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Second-Order Duality For a Class of Nondifferentiable Continuous Programming Problems

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Abstract. A dual problem associated with a class of non-differentiable continuous programming problems is formulated. Under the second-order pseudo-invexity, various duality theorems are validated for this pair of dual problems. A pair of dual problems with natural boundary values is constructed and the proofs of its various duality results are merely indicated. Further, it is shown that our results can be viewed as dynamic generalizations of corresponding (static) second-order duality theorems for a class of nondifferentiable nonlinear programming problems existing in the literature.

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1. Introduction

Second-order duality in mathematical programming has been extensively investigated in the literature. A second-order dual formulation for a non-linear programming problem was introduced by Mangasarian [5]. Later Mond [6] established various duality theorems under a condition which is called "Second-order convexity". This condition is much simpler than that used by Mangasarian [5]. In [9], Mond and Weir reconstructed the second-order duals and higher order dual models to drive usual duality results. It is remarked here that second-order dual to a mathematical programming problem presents a tighter bound and because of which it enjoys computational advantage over a first order dual.

Duality and optimality for continuous programming have been widely investigated by many authors in the recent past notably, Mond and Hanson [7], Bector, Chandra and Husain [1], Mond and Husain [8] and Chen [3] and other cited references in these expositions.

Chen [3] was the first to identify second-order dual formulated for a constrained variational problem and established various duality results under an involved invexity- like assumptions. Recently, Husain et al [4] have presented Mond-Weir type duality for the problem

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of [3] and by introducing continuous-time version of second-order invexity and generalized second-order invexity, validated various duality results.

In this paper we formulate a Wolfe type second order dual to a class of nondifferentiability continuous programming problems where nondifferentiability enters due to the square root of a certain quadratic form appearing in the integrand of the objective functional. The popularity of this type of problems seems to originate from the fact that, even though the objective function and or / constraint functions are non-smooth, a simple representation of the dual problem may be found. The theory of non-smooth mathematical programming deals with more general type of functions by means of generalized sub- differentials. However, square root of positive semi-definite quadratic form is one of the few cases of the nondifferentiable functions for which one can write down the sub-or quasi-differentials explicitly. Here, various duality theorems for this pair of Wolfe type dual problems are validated under second order pseudo-invexity condition. A pair of Wolfe type dual variational problems with natural boundary values rather than fixed end points is presented and the proofs of its duality results are indicated. It is also shown that our second-order duality results can be considered as dynamic generalizations of corresponding (Static) second-order duality results established for nondifferentiable nonlinear programming problem, considered by Zhang and Mond [10].

2. Definitions and Related Pre-requistes

Let $I=[a,b]$ be a real interval, $\phi : I\times R^n\times R^n \longrightarrow R$ and $\psi : I\times R^n\times R^n \longrightarrow R^m$ be twice continuously differentiable functions. In order to consider $\phi(t,x(t),\dot{x}(t))$ where $x : I \longrightarrow R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x and $\phi_{\dot{x}}$, the first order of ϕ with respect to $x(t)$ and $\dot{x}(t)$, respectively, that is,

$$\phi_x=\left[\frac{\partial\phi}{\partial x^1},\frac{\partial\phi}{\partial x^2},\ldots,\frac{\partial\phi}{\partial x^n}\right]^T, \qquad \phi_{\dot{x}}=\left[\frac{\partial\phi}{\partial \dot{x}^1},\frac{\partial\phi}{\partial \dot{x}^2},\ldots,\frac{\partial\phi}{\partial \dot{x}^n}\right]^T$$

Denote by ϕ_{xx} the Hessian matrix of ϕ, and ψ_x the $m\times n$ Jacobian matrix respectively, that is, with respect to $x(t)$, that is, $\phi_{xx}=(\frac{\partial^2\phi}{\partial x^i\partial x^j})$, $i,j=1,2,...n$, ψ_x the $m\times n$ Jacobian matrix

$$\psi_x=\begin{pmatrix} \frac{\partial\psi^1}{\partial x^1} & \frac{\partial\psi^1}{\partial x^2} & \cdots & \frac{\partial\psi^1}{\partial x^n} \\ \frac{\partial\psi^2}{\partial x^1} & \frac{\partial\psi^2}{\partial x^2} & \cdots & \frac{\partial\psi^2}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\psi^m}{\partial x^1} & \frac{\partial\psi^m}{\partial x^2} & \cdots & \frac{\partial\psi^m}{\partial x^n} \end{pmatrix}_{n\times n}$$

The symbols $\phi_{\dot{x}}, \phi_{\dot{x}x}, \phi_{x\dot{x}}$ and $\psi_{\dot{x}}$ have analogous representations.

