda^2-a^2}. \end{aligned} \tag{2}$$

Note that this model maintains the properties of zero mean and unit variance of the standardized Student-t distribution with a skew parameter λ.

In the statistical literature, the Student-t random variable is constructed with the following principle. Let X be a standard normal random variable which is independent of a χ^2_ν random variable, Y, then the ratio $\frac{X}{\sqrt{Y/\nu}}$ is defined as the Student-t random variable. If X is a skew normal model, then the ratio $\frac{X}{\sqrt{Y/\nu}}$ is defined as a skew-t random variable.

However, the original definition of model (1) as a skew-t model is due to the following consideration [10]:"*To allow for a richer set of behavior, we may need a more flexible family of densities. A minimal desirable extension is to allow for skewness. In order to keep in the ARCH tradition, it is also important to have density functions which can be easily parameterized so that the innovations are mean zero and unit variance*."

Thus the beginning of model (1) did not follow the standard framework on the definition of the (skew) Student-t statistic: a (skew) normal random variable divided by the square root of an independent and standardized χ^2 random variable. In fact, Model (1) is purely a skew model that keeps the ARCH tradition in analyzing economics data.

There are several versions of skew-t models defined in the statistical literature. For example, when X follows a skew normal model

$$f_X(x,\lambda)=2\phi(x)\Phi(\lambda x), \tag{3}$$

where ϕ and Φ are the pdf and cdf of the standard normal distribution, respectively, the ratio $\frac{X}{\sqrt{Y/\nu}}$ is defined as a skew-t model.

The skew normal model (3) has been thoroughly discussed in the literature. For example, among many other publications, [3] discuss a skew normal model to fit the stock market data;

and [9] introduce a multivariate skew normal model, and [1] propose a matrix variate skew normal model, to list just a few. Unfortunately, all currently available skew normal models do not lead to the skew-t model defined in (1). This raises a question on the plausibility of model (1) being named a skew-t model. Does the corresponding skew normal model exist for the definition of the skew-t model in (1)?

The three criteria (skewness, zero mean and unit variance) of [10] essentially exclude the use of any skew-t model derived from the currently existing skew normal model (3). It is awkward to claim that model (1) is a skew-t model if the corresponding skew normal model does not exist. Given a skew normal model, it is straightforward to define a skew-t. However, Given a skew-t model, it is not easy to retrospectively define the skew normal model that fits in to the already specified skew-t model. Under this scenario, it is critical to search for a skew normal model with which Model (1) can be derived following the principle of the Student -t statistic. This partly motivates the investigation in this paper.

The rest of the paper is organized in the following way. After defining a new skew normal model in Section 2, we derive the corresponding skew-t model in Section 3, which coincides with the skew-t model discussed in [10] and [7]. The new skew normal model opens a new research area for modeling data with excess kurtosis and asymmetry. An extension of this skewness methodology is provided in Section 4, which is followed by concluding remarks in Section 5.

2. The Density of a New Skew Normal Model

Consider the following distribution family for any $\lambda \in (0,1)$,

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-\lambda)^2}} & \text{if } x \le 0 \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1+\lambda)^2}} & \text{if } x > 0. \end{cases} \tag{4}$$

This model is different from the skew normal model discussed in the literature. It skews the normal model in the way of adjusting the left-side and right-side of the density by changing the skew parameter λ.

First, notice that $f(x) \ge 0$ and

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1+\lambda)^2}} dx + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-\lambda)^2}} dx \\ &= \frac{1+\lambda}{2} + \frac{1-\lambda}{2} \\ &= 1. \end{aligned} \tag{5}$$

