EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 14-19
ISSN 1307-5543 – www.ejpam.com

$(\alpha,\beta,\delta)-$Neighborhood for Certain Analytic Functions with Negative Coefficients

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Abstract. In this paper, we introduce $(\alpha,\beta,\delta)-$neighborhoods of analytic functions with negative coefficients. Furthermore, we obtain some interesting results for functions belonging to this neighborhoods.

2000 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Analytic functions, Neighborhood

1. Introduction and definitions

Let $\mathscr{T}$ denote the class of functions of the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0). \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$.For a function $f(z) \in \mathscr{T}$, we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$\begin{aligned} D^k f(z) &= D(D^{k-1} f(z)) \\ &= z - \sum_{n=2}^{\infty} n^k a_n z^n \qquad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \end{aligned}$$

The differential operator D^k was introduced by Sălăgean [12].

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Following a recent investigation by Frasin and Darus [6] [see also 1], if $f(z) \in \mathscr{T}$ and $\mu \geq 0$, then we define the (k,μ)-neighborhood for the function $f(z)$ by

$$\mathscr{N}_{\mu}^{k}(f) = \{g \in \mathscr{T}:\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{k+1} \left|a_n - b_n\right| \leq \mu\}. \tag{2}$$

In particular, for the identity function $e(z) = z$, we immediately have

$$\mathscr{N}_{\mu}^{k}(e) = \{g \in \mathscr{T}:\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{k+1} \left|b_n\right| \leq \mu\}, \tag{3}$$

We observe that $\mathscr{N}_{\mu}^{0}(f) \equiv \mathscr{N}_{\mu}(f)$ and $\mathscr{N}_{\mu}^{1}(f) \equiv \mathscr{M}_{\mu}(f)$, where $\mathscr{N}_{\mu}^{k}(f)$ and $\mathscr{M}_{\mu}(f)$ denote, respectively, the μ-neighborhoods of f as defined by Ruscheweyh [11] and Silverman [13]. For further details about the neighborhood of analytic functions see (as examples) the papers in [2, 3, 4, 7, 5, 8, 9].

Very recently, Orhan *et al.* [10], introduced new definition of (α,δ)-neighborhood for analytic function $f(z)$ in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{4}$$

In this paper, we introduce the following new definition of (α,β,δ)-neighborhood for a function given by 1.

Definition 1. *A function $f(z) \in \mathscr{T}$ is said to be (α,β,δ)-neighborhood for $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathscr{T}$ if it satisfies*

$$\left|e^{i\alpha}(D^k f(z))' - e^{i\beta}(D^k g(z))'\right| < \delta \qquad (z \in \mathscr{U}) \tag{5}$$

for some $-\pi \leq \alpha, \beta \leq \pi$ and $\delta > \sqrt{2(1-\cos(\alpha-\beta))}$.

We denote this neighborhood by $(\alpha,\beta,\delta) - \mathscr{N}(g)$.

Now we show some results for functions belonging to $(\alpha,\beta,\delta) - \mathscr{N}(g)$.

2. Main results

In our first theorem, we introduce a sufficient condition to be in $(\alpha,\beta,\delta) - \mathscr{N}(g)$.

Theorem 1. *If $f(z) \in \mathscr{T}$ satisfies*

$$\sum_{n=2}^{\infty} n^{k+1} \left|e^{i\alpha} a_n - e^{i\beta} b_n\right| \leq \delta - \sqrt{2(1-\cos(\alpha-\beta))} \tag{6}$$

for some $-\pi \leq \alpha, \beta \leq \pi$ and $\delta > \sqrt{2(1-\cos(\alpha-\beta))}$ then $f(z) \in (\alpha,\beta,\delta) - \mathscr{N}(g)$.

Proof. We observe that

$$\begin{aligned}\left|e^{i\alpha}(D^k f(z))' - e^{i\beta}(D^k g(z))'\right| &= \left|e^{i\alpha} - e^{i\beta} - \sum_{n=2}^{\infty} n^{k+1}(e^{i\alpha}a_n - e^{i\beta}b_n)z^{n-1}\right| \\ &\le \left|e^{i\alpha} - e^{i\beta}\right| + \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| |z|^{n-1} \\ &\le \sqrt{2(1-\cos(\alpha-\beta))} + \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right|.\end{aligned}$$

If

$$\sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| \le \delta - \sqrt{2(1-\cos(\alpha-\beta))},$$

then we have $\left|e^{i\alpha}(D^k f(z))' - e^{i\beta}(D^k g(z))'\right| < \delta$ $(z \in \mathscr{U})$. This shows that $f(z) \in (\alpha, \beta, \delta) - \mathscr{N}(g)$.

Corollary 1. *Let $f(z) \in \mathscr{T}$. Then for $0 < \mu \le \delta$, we have $\mathscr{N}_\mu^k(g) \subseteq (\alpha, \alpha, \delta) - \mathscr{N}_\delta^k(g)$.*

Proof. Assuming that $f(z) \in \mathscr{N}_\mu^k(g)$. We find from the definition (2) that

$$\sum_{n=2}^{\infty} n^{k+1}\left|a_n - b_n\right| \le \mu.$$

Now

$$\begin{aligned}\sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\alpha}b_n\right| &= \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}\right|\left|a_n - b_n\right| \\ &= \sum_{n=2}^{\infty} n^{k+1}\left|a_n - b_n\right| \\ &\le \delta.\end{aligned}$$

Thus by Theorem 1, we have $f(z) \in (\alpha, \alpha, \delta) - \mathscr{N}(g)$.