Designate by X the space of piecewise smooth functions $x : I \longrightarrow R^n$, with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by

$$u=Dx \Longleftrightarrow x(t)=\int_a^t u(s)ds,$$

Thus $\frac{d}{dt} = D$ except at discontinuities.

We incorporate the following definitions which are required in the subsequent analysis.

Definition 1 (Second order Invex). *If there exists a vector function* $\eta = \eta(t, x, \bar{x}) \in R^n$ *where* $\eta : I \times R^n \times R^n \longrightarrow R^n$ *with* $\eta = 0$ *at* $t = a$ *and* $t = b$*, such that for a scalar function* $\phi(t, x, \dot{x})$*, the functional* $\int_I \phi(t, x, \dot{x})dt$ *where* $\phi : I \times R^n \times R^n \longrightarrow R$ *satisfies*

$$\int_I \phi(t, x, \dot{x})dt - \int_I \{\phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p^T(t)Gp(t)\}dt \geq \int_I \{\eta^T \phi_x(t, \bar{x}, \dot{\bar{x}}) + (D\eta)^T \phi_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \eta^T Gp(t)\}dt$$

then $\int_I \phi(t, x, \dot{x})dt$ *is second-order invex with respect to* η *where* $G = \phi_{xx} - 2D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}} - D^3\phi_{\dot{x}\ddot{x}}$ *and* $p \in C(I, R^n)$*, the space of continuous n-dimensional continuous vector functions.*

Definition 2 (Second order Pseudoinvex). *If the functional* $\int_I \phi(t, x, \dot{x})dt$ *satisfies*

$$\int_I \{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T Gp(t)\}dt \geq 0 \Longrightarrow \int_I \phi(t, x, \dot{x})dt \geq \int_I \{\phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p(t)^T Gp(t)\}dt,$$

then $\int_I \phi(t, x, \dot{x})dt$ *is said to be second-order pseudoinvex with respect to* η*.*

Definition 3 (Second order Quasi-invex). *If the functional* $\int_I \phi(t, x, \dot{x})dt$ *satisfies*

$$\int_I \phi(t, x, \dot{x})dt \leq \int_I \{\phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p(t)^T(t)Gp(t)\}dt \Longrightarrow \int_I \{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G(t)p(t)\}dt \leq 0,$$

then $\int_I \phi(t, x, \dot{x})dt$ *is said to be second-order quasi-invex with respect to* η*.*

Remark 1. *If* ϕ *does not depend explicitly on* t*, then the above definitions reduce to those given in [6] for static cases.*

Consider the following class of non-differentiable continuous programming problem studied in [2]:

$$\begin{aligned} (\mathbf{P}+) \quad \text{Minimize} \quad & \int_I \{f(t, x(t), \dot{x}(t)) + (x(t)^T B(t)x(t))^{1/2}\}dt \\ \text{subject to} \quad & x(a) = 0 = x(b), \\ & g(t, x(t), \dot{x}(t)) \leq 0, t \in I, \\ & h(t, x(t), \dot{x}(t)) = 0, t \in I \end{aligned}$$

where

(i) f, g and h are twice differentiable functions from $I \times R^n \times R^n$ into R, R^m and R^k respectively, and

(ii) $B(t)$ is a positive semidefinite $n \times n$ matrix with $B(.)$ continuous on I.

The following proposition gives the Fritz John type of optimality conditions which are derived by Chandra, Craven and Husain [2]:

Proposition 1 (Fritz-John Conditions). *If (VP+) attains a local minimum at $\bar{x} \in X$ and if $h_x(\cdot, \bar{x}(\cdot), \dot{\bar{x}}(\cdot))$ maps X onto a closed subspace of $C(I, R^p)$, then there exist Lagrange multipliers $\tau \in R_+$, piecewise smooth $\bar{y} : I \to R^m$ and $\bar{\lambda} : I \to R^k$, not all zero, and also piecewise smooth $\bar{z} : I \to R^n$ satisfying for all $t \in I$,*

$$\tau f_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{z}(t)^T B(t) + \bar{y}(t)^T g_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{\mu}(t)^T h_x(t, \bar{x}(t), \dot{\bar{x}}(t))$$
$$= D[\tau f_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{\mu}(t)^T h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))] \quad t \in T,$$
$$\bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0, \quad t \in I$$
$$\bar{z}(t)^T B(t) \bar{z}(t) \leq 1, \quad t \in I$$
$$\bar{x}(t)^T B(t) \bar{z}(t) = (\bar{x}(t)^T B(t) \bar{x}(t))^{1/2}, \quad t \in I$$