Now the mean of model (4) becomes

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$
\begin{aligned}
&= \int_0^\infty \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1+\lambda)^2}} dx + \int_{-\infty}^0 \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-\lambda)^2}} dx \\
&= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2(1+\lambda)^2}} dy - \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2(1-\lambda)^2}} dy \\
&= \frac{1}{2\sqrt{2\pi}} e^{-\frac{y}{2(1+\lambda)^2}} (1+\lambda)^2(-2)|_0^\infty - \frac{1}{2\sqrt{2\pi}} e^{-\frac{y}{2(1-\lambda)^2}} (1-\lambda)^2(-2)|_0^\infty \\
&= \frac{1}{\sqrt{2\pi}}(1+\lambda)^2 - \frac{1}{\sqrt{2\pi}}(1-\lambda)^2 \\
&= \frac{4}{\sqrt{2\pi}}\lambda. \qquad (6)
\end{aligned}
$$

In order to evaluate the variability, the second moment of the new random variable can be computed as follows.

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^\infty x^2 f(x) dx \\
&= \int_0^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1+\lambda)^2}} dx + \int_{-\infty}^0 \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-\lambda)^2}} dx \\
&= \int_0^\infty \frac{(1+\lambda)^3 y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \int_{-\infty}^0 \frac{(1-\lambda)^3 y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= (6\lambda^2+2)\int_0^\infty \frac{y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= 3\lambda^2+1, \qquad (7)
\end{aligned}
$$

because

$$\int_0^\infty \frac{y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1/2.$$

Thus the variance of the new skew normal model is

$$V(X) = E(X^2) - (EX)^2 = 3\lambda^2 + 1 - (\frac{4}{\sqrt{2\pi}}\lambda)^2. \qquad (8)$$

Now denote $E(X) = \eta$, $V(X) = \tau^2$, and $Z = \frac{X-\eta}{\tau}$, we get

$$E(Z) = 0 \quad \text{and} \quad V(Z) = 1. \qquad (9)$$

The density of Z reads

$$f(z;\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tau z+\eta)^2}{2(1-\lambda)^2}} & \text{if } z \le -\frac{\eta}{\tau} \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tau z+\eta)^2}{2(1+\lambda)^2}} & \text{if } z > \frac{\eta}{\tau} \end{cases} \qquad (10)$$

where

$$\begin{aligned} \eta &= \frac{4}{\sqrt{2\pi}}\lambda \\ \tau &= \sqrt{1+3\lambda^2-\frac{8}{\pi}\lambda^2}. \end{aligned} \tag{11}$$

Notice the similarity of this density compared with the skew-t density described in (1) and (2). If, for an independent χ^2 random variable, the skew normal defined above leads to the skew-t density in (1), we can then establish the connection. With this idea in mind, in what follows, we will set up the connection between the skew normal model (9)-(10) and the skew-t model (1)-(2).

3. The New Skew-normal and Skew-t models

In this section, we will derive the relation between the skew normal model (9)-(10) and the skew-t model (1)-(2), notice that both of the two models are skew with zero mean and unit variance.

Theorem 1. *Let X be a random variable following the skew normal density of (4), let Q be a χ^2 random variable with degrees of freedom ν, and define*

$$T = \frac{X}{\sqrt{\frac{Q}{\nu}}}.$$

Then $E(\sqrt{\frac{\nu-2}{\nu}}T) = a$, $V(\sqrt{\frac{\nu-2}{\nu}}T) = b$, and $T^* = \frac{\sqrt{\frac{\nu-2}{\nu}}T-a}{b}$ has the density given in (1)-(2), where a and b are the constants given in (2).

Proof. Since X and V are independent, the joint density between X and Q reads

$$f(x,q) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2(1-\lambda)^2}}\frac{1}{\Gamma(\nu/2)}2^{-\nu/2}q^{\frac{1}{2}-1}e^{-q/2} & \text{if } x \le 0, q > 0 \\ \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2(1+\lambda)^2}}\frac{1}{\Gamma(\nu/2)}2^{-\nu/2}q^{\frac{1}{2}-1}e^{-q/2} & \text{if } x > 0, q > 0. \end{cases} \tag{12}$$

