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Inequalities Involving Certain Integral Operator

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Abstract. The object of this paper is to give several strict inequalities associated with the operator $I^{\lambda}_{p,n}(a,b;c)$ $(a,b \in R \setminus \mathbb{Z}_0^-,\ p,n \in \mathbb{N} = \{1,2,\ldots\},\ \lambda > -p)$ defined by X.-L. Fu and M.-S. Liu, Some subclasses of analytic functions involving the generalized Noor integral operator [see 3].

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1. Introduction

Let $\mathscr{A}_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p,n \in \mathbb{N} = \{1,2,\ldots.\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$. For functions f given by (1) and $g \in \mathscr{A}_n(p)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (z \in U) \tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z)\ .$$

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For real or complex numbers a, b, c other than $0, -1, -2, \ldots$, the Gaussian hypergeometric series is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \ , \tag{3} $$

where

$$ (d)_k = \begin{cases} 1 & (k=0; d \in \mathbb{C} \setminus \{0\}), \\ d(d+1)\ldots(d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}), \end{cases} $$

we note that the series (3) converges absolutely for all $z \in U$ so that it represents an analytic function in U (see [8]).

With the aid of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$, let us consider a family of linear operators $I^{\lambda}_{p,n} : \mathscr{A}_n(p) \to \mathscr{A}_n(p)$ as follows:

$$ \begin{aligned} I^{\lambda}_{p,n}(a,b;c)f(z) &= z^p + \sum_{k=n}^{\infty} \frac{(c)_k (\lambda+p)_k}{(a)_k (b)_k} a_{k+p} z^{k+p} \\ &= z^p \ {}_2F_1(c,1;a;z) * z^p \ {}_2F_1(\lambda+p,1;b;z) \\ & \quad (a,b,c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; z \in U) \ . \end{aligned} \tag{4} $$

The operator $I^{\lambda}_{p,n}$ was introduced and studied by Fu and Liu [3].

We note that:

(i) $I^{n}_{1,1}(a,n+1;a)f(z) = I_n f(z)$ $(n > -1)$, where I_n is the Noor integral operator of $n-th$ order [see 6];

(ii) $I^{\lambda}_{1,1}(\mu+2,1;1)f(z) = I_{\mu,\lambda}f(z)$ $(\mu > -2, \lambda > -1)$, where $I_{\mu,\lambda}$ is the Choi–Saigo–Srivastava operator [see 2];

(iii) $I^{\lambda}_{p,1}(\lambda+p+1,b;b)f(z) = F_{\lambda,p}(f)(z)$ $(\lambda > -p)$, where $F_{\lambda,p}(f)(z)$ is the generalized Bernardi–Libera–Livingston operator [see 2];

(iv) $I^{\lambda}_{p,1}(a,1;c)f(z) = I^{\lambda}_{p}(a,c)f(z)$ $(a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p)$, where $I^{\lambda}_{p}(a,c)$ is the Cho–Kwon–Srivastava operator [see 1];

(v) $I^{1}_{p,1}(n+p,c;c)f(z) = I_{n,p}f(z)$ $(n > -p)$, where $I_{n,p}$ is the Noor integral operator of $(n+p-1)-th$ order (see Liu and Noor [4] and Patel and Cho [7]).

Also it is easily to show that [see 3]:

$$ I^{\lambda}_{p,n}(a,\lambda+p;a)f(z) = I^{1}_{p,n}(p+1,b;b)f(z) = f(z) \text{ and } I^{1}_{p,n}(a,p;a)f(z) = \frac{zf'(z)}{p}, $$

$$ z\left(I^{\lambda}_{p,n}(a,b;c)f(z)\right)' = (\lambda+p)I^{\lambda+1}_{p,n}(a,b;c)f(z) - \lambda I^{\lambda}_{p,n}(a,b;c)f(z) \tag{5} $$

and

$$ z\left(I^{\lambda}_{p,n}(a+1,b;c)f(z)\right)' = aI^{\lambda}_{p,n}(a,b;c)f(z) - (a-p)I^{\lambda}_{p,n}(a+1,b;c)f(z). \tag{6} $$