If $h_x(\cdot, \bar{x}(\cdot), \dot{\bar{x}}(\cdot))$ is subjective, then τ and $\bar{y}$ are not both zero.

Lemma 1 (Schwartz inequality). *It states that*

$$x(t)^T B(t) z(t) \leq (x(t)^T B(t) x(t))^{1/2} (z(t)^T B(t) z(t))^{1/2}, \quad t \in I \tag{1}$$

with equality in (1) if (and only if)

$$B(t)(x(t) - q(t) z(t)) = 0 \text{ for some } q(t) \in R.$$

Remark 2. *The Fritz John necessary optimality conditions in Proposition 1 for (P+), become the Karush-Kuhn-Tucker type optimality conditions if $\tau = 1$. It suffices for $\tau = 1$, that the following Slater's condition holds:*

$$g(t, \bar{x}(t), \dot{\bar{x}}(t)) + g_x(t, \bar{x}(t), \dot{\bar{x}}(t)) v(t) + g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) \dot{v}(t) < 0, \; v(t) \in X, t \in I.$$

3. Second-Order Duality

Consider the following continuous programming problem (CP) by ignoring the equality constraint $h(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0$, $t \in I$ in the problem (P+):

$$(\textbf{CP}) \quad \text{Minimize} \quad \int_I \{f(t, x(t), \dot{x}(t)) + (x(t)^T B(t) x(t))^{1/2}\} dt$$
$$\text{subject to} \quad x(a) = 0 = x(b), \tag{2}$$
$$g(t, x(t), \dot{x}(t)) \leq 0, t \in I \tag{3}$$

Analogously to the second-order dual problem introduced by Mangasarian [5] for a nonlinear programming problem, we consider the following second order dual continuous programming problem (CD) for (CP).

$$(\mathbf{CD})\quad \text{Maximize} \quad \int_I \{f(t,u(t),\dot u(t)) + u(t)^T B(t)z(t) + y(t)^T g(t,u(t),\dot u(t)) - \frac{1}{2}p(t)^T H p(t)\}dt$$

$$\text{subject to} \quad u(a) = 0 = u(b), \tag{4}$$

$$f_u(t,u(t),\dot u(t)) + B(t)z(t) + y(t)^T g_u(t,u(t),\dot u(t))$$
$$-D(f_{\dot u}(t,u(t),\dot u(t)) + y(t)^T \tag{5}$$
$$z(t)^T B(t)z(t) \le 1,\ t \in I \tag{6}$$
$$y(t) \ge 0,\ t \in I \tag{7}$$

where

$$H = f_{uu}(t,u,\dot u) + (y(t)^T g_u(t,u,\dot u))_u - 2D[f_{u\dot u}(t,u,\dot u) + (y(t)^T g_u(t,u,\dot u))_{\dot u}]$$
$$+ D^2[f_{\dot u\dot u}(t,u,\dot u) + (y(t)^T g_{\dot u}(t,u,\dot u))_{\dot u}] - D^3[f_{\dot u\ddot u}(t,u,\dot u) + (y(t)^T g_{\dot u}(t,u,\dot u))_{\ddot u}]$$