Now, let

$$\begin{cases} t &= \frac{x}{\sqrt{\frac{q}{\nu}}} \\ u &= q, \end{cases}$$

We have the determinant of the Jacobian for the transformation from (x,q) to (t,u) as

$$\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial q}{\partial t} & \frac{\partial q}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{\nu}} & \frac{1}{2}\frac{t}{\sqrt{\nu}}u^{-1/2} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{\nu}}$$

Thus

$$f_T(t) = \int_0^\infty f(t\sqrt{\frac{u}{v}}, u)(\frac{u}{v})^{\frac{1}{2}} du, \tag{13}$$

where $f(x, v)$ is the joint density function in (11). When $t > 0$, Equation (12) becomes

$$\begin{aligned}
f_T(t) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(v/2)}2^{-v/2}\int_0^\infty e^{-\frac{t^2u}{2v(1+\lambda)^2}}u^{\frac{1}{2}-1}e^{-u/2}(\frac{u}{v})^{1/2}du \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(v/2)}2^{-v/2}(\frac{1}{v})^{1/2}\int_0^\infty e^{-\frac{1}{2}(\frac{t^2}{(1+\lambda)^2v}+1)u}u^{\frac{v-1}{2}}du \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(v/2)}2^{-v/2}(\frac{1}{v})^{1/2}\int_0^\infty u^{\frac{v+1}{2}-1}e^{-\frac{1}{2}(\frac{t^2}{(1+\lambda)^2v}+1)u}du \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(v/2)}2^{-v/2}(\frac{1}{v})^{1/2}(\frac{1}{2}(\frac{t^2}{(1+\lambda)^2v}+1))^{-\frac{v+1}{2}}\Gamma(\frac{v+1}{2}) \\
&= \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi v}}(\frac{t^2}{(1+\lambda)^2v}+1)^{-\frac{v+1}{2}}.
\end{aligned} \tag{14}$$

Similarly, when $t < 0$, Equation (12) becomes

$$f_T(t) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi v}}[\frac{t^2}{(1-\lambda)^2v}+1]^{-\frac{v+1}{2}}. \tag{15}$$

Combining (14) and (15) yields

$$f_T(t;\lambda) = \begin{cases} \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi v}}(\frac{t^2}{(1-\lambda)^2v}+1)^{-\frac{v+1}{2}} & \text{if } t < 0 \\ \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi v}}(\frac{t^2}{(1+\lambda)^2v}+1)^{-\frac{v+1}{2}} & \text{if } t > 0. \end{cases} \tag{16}$$

Now, consider $Y = \sqrt{\frac{v-2}{v}}T$, due to (16), the density of Y reads

$$f_Y(y;\lambda) = \begin{cases} \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi(v-2)}}(\frac{t^2}{(1-\lambda)^2(v-2)}+1)^{-\frac{v+1}{2}} & \text{if } t < 0 \\ \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi(v-2)}}(\frac{t^2}{(1+\lambda)^2(v-2)}+1)^{-\frac{v+1}{2}} & \text{if } t > 0. \end{cases} \tag{17}$$

Denote

$$c = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)\sqrt{\pi(v-2)}},$$

similar to the derivation in (6)-(8), we have

$$\begin{aligned}
E(Y) &= 4\lambda c\frac{v-2}{v-1} = a \\
V(Y) &= \sqrt{1+3\lambda^2-a^2} = b.
\end{aligned} \tag{18}$$

Standardizing the random variable Y by setting $T^* = \frac{Y-a}{b}$ gets the density in equations (1)-(2). This completes the proof of this theorem.

4. An Extension on the Skewness Formulation

The formulation of the skew normal model (4) shares the same principle as the formulation of the skew-t model in [10]. This way of modeling skewness is more flexible because it models the shapes of the positive half and the negative half using skew factors $\frac{1}{1+\lambda}$ and $\frac{1}{1-\lambda}$, respectively. The following proposition extends this method of skewness formulation into a general setting. First, notice that the skewness formulation discussed in this paper is applicable to any density function truncated at its median.