By using the operator $I^{\lambda}_{p,n}(a,b;c)$, we define the following classes of functions:

Definition 1. *Let Φ be the set of complex-valued functions $\varphi(r,s,t)$,*

$$\varphi(r,s,t):\mathbb{C}^3\to\mathbb{C}\quad (\mathbb{C} \text{ is the complex plane})$$

such that

1. $\varphi(r,s,t)$ *is continuous in a domain* $D\subset\mathbb{C}^3$*;*
2. $(0,0,0)\in D$ *and* $\left|\varphi(0,0,0)\right|<1$*;*
3. $\left|\varphi\left(e^{i\theta},f(\lambda,\zeta,\theta,p),g(\lambda,\zeta,\theta,p,M)\right)\right|>1$

whenever

$$\left(e^{i\theta},f(\lambda,\zeta,\theta,p),g(\lambda,\zeta,\theta,p,M)\right)\in D,$$

with $\mathfrak{Re}\left\{e^{-i\theta}M\right\}\geq\zeta(\zeta-1)$*, for all* $\theta\in\mathbb{R}$*, and for all* $\zeta\geq p\geq1$*, where*

$$f(\lambda,\zeta,\theta,p)=\left(\frac{\zeta+\lambda}{\lambda+p}\right)e^{i\theta}$$

and

$$g(\lambda,\zeta,\theta,p,M)=\frac{(\lambda+1)(\lambda+2\zeta)e^{i\theta}+M}{(\lambda+p)(\lambda+p+1)}.$$

Definition 2. *Let $\mathscr{H}$ be the set of complex-valued functions $h(r,s,t)$;*

$$h(r,s,t):\mathbb{C}^3\to\mathbb{C}$$

such that

1. $h(r,s,t)$ *is continuous in a domain* $D\subset\mathbb{C}^3$*;*
2. $(1,1,1)\in D$ *and* $\left|g(1,1,1)\right|<J\ (J>1)$*;*
3. $\left|h\left(Je^{i\theta},f(\lambda,\zeta,\theta,p,J),g(\lambda,\zeta,\theta,p,J)\right)\right|\geq J$

whenever

$$\left(Je^{i\theta},f(\lambda,\zeta,\theta,p,J),g(\lambda,\zeta,\theta,p,J,L)\right)\in D,$$

with $\mathfrak{Re}\{L\}\geq\zeta(\zeta-1)$ *for all* $\theta\in\mathbb{R}$ *and for all* $\zeta\geq\dfrac{J-1}{J+1}$*, where*

$$f(\lambda,\zeta,\theta,p,J)=\frac{1+\zeta+(\lambda+p-1)Je^{i\theta}}{(\lambda+p)}$$

and

$$g(\lambda,\zeta,\theta,p,J,L)=\frac{1}{(\lambda+p+1)}\left\{2+\zeta+(\lambda+p-1)Je^{i\theta}+\frac{\zeta-\zeta^2+(\lambda+p+1)\zeta Je^{i\theta}+L}{\zeta+(\lambda+p-1)Je^{i\theta}}\right\}.$$

2. Main Results

We recall the following lemma due to Miller and Mocanu [5].

Lemma 1. *Let* $w(z) = a + w_\nu z^\nu + \ldots$ *be regular in* U *with* $\nu \in \mathbb{N}$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $\left|w(z_0)\right| = \max_{|z| \le |z_0|} |w(z)|$, *then*

$$z_0 w'(z_0) = \zeta w(z_0) \tag{7}$$

and

$$\mathfrak{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \ge \zeta, \tag{8}$$

where ζ *is a real number and*

$$\zeta \ge \nu \frac{\left|w(z_0) - a\right|^2}{\left|w(z_0)\right|^2 - |a|^2} \ge \nu \frac{\left|w(z_0)\right| - |a|}{\left|w(z_0)\right| + |a|}. \tag{9}$$

Note that if $\nu = 0$ then the condition (9) becomes $\zeta \ge \nu \ge 1$.