Theorem 1 (Weak duality). *Let $x(t) \in X$ be a feasible solution of (CP) and $u(t), y(t), z(t)$ be a feasible solution of (CD). If*

$$\int_I \{f(t,.,.) + (\cdot)^T B(t)z(t) + y(t)^T g(t,.,.)\}dt$$

is second-order pseudo-invex with respect to $\eta = \eta(t,x,u)$, then

$$\inf(CP) \ge \sup(CD).$$

Proof. From (5), we have

$$\int_I \eta^T\{f_u(t,u(t),\dot u(t)) + B(t)z(t) + y(t)^T g_u(t,u(t),\dot u(t))$$
$$- D(f_{\dot u}(t,u(t),\dot u(t)) + y(t)^T g_{\dot u}(t,u(t),\dot u(t)))\}dt + \int_I \eta^T H p(t)dt$$
$$= \int_I [\eta^T\{f_u(t,u(t),\dot u(t)) + B(t)z(t) + y(t)^T g_u(t,u(t),\dot u(t))$$
$$+ (D\eta)^T(f_{\dot u}(t,u(t),\dot u(t)) + y(t)^T g_{\dot u}(t,u(t),\dot u(t))) + \eta^T H p(t)\}]dt$$
$$- \eta^T(f_{\dot u}(t,u(t),\dot u(t)) + y(t)^T g_{\dot u}(t,u(t),\dot u(t)))|_{t=a}^{t=b},$$

by integration by parts. Using the boundary conditions (2) and (4), we have

$$\int_I [\eta^T\{f_u(t,u(t),\dot u(t)) + B(t)z(t) + y(t)^T g_u(t,u(t),\dot u(t))$$
$$+ (D\eta)^T(f_{\dot u}(t,u(t),\dot u(t)) + y(t)^T g_{\dot u}(t,u(t),\dot u(t))) + \eta^T H p(t)\}]dt = 0.$$

This, in view of second-order pseudo-invexity of $\int_I \{f(t,.,.)+(\cdot)^T B(t)z(t)+y(t)^T g(t,.,.)\}dt$ yields

$$\int_I \{f(t,x,\dot{x})+x(t)^T B(t)z(t)+y(t)^T g(t,x,\dot{x})\}dt$$
$$\geq \int_I \{f(t,u,\dot{u})+u(t)^T B(t)z(t)+y(t)^T g(t,u,\dot{u})-\frac{1}{2}p(t)^T H p(t)\}dt$$

Because of Schwartz's inequality (1) along with (5), (6) and (2), this implies

$$\int_I \{f(t,x(t),\dot{x}(t))+(x(t)^T B(t)x(t))^{1/2}\}dt$$
$$\geq \int_I \{f(t,u,\dot{u})+u(t)^T B(t)z(t)+y(t)^T g(t,u,\dot{u})-\frac{1}{2}p(t)^T H p(t)\}dt$$

yielding,

$$\inf(CP) \geq \sup(CD).$$

Theorem 2 (Strong duality). *If $\bar{x}(t) \in X$ is an optimal solution of (CP) and is also normal, then there exist piecewise smooth functions $y \to R^m$ and $z \to R^n$ such that $(\bar{x}(t),\bar{y}(t),\bar{z}(t),p(t)=0)$ is a feasible solution of (CD) and the two objective values are equal. Furthermore, if the hypotheses of Theorem 1 hold, then $(\bar{x}(t),\bar{y}(t),\bar{z}(t),p(t))$ is an optimal of (CD).*

Proof. From Proposition 1, there exist piecewise smooth functions $\bar{y}: I \to R^m$ and $\bar{z}: I \to R^n$ such that (for $t \in I$)

$$(f_x(t,\bar{x},\dot{\bar{x}})+B(t)\bar{z}(t)+\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}))-D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}})+\bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}))=0, \quad (8)$$
$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}})=0, \quad (9)$$
$$\bar{x}(t)^T B(t)\bar{z}(t)=(\bar{x}(t)^T B(t)x(t))^{1/2}, \quad (10)$$
$$\bar{z}(t)^T B(t)\bar{z}(t) \leq 1, \quad (11)$$
$$y(t) \geq 1. \quad (12)$$

Hence $(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{p}(t)=0)$ satisfies the constraints of (CD) and the objective values are equal. Furthermore, for every feasible solution $(u(t),y(t),z(t),p(t)=0)$, from the above conditions and using (9), (10) and $\bar{p}(t)=0$ we have

$$\int_I \{f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T B(t)\bar{z}(t)+\bar{y}(t)^T g(t,\bar{x}(t),\dot{\bar{x}}(t))-\frac{1}{2}\bar{p}(t)^T H \bar{p}(t)\}dt$$
$$=\int_I \{f(t,\bar{x},\dot{\bar{x}})+(\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}\}dt$$

$$\geq \int_I \{f(t,u,\dot{u}) + u(t)^T B(t)z(t) + y(t)^T g(t,u,\dot{u}) - \frac{1}{2}p(t)^T H p(t)\}dt$$

So, $(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{p}(t))$ is an optimal solution of (CD).