Lemma 1. *Let d be the median of a random variable X with a density $g(x)$, then the skewed function $f(x;\lambda)$ is a density with skew factor λ, where*

$$f(x;\lambda) = \begin{cases} g(\frac{x}{1-\lambda} & \textit{if } x \leq d \\ g(\frac{x}{1+\lambda} & \textit{if } x > d. \end{cases} \tag{19}$$

Proof. Since $g(x)$ is a density function, $g(x) \geq 0$ and $\int_R g(x)dx = 1$. Notice that d is the median of $g(x)$, namely

$$\int_{-\infty}^{d} g(x)dx = \int_{d}^{\infty} g(x)dx = \frac{1}{2},$$

we have $f(x;\lambda) \geq 0$ and

$$\begin{aligned} \int_{-\infty}^{\infty} f(x;\lambda)dx &= \int_{d}^{\infty} g(\frac{x}{1+\lambda})dx + \int_{-\infty}^{d} g(\frac{x}{1-\lambda})dx \\ &= \frac{1+\lambda}{2} + \frac{1-\lambda}{2} \\ &= 1. \end{aligned} \tag{20}$$

This completes the proof of Lemma 1.

With Lemma 1, the skewness formulation in the skew normal model defined in Section 2 and the skew-t model defined in [10] is a special case in which the median of the density function $g(x)$ is zero.

Proposition 1. *For any symmetric density $g(x)$ of a random variable X with the mean value $\mu = 0$, let $f(x;\lambda)$ be the skewed density of $g(x)$ truncated at the median of X with the skew parameter λ. Denote Y the random variable corresponding to the density $f(x;\lambda)$, we have*

$$E(Y) = 4\lambda\eta, \tag{21}$$

where

$$\eta = \int_{0}^{\infty} y g(y) dy. \tag{22}$$

Proof. By Lemma 1, $f(x;\lambda)$ is a density. Since the density $g(x)$ is symmetric with mean zero, the median of X is zero. Now the mean of the new skew random variable can be evaluated as follows.

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} yf(y)dy \\
&= \int_{-\infty}^{0} yg(\frac{y}{1-\lambda})dy + \int_{0}^{\infty} yg(\frac{y}{1+\lambda})dy \\
&= \int_{-\infty}^{0} (1-\lambda)^2 xg(x)dx + \int_{0}^{\infty} (1+\lambda)^2 xg(x)dx \\
&= (1+\lambda^2)\int_{-\infty}^{\infty} xg(x)dx + \int_{0}^{\infty} 2\lambda xg(x)dx - \int_{-\infty}^{0} 2\lambda xg(x)dx \\
&\quad \text{by the symmetry of} \quad g(x) \\
&= (1-\lambda)^2\mu + 4\lambda \int_{d}^{\infty} xg(x)dx. \\
&= 4\lambda\eta.
\end{aligned}$$

This completes the proof of this proposition.

Obviously, Equation (6) is a special case of Proposition 1 when $g(x)$ is the density of the standard normal density.

We have considered skewness and mean in the above discussion for any symmetric density $f(x)$. In terms of the three standardizing conditions to keep the ARCH tradition for economics data, we need to consider the second moment of the new random variable corresponding to the skew density $f(x;\lambda)$.

Proposition 2. *For any symmetric density $g(x)$ of a random variable X with the mean value $\mu = 0$, let $f(x;\lambda)$ be the skewed density of $g(x)$ truncated at the median of X with the skew parameter λ. Denote Y the random variable corresponding to the density $f(x;\lambda)$, we have*