Theorem 1. *Let* $\varphi(r,s,t) \in \Phi$ *and let* f *in the class* $\mathscr{A}_n(p)$ *satisfy*

$$\left(I_{p,n}^{\lambda}(a,b;c)f(z), I_{p,n}^{\lambda+1}(a,b;c)f(z), I_{p,n}^{\lambda+2}(a,b;c)f(z)\right) \in D \subset \mathbb{C}^3 \tag{10}$$

and

$$\left|\varphi\left(I_{p,n}^{\lambda}(a,b;c)f(z), I_{p,n}^{\lambda+1}(a,b;c)f(z), I_{p,n}^{\lambda+2}(a,b;c)f(z)\right)\right| < 1 \tag{11}$$

for $a,b,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$ *and* $z \in U$. *Then we have*

$$\left|I_{p,n}^{\lambda}(a,b;c)f(z)\right| < 1 \quad (z \in U)\,. \tag{12}$$

Proof. We define the function w by

$$w(z) = I_{p,n}^{\lambda}(a,b;c)f(z) \quad \left(a,b,c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; p \in \mathbb{N}\right) \tag{13}$$

for f belonging to the class $\mathscr{A}_n(p)$. Then, it follows that $w \in \mathscr{A}_n(p)$ and $w(z) \ne 0$ for $z \in U \setminus \{0\}$. With the aid of (5), we have

$$I_{p,n}^{\lambda+1}(a,b;c)f(z) = \frac{1}{(\lambda+p)}\left[zw'(z) + \lambda w(z)\right] \tag{14}$$

and

$$I_{p,n}^{\lambda+2}(a,b;c)f(z) = \frac{z^2 w'(z) + 2(\lambda+1)zw'(z) + \lambda(\lambda+1)w(z)}{(\lambda+p)(\lambda+p+1)}. \tag{15}$$

Suppose that $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1; \theta \in \mathbb{R})$ and

$$w(z_0) = \max_{|z| \le |z_0|} |w(z)| = 1. \tag{16}$$

Then, letting $w(z_0) = e^{i\theta}$ and using (7) of Lemma 1, we obtain

$$I^{\lambda}_{p,n}(a,b;c)f(z_0) = e^{i\theta}, \tag{17}$$

$$I^{\lambda+1}_{p,n}(a,b;c)f(z_0) = \frac{1}{(\lambda+p)}\left[z_0 w'(z_0) + \lambda w(z_0)\right] = \frac{(\zeta+\lambda)}{(\lambda+p)}e^{i\theta} \tag{18}$$

and

$$\begin{aligned} I^{\lambda+2}_{p,n}(a,b;c)f(z_0) &= \frac{1}{(\lambda+p)(\lambda+p+1)}\left[(\lambda+1)(\lambda+2\zeta)e^{i\theta} + z_0^2 w'(z_0)\right] \\ &= \frac{(\lambda+1)(\lambda+2\zeta)e^{i\theta} + M}{(\lambda+p)(\lambda+p+1)}, \end{aligned} \tag{19}$$

where $M = z_0^2 w''(z_0)$ and $\zeta \geq p \geq 1$. Further, an application of (8) in Lemma 1, gives

$$\mathfrak{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \mathfrak{Re}\left\{\frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}}\right\} \geq \zeta - 1, \tag{20}$$

or

$$\mathfrak{Re}\left\{e^{-i\theta}M\right\} \geq \zeta(\zeta-1) \quad (\theta \in \mathbb{R}; \zeta \geq 1). \tag{21}$$