Theorem 3 (Converse duality). *Assume that f and g are thrice continuously differentiable and $(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{p}(t))$ be an optimal solution of (CD). Let the following conditions hold:*

(i) The Hessian matrix H is non-singular, and

(ii)

$$\begin{aligned}
&(\psi(t)^T H\psi(t))_x + 2(D(\psi(t)^T H\psi(t))_{\dot{x}} - \psi(t)^T D(H\psi(t))_{\dot{x}}) \\
&\quad - (D^2(\psi(t)^T H\psi(t))_{\ddot{x}} - \psi(t)^T D^2(H\psi(t))_{\ddot{x}}) \\
&\quad + (D^3(\psi(t)^T H\psi(t))_{\dddot{x}} - \psi(t)^T D^3(H\psi(t))_{\dddot{x}}) \\
&\quad - (D^4(\psi(t)^T H\psi(t))_{\ddddot{x}} - \psi(t)^T D^4(H\psi(t))_{\ddddot{x}}) = 0, \ t \in I \\
&\Longrightarrow \psi(t) = 0, \ t \in I
\end{aligned}$$

Then $x(t)$ is feasible for (CP), $\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0$, $t \in I$. In addition, if the hypotheses in Theorem 1 hold, then $\bar{x}(t)$ is an optimal solution.

Proof. Since $(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{p}(t))$ is an optimal solution for (CD), by Proposition 1, there exist Lagrange multiplier $\tau \in R$, and piecewise smooth $\theta : I \to R^n$, $\mu : I \to R^m$ and $\alpha : I \to R^n$ such that following conditions hold at the feasible point of (CD).

$$\begin{aligned}
&\tau\,[(f_x + B(t)z(t) + y(t)^T g_x) - D(f_{\dot{x}} + y(t)^T g_{\dot{x}}) - \frac{1}{2}(p(t)^T H p(t))_x + D(p(t)^T H p(t))_{\dot{x}} \\
&\quad - \frac{1}{2}D^2(p(t)^T H p(t))_{\ddot{x}} + \frac{1}{2}D^3(p(t)^T H p(t))_{\dddot{x}} - \frac{1}{2}D^4(p(t)^T H p(t))_{\ddddot{x}}] \\
&+\theta(t)^T [f_{xx} - (y(t)^T g_x)_x - 2D(f_{\dot{x}x} + (y(t)^T g_x)_{\dot{x}}) - D^2(f_{\dot{x}\dot{x}} + y(t)^T g_{\dot{x}\dot{x}}) \\
&\quad + D^3(f_{\dot{x}\ddot{x}} + (y(t)^T g_{\dot{x}})_{\ddot{x}} + (Hp(t))_x - D(Hp(t))_{\dot{x}} + D^2(Hp(t))_{\ddot{x}} \\
&\quad - D^3(Hp(t))_{\dddot{x}} + D^4(Hp(t))_{\ddddot{x}}]
\end{aligned} \tag{13}$$

$$\tau(f^j - \frac{1}{2}p(t)^T g^j_{xx} p(t)) + \theta(t)^T (g^j_{xx} - 2Dg^j_{x\dot{x}} + D^2 g^j_{\dot{x}\dot{x}})p(t) + \mu^j(t) = 0,$$
$$j = 1, 2, \ldots, m. \tag{14}$$

$$\tau\,\bar{x}(t)^T B(t) + \theta(t)^T B(t) - 2\alpha(t)B(t)z(t) = 0, \tag{15}$$

$$(\theta(t) - \tau p(t))H = 0, \tag{16}$$

$$f_x + B(t)z(t) + \bar{y}(t)^T g_x - D(f_{\dot{x}} + \bar{y}(t)g_{\dot{x}}) + Hp(t) = 0, \tag{17}$$