Corollary 2. *If $f(z) \in \mathscr{T}$ satisfies*

$$\sum_{n=2}^{\infty} n^{k+1}\left|\left|a_n\right| - \left|b_n\right|\right| \le \delta - \sqrt{2(1-\cos(\alpha-\beta))} \tag{7}$$

for some $-\pi \le \alpha, \beta \le \pi$, $\delta > \sqrt{2(1-\cos(\alpha-\beta))}$ and $\arg a_n - \arg b_n = \beta - \alpha$ $(n = 2, 3, 4, \ldots)$, then $f(z) \in (\alpha, \beta, \delta) - \mathscr{N}(g)$.

Proof. Let $\arg a_n - \arg b_n = \beta - \alpha$ and $\arg a_n = \theta_n$. Then $\arg b_n = \theta_n + \alpha - \beta$. Therefore,

$$e^{i\alpha}a_n - e^{i\beta}b_n = \left|a_n\right| e^{i(\alpha+\theta_n)} - \left|b_n\right| e^{i(\alpha+\theta_n)},$$

which implies

$$\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| = \left|\left|a_n\right| - \left|b_n\right|\right|. \tag{8}$$

From the hypotheses (7) and (8), we get (6). Thus by Theorem 1, it follows that $f(z) \in (\alpha,\beta,\delta) - \mathcal{N}(g)$.

Furthermore, from Theorem 1, we easily get

Corollary 3. *If $f(z) \in \mathcal{T}$ satisfies*

$$\sum_{n=2}^{\infty} n^{k+1}(\left|a_n\right| + \left|b_n\right|) \leq \delta - \sqrt{2(1-\cos(\alpha-\beta))} \tag{9}$$

for some $-\pi \leq \alpha, \beta \leq \pi$ and $\delta > \sqrt{2(1-\cos(\alpha-\beta)}$ then $f(z) \in (\alpha,\beta,\delta) - \mathcal{N}(g)$.

Next, we prove

Theorem 2. *Iff $(z) \in (\alpha,\beta,\delta) - \mathcal{N}(g)$ and $\arg(e^{i\alpha}a_n - e^{i\beta}b_n) = (n-1)\varphi$ $(n = 2,3,4,\ldots)$, then*

$$\sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| > \delta + \cos\alpha - \cos\beta. \tag{10}$$

Proof. Let $f(z) \in (\alpha,\beta,\delta) - \mathcal{N}(g)$ and $\arg z = -\varphi$. Then for all $z \in \mathcal{U}$, we have

$$\begin{aligned}
\left|e^{i\alpha}(D^k f(z))' - e^{i\beta}(D^k g(z))'\right| &= \left|(e^{i\alpha} - e^{i\beta}) - \sum_{n=2}^{\infty} n^{k+1}(e^{i\alpha}a_n - e^{i\beta}b_n)z^{n-1}\right| \\
&= \left|(e^{i\alpha} - e^{i\beta}) - \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| e^{i(n-1)\varphi}\,|z|^{n-1}\,e^{-i(n-1)\varphi}\right| \\
&= \left|(e^{i\alpha} - e^{i\beta}) - \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| |z|^{n-1}\right| \\
&= \Bigg([(\cos\alpha - \cos\beta) - \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| |z|^{n-1}]^2 + \\
&\quad (\sin\alpha - \sin\beta)^2\Bigg)^{1/2} \\
&< \delta
\end{aligned}$$

for $z \in \mathcal{U}$. This implies that

$$(\cos\alpha - \cos\beta) - \sum_{n=2}^{\infty} n^{k+1}\left|e^{i\alpha}a_n - e^{i\beta}b_n\right| |z|^{n-1} < \delta$$

for $z \in \mathcal{U}$. Letting $|z| \to 1^-$, we have

$$\sum_{n=2}^{\infty} n^{k+1} \left| e^{i\alpha} a_n - e^{i\beta} b_n \right| > \delta + \cos\alpha - \cos\beta.$$

Finally, we prove

Theorem 3. *If $f(z) \in \mathcal{T}$ satisfies*

$$\sum_{n=2}^{\infty} n^{k+1} \left| e^{i\alpha} a_n - e^{i\beta} b_n \right| < \mu - \sqrt{2(1 - \cos(\alpha - \beta))} \tag{11}$$

for some $-\pi \leq \alpha, \beta \leq \pi$ and $\mu > \sqrt{2(1 - \cos(\alpha - \beta))}$, then

$$\operatorname{Re}\left(\frac{e^{i\alpha}(D^k f(z))'}{e^{i\beta}(D^k g(z))'} \right) > 0 \tag{12}$$

where $g(z) \in \mathcal{N}^k_{1-\mu}(e)$.

Proof. Note that

$$\begin{aligned} \left| \frac{e^{i\alpha}(D^k f(z))'}{e^{i\beta}(D^k g(z))'} - 1 \right| &\leq \frac{\sqrt{2(1 - \cos(\alpha - \beta))} + \sum_{n=2}^{\infty} n^{k+1} \left| e^{i\alpha} a_n - e^{i\beta} b_n \right|}{1 - \sum_{n=2}^{\infty} n^{k+1} b_n} \\ &\leq \frac{\sqrt{2(1 - \cos(\alpha - \beta))} + \sum_{n=2}^{\infty} n^{k+1} \left| e^{i\alpha} a_n - e^{i\beta} b_n \right|}{\mu}. \end{aligned}$$

Hence by the condition (11), we have

$$\left| \frac{e^{i\alpha}(D^k f(z))'}{e^{i\beta}(D^k g(z))'} - 1 \right| < 1.$$

This evidently proves Theorem 3.

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