Since $\varphi(r,s,t) \in \Phi$, we also have

$$\begin{aligned} &\left|\varphi\left(I^{\lambda}_{p,n}(a,b;c)f(z), I^{\lambda+1}_{p,n}(a,b;c)f(z), I^{\lambda+2}_{p,n}(a,b;c)f(z)\right)\right| \\ = &\left|\varphi\left(e^{i\theta}, \frac{\zeta+\lambda}{\lambda+p}e^{i\theta}, \frac{1}{(\lambda+p)(\lambda+p+1)}\left[(\lambda+1)(\lambda+2\zeta)e^{i\theta} + M\right]\right)\right| > 1 \end{aligned} \tag{22}$$

which contradicts the condition (11) of Theorem 1. Therefore, we conclude that

$$|w(z)| = \left|I^{\lambda}_{p,n}(a,b;c)f(z)\right| < 1 \quad (z \in U). \tag{23}$$

This completes the proof of Theorem 1.

Corollary 1. *Let $\varphi_1(r,s,t) = s$ and let $f \in \mathscr{A}_n(p)$ satisfy the conditions in Theorem 1 for $a,b,c \in \mathbb{R}\setminus\mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$ and $z \in U$. Then*

$$\left|I^{\lambda+i}_{p,n}(a,b;c)f(z)\right| < 1 \ (i = 0,1,2,\ldots; a,b,c \in \mathbb{R}\setminus\mathbb{Z}_0^-; \lambda > -p; p \in \mathbb{N}; z \in U).$$

Note that $\varphi_1(r,s,t) = s$ is in Φ, with the aid of Theorem 1, we have

$$\left|I^{\lambda}_{p,n}(a,b;c)f(z)\right| < 1 \Rightarrow \left|I^{\lambda+1}_{p,n}(a,b;c)f(z)\right| < 1$$

$$\Rightarrow \left|I^{\lambda+i}_{p,n}(a,b;c)f(z)\right| < 1 \ (i = 0,1,2,\ldots; a,b,c \in \mathbb{R}\setminus\mathbb{Z}_0^-; \lambda > -p; p \in \mathbb{N}; z \in U).$$

Theorem 2. *Let $h(r,s,t)\in\mathscr{H}$, and let $f\in\mathscr{A}_n(p)$ satisfy*

$$\left(\frac{I_{p,n}^{\lambda}(a,b;c)f(z)}{I_{p,n}^{\lambda-1}(a,b;c)f(z)},\frac{I_{p,n}^{\lambda+1}(a,b;c)f(z)}{I_{p,n}^{\lambda}(a,b;c)f(z)},\frac{I_{p,n}^{\lambda+2}(a,b;c)f(z)}{I_{p,n}^{\lambda+1}(a,b;c)f(z)}\right)\in D\subset\mathbb{C}^3 \tag{24}$$

and

$$\left|h\left(\frac{I_{p,n}^{\lambda}(a,b;c)f(z)}{I_{p,n}^{\lambda-1}(a,b;c)f(z)},\frac{I_{p,n}^{\lambda+1}(a,b;c)f(z)}{I_{p,n}^{\lambda}(a,b;c)f(z)},\frac{I_{p,n}^{\lambda+2}(a,b;c)f(z)}{I_{p,n}^{\lambda+1}(a,b;c)f(z)}\right)\right|<J \tag{25}$$

for some a, b, c, λ, p, n, J $\left(a,b,c\in\mathbb{R}\setminus\mathbb{Z}_0^-;\lambda>1;p,n\in\mathbb{N};J>1\right)$ and for all $z\in U$. Then we have

$$\left|\frac{I_{p,n}^{\lambda}(a,b;c)f(z)}{I_{p,n}^{\lambda-1}(a,b;c)f(z)}\right|<J\quad(z\in U). \tag{26}$$