$$\alpha(t)(1 - \bar{z}(t)^T B(t)\bar{z}(t)) = 0, \tag{18}$$

$$\bar{\mu}(t)^T \bar{y}(t) = 0, \tag{19}$$

$$(\tau, \alpha(t), \mu(t)) \geq 0, \tag{20}$$

$$(\tau, \alpha(t), \mu(t), \theta(t)) \neq 0, \tag{21}$$

where $t \in I$. By the nonsingularity of H, eqref11 yields,

$$\theta(t) + \tau \bar{p}(t) = 0, \quad t \in I \tag{22}$$

If $\tau = 0$, (22) implies $\theta(t) = 0$ $t \in I$. From (14), we have $\mu(t) = 0$, $t \in I$. The relation (15) together with (18) gives $\alpha(t) = 0$. Hence $(\tau, \alpha(t), \theta(t), \mu(t)) = 0$, $t \in I$, contradicting (21). Consequently $\tau > 0$. From (22) and $\tau > 0$, (13) becomes,

$$\begin{aligned}
&[(f_x + B(t)z(t) + y(t)^T g_x) - D(f_{\dot{x}+y(t)^T g_{\dot{x}}}) - \frac{1}{2}(p(t)^T Hp(t))_x + D(p(t)^T Hp(t))_{\dot{x}} \\
&- \frac{1}{2}D^2(p(t)^T Hp(t))_{\ddot{x}} + \frac{1}{2}D^3(p(t)^T Hp(t))_{\dddot{x}} - \frac{1}{2}D^4(p(t)^T Hp(t))_{\ddddot{x}}] \\
&+ p(t)^T[(f_{xx} - (y(t)^T g_x)_x) - D(f_{\dot{x}x} + (y(t)^T g_x)_{\dot{x}}) - D(f_{\dot{x}x} + (y(t)^T g_{\dot{x}})_x) \\
&- D(D(f_{\dot{x}\dot{x}} + y(t)^T g_{\dot{x}\dot{x}})) + D^2(D(f_{\dot{x}\ddot{x}} + (y(t)^T g_{\dot{x}})_x)) \\
&+ (Hp(t))_x - D(Hp(t))_{\dot{x}} + D^2(Hp(t))_{\ddot{x}} - D^3(Hp(t))_{\dddot{x}} + D^4(Hp(t))_{\ddddot{x}}] = 0.
\end{aligned}$$

Using the expression for H, this gives

$$\begin{aligned}
&[(f_x + B(t)z(t) + y(t)^T g_x) - D(f_{\dot{x}+y(t)^T g_{\dot{x}}}) + Hp(t) - \frac{1}{2}(p(t)^T Hp(t))_x + D(p(t)^T Hp(t))_{\dot{x}} \\
&- \frac{1}{2}D^2(p(t)^T Hp(t))_{\ddot{x}} + \frac{1}{2}D^3(p(t)^T Hp(t))_{\dddot{x}} - \frac{1}{2}D^4(p(t)^T Hp(t))_{\ddddot{x}}] \\
&+ p(t)^T[(Hp(t))_x - D(Hp(t))_{\dot{x}} + D^2(Hp(t))_{\ddot{x}} - D^3(Hp(t))_{\dddot{x}} + D^4(Hp(t))_{\ddddot{x}}] = 0.
\end{aligned}$$

This, by using (17), reduces to

$$\begin{aligned}
&(p(t)^T Hp(t))_x + 2(D(p(t)^T Hp(t))_{\dot{x}} - p(t)^T D(Hp(t))_{\dot{x}}) - (D^2(p(t)^T Hp(t))_{\ddot{x}} \\
&- 2p(t)^T D^2(Hp(t))_{\ddot{x}}) + (D^3(p(t)^T Hp(t))_{\dddot{x}} - 2p(t)^T D^3(Hp(t))_{\dddot{x}}) \\
&- (D^4(p(t)^T Hp(t))_{\ddddot{x}} - 2p(t)^T D^4(Hp(t))_{\ddddot{x}}) = 0, \ t \in I,
\end{aligned}$$

which, because of the hypothesis (ii) implies $\bar{p}(t) = 0$ $t \in I$. From (14), we have

$$\tau g^j + \mu^j(t) = 0, \ t \in I, \ j = 1, 2, \dots, m \tag{23}$$

This because of $\tau > 0$, yields

$$g^j(t, \bar{x}, \dot{\bar{x}}) \leq 0, \ t \in I$$

The relation (23) along with (19) and $\tau > 0$ gives

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0, \ t \in I \tag{24}$$

Using $\theta(t) = 0$, $t \in I$ and $\tau > 0$, (15) yields

$$B(t)\bar{x}(t)^T = 2(\frac{\alpha(t)}{\tau})B(t)\bar{z}(t), \ t \in I \tag{25}$$

which is the required condition for the equality in Schwartz inequality, i.e.,

$$\bar{x}(t)^T B(t)\bar{z}(t) = (\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}(\bar{z}(t)^T B(t)\bar{z}(t))^{1/2},\ t \in I \tag{26}$$