$$E(Y^2) = 2(1+3\lambda^2)\psi, \tag{23}$$

where

$$\psi = \int_{0}^{\infty} y^2 g(y)dy. \tag{24}$$

Proof. Similar to the proof of the proposition 1, we have

$$\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} x^2 f(x;\lambda)dx \\
&= \int_{-\infty}^{0} x^2 g(\frac{x}{1-\lambda})dx + \int_{0}^{\infty} x^2 g(\frac{x}{1+\lambda})dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0}(1-\lambda)^3 y^2 g(y)dy + \int_{0}^{\infty}(1+\lambda)^3 y^2 g(y)dy \\
&= (1-3\lambda+3\lambda^2-\lambda^3)\int_{-\infty}^{\infty} y^2 g(y)dy + (1+3\lambda+3\lambda^2+\lambda^3)\int_{0}^{\infty} y^2 g(y)dy \\
&= 2(1+3\lambda^2)\int_{0}^{\infty} y^2 g(y)dy \\
&= 2(1+3\lambda^2)\psi.
\end{aligned}
$$

This completes the proof of Proposition 2.

Notice that Equation (7) is a special case of Proposition 2. Summarizing above discussions, by (21) and (23), we have the following result for the skewness formulation of any symmetric density.

Proposition 3. *For any symmetric density $g(x)$ of a random variable X with the mean value $\mu = 0$, let $f(x;\lambda)$ be the skewed density of $g(x)$ truncated at the median of X with the skew parameter λ. Denote Y the random variable corresponding to the density $f(x;\lambda)$. Let*

$$
Y^* = \frac{Y - 4\lambda\eta}{\sqrt{2(1+\lambda^2)\psi - 16\lambda^2\eta^2}},
$$

then Y^* satisfies the three criteria for the ARCH modeling (skew, zero mean and unit variance), where ψ and η are given in (22) and (24).

5. Discussion

This paper discusses a general method of formulation for statistical analysis of data with excess kurtosis and asymmetry. The new way of modeling skewness is application oriented. It stems from the application in analyzing bivariate time series for economics data. Compared with existing skew models such as the generalized Gaussian model [11] or the model of skew normal sample mean [2], the new skew normal family enjoys more freedoms due to the explicit form taking two different skew factors in two halves of the real line. There are many follow-up inference results associated with this new model. For example, the goodness of fit test for the new skew normal model can be developed similarly to the method of [8].

The new skew-normal and skew-t models are essentially a new univariate model, which may be further extended to a new multivariate-t models such as the Gupta and Chen [9] formulation or the Fernandez and Steel [5] formulation.

References

[1] J Chen and A Gupta. Matrix variate skew normal distributions. *Statistics*, 39(3):247–253, 2005.

[2] J Chen, A Gupta, and T Nguyen. The density of the skew normal sample mean and its applications. *Journal of Statistical Computation and Simulation*, 74(7):487–494, 2004.

[3] J Chen, A Gupta, and C Troskie. Distribution of stock returns when the market is up (down). *Communications in Statistics*, 32:1541–1558, 2003.

[4] R Engle. Autoregressive conditional heteroscedasticity with estimates of the variance of UK inflation. *Econometrica*, 50:987–1007, 1982.

[5] C Fernandez and M Steel. Multivariate student-t regression models: Pitfalls and inference. *Biometrika*, 86:153–167, 1999.

[6] C Granger. Implications of aggregation with common factors. *Economic Theory*, 3:208–222, 1987.

[7] C Granger, T Terasvirta, and A Patton. Common factors in conditional distributions for bivariate time series. *Journal of Econometrics*, 132:43–57, 2006.

[8] A Gupta and J Chen. Goodness of fit test for the skew-normal distribution. *Communication in Statistics*, 30(4):907–930, 2001.

[9] A Gupta and J Chen. A class of multivariate skew-normal models. *The Annals of the Institute of Statistical Mathematics*, 56(2):305–315, 2004.

[10] B Hansen. Autoregressive conditional density estimation. *International Economic Review*, 35:705–730, 1994.

[11] T Nguyen, J Chen, A Gupta, and K Dinh. A proof of the conjecture on positive skewness of generalized inverse gaussian distributions. *Biometrika*, 90:245–250, 2003.

[12] J Stock and M Watson. New indexes of coincident and leading economic indicators. *NBER Macroeconomics Annual*, 4, 1989.