Proof. We define the function w by

$$w(z)=\frac{I_{p,n}^{\lambda}(a,b;c)f(z)}{I_{p,n}^{\lambda-1}(a,b;c)f(z)} \tag{27}$$

for f belonging to the class $\mathscr{A}_n(p)$. Then, it follows that w is either analytic or meromorphic in U, $w(0)=1$, and $w(z)\neq 1$. With the aid of the identity (5), we have

$$\frac{I_{p,n}^{\lambda+1}(a,b;c)f(z)}{I_{p,n}^{\lambda}(a,b;c)f(z)}=\frac{1}{(\lambda+p)}\left[1+(\lambda+p-1)w(z)+\frac{zw'(z)}{w(z)}\right] \tag{28}$$

and

$$\begin{aligned}\frac{I_{p,n}^{\lambda+2}(a,b;c)f(z)}{I_{p,n}^{\lambda+1}(a,b;c)f(z)} &= \frac{1}{(\lambda+p+1)}\left\{2+(\lambda+p-1)w(z)+\frac{zw'(z)}{w(z)}\right.\\ &\left.+\frac{(\lambda+p-1)zw'(z)+\frac{zw'z)}{w(z)}+\frac{z^2w'(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{1+(\lambda+p-1)w(z)+\frac{zw'(z)}{w(z)}}\right\}.\end{aligned} \tag{29}$$

Suppose that $z_0=r_0e^{i\theta}(0<r_0<1;\theta\in\mathbb{R})$ and $\left|w(z_0)\right|=\max_{|z|\leq|z_0|}|w(z)|=J$. Letting $w(z_0)=Je^{i\theta}$ and using Lemma 1 with $a=\nu=1$, we see that

$$\frac{I_{p,n}^{\lambda+1}(a,b;c)f(z_0)}{I_{p,n}^{\lambda}(a,b;c)f(z_0)}=\frac{1}{(\lambda+p)}\left[1+\zeta+(\lambda+p-1)Je^{i\theta}\right] \tag{30}$$

and

$$\frac{I_{p,n}^{\lambda+2}(a,b;c)f(z)}{I_{p,n}^{\lambda+1}(a,b;c)f(z)}=\frac{1}{(\lambda+p+1)}\left\{2+\zeta+(\lambda+p-1)Je^{i\theta}+\frac{\zeta-\zeta^2+(\lambda+p-1)\zeta Je^{i\theta}+L}{\zeta+(\lambda+p-1)Je^{i\theta}}\right\}, \tag{31}$$

where $L = \dfrac{z_0^2 w''(z_0)}{w'(z_0)}$ and $\zeta \geq \dfrac{J-1}{J+1}$. Further, an application of (16) in Lemma 1 gives

$$\mathfrak{Re}\{L\} \geq \zeta(\zeta - 1).$$

Since $h(r, s, t) \in \mathscr{H}$, we have

$$\begin{aligned} & \left| h\left(\frac{I_{p,n}^{\lambda}(a,b;c)f(z_0)}{I_{p,n}^{\lambda-1}(a,b;c)f(z_0)}, \frac{I_{p,n}^{\lambda+1}(a,b;c)f(z_0)}{I_{p,n}^{\lambda}(a,b;c)f(z_0)}, \frac{I_{p,n}^{\lambda+2}(a,b;c)f(z_0)}{I_{p,n}^{\lambda+1}(a,b;c)f(z_0)} \right) \right| \\ = & \left| h\left(Je^{i\theta}, \frac{1+\zeta+(\lambda+p-1)Je^{i\theta}}{(\lambda+p} + \frac{1}{(\lambda+p+1)} \left\{ 2+\zeta+(\lambda+p-1)Je^{i\theta} \right.\right.\right. \\ & + \left.\left.\left. \frac{\zeta-\zeta^2+(\lambda+p-1)\zeta Je^{i\theta}+L}{\zeta+(\lambda+p-1)Je^{i\theta}} \right\} \right) \right| \geq J, \end{aligned} \tag{32}$$

which contradicts condition (25) of Theorem 2. Therefore, we conclude that

$$|w(z)| = \left| \frac{I_{p,n}^{\lambda}(a,b;c)f(z)}{I_{p,n}^{\lambda-1}(a,b;c)f(z)} \right| < J \tag{33}$$

for some $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 1$, p, $n \in \mathbb{N}$, $J > 1$ and for all $z \in U$. This completes the proof of Theorem 2.

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