If $\alpha(t) > 0$, $t \in I$ (18) gives $\bar{z}(t)^T B(t)\bar{z}(t) = 1$, and so (25) implies

$$\bar{x}(t)^T B(t)\bar{z}(t) = (\bar{x}(t)^T B(t)\bar{x}(t))^{1/2},\ t \in I$$

If $\alpha(t) = 0$, $t \in I$, (25) implies $B(t)\bar{x}(t)$, $t \in I$. So we still get

$$\bar{x}(t)^T B(t)\bar{z}(t) = (\bar{x}(t)^T B(t)\bar{x}(t))^{1/2},\ t \in I \tag{27}$$

Therefore from (24), (27) and $\bar{p}(t) = 0$, $t \in I$, we have

$$\int_T \{f(t,\bar{x},\dot{\bar{x}}) + (\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}\}dt =$$
$$\int_T \{f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T B(t)\bar{z}(t) + \bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{p}(t)^T H\bar{p}(t)\}dt$$

Thus, by the application of Theorem 1 the optimality of $\bar{x}(t)$ for (CP) follows.

4. Natural Boundary Values

In this section, we formulate a pair of non differentiable dual variational problems with natural boundary values rather than fixed end points.

$$(\mathbf{CP_0}) \quad \text{Minimize} \quad \int_I \{f(t,x(t),\dot{x}(t)) + (\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}\}dt$$
$$\text{subject to} \quad g(t,x,\dot{x}) \leq 0, \qquad t \in I$$

$$(\mathbf{CD_0}) \quad \text{Maximize} \quad \int_I \{f(t,x(t),\dot{x}(t)) + x(t)^T B(t)z(t) + y(t)^T g(t,x,\dot{x}) - \frac{1}{2}p(t)^T H p(t)\}dt$$
$$\text{subject to} \quad (f_x(t,x,\dot{x}) + B(t)\bar{z}(t) + \bar{y}(t)^T g_x(t,x,\dot{x}))$$
$$-D(f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x})) + H p(t) = 0,\ t \in I$$
$$z(t)^T B(t)z(t) \leq 1, \qquad t \in I$$
$$y(t) \geq 0, \qquad t \in I$$
$$f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x})|_{t=a} = 0$$
$$f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x})|_{t=b} = 0$$

We shall not repeat the proofs of Theorems 1-3, as these follow on the lines of the analysis of the preceding section with slight modifications.

5. Non-Differentiable Nonlinear Programming Problems

If all functions in the problems (CP_0) and (CD_0) are independent of t and $b-a=1$, then these problems will reduce to following nondifferentiable dual variational problems, treated by Zhang and Mond [10].

$$
\begin{aligned}
(\mathbf{NP})\quad & \text{Minimize} && f(x)+(x^TBx)^{1/2}\\
& \text{subject to} && g(x)\le 0,
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ND})\quad & \text{Maximize} && f(x)+x^TBz+y^Tg(x)-\frac{1}{2}p^T\nabla^2(f(x)+y^Tg(x))p\\
& \text{subject to} && \nabla(f(x)+x^TBz+y^Tg(x))+\nabla^2(f(x)+y^Tg(x))p=0,\\
& && z^TBz\le 1,\quad y\ge 0
\end{aligned}
$$

where

$$\nabla(f(x)+x^TBz+y^Tg(x))=f_x(x)+Bz+y^Tg_x(x)$$

and

$$\nabla^2(f(x)+y^Tg(x))=f_{xx}(x)+(y^Tg_x(x))_{\dot{x}}$$

6. Conclusion

In this exposition, we have discussed a class of nondifferentiable continuous programming problems treated in [2] and formulated Wolfe type Second-order dual variation problem which is analogous to the second-order dual problem constructed by Zhang and Mond [10] for a nondifferentiable nonlinear programming problem. Under second-order pseudoinvexity we established weak, strong and converse duality theorems. When functions, occurring in the formulations of the problems, do not depend explicitly on t, our results reduce to those of [10]. Thus our results become dynamic generalizations of the results in [10]. The problems of this research can be revisited in multiobjective setting.

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