



Honorary Invited Paper

Theory of Fractional Differential Equations in a Banach Space

V. Lakshmikantham^{1,*}, J. Vasundhara Devi²

¹ Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901 USA

² GVP Institute for Advanced Studies, Visakhapatnam, India

Abstract. In this paper, the basic theory of fractional differential equations in a Banach Space is discussed including flow invariance and theory of inequalities in cones.

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1. Introduction

The concept of fractional derivative and the corresponding fractional Volterra integral are non-local [14] compared to the usual standard notions. As a result, the change of initial value in time from zero to any $t_0 > 0$ changes the fractional dynamic systems which needs to be considered depending on the requirement of the properties of solutions of such equations. Recently [9]- [12], we have investigated the fundamental theory of the initial value problem for fractional differential equations involving Riemann-Liouville differential operators of arbitrary order $0 < q < 1$, because such dynamic systems are important in modeling a variety of real world problems [1]- [6], [13]- [15]. We followed the classical approach of the theory of differential equations of integer order in order to compare and contrast the differences and intricacies that might result in the development [7].

In this paper, we discuss the theory of fractional differential equations in a Banach Space parallel to [8] utilizing the initial time $t_0 \geq 0$. We prove general existence and uniqueness, continuous dependence, fractional differential inequalities in cones and flow invariance. Although the developed theory includes as a special case, fractional differential systems in R^n , it does not cover the case when each component or a group of components of the vector in R^n , has a different arbitrary order. This case needs a different consideration such as employed in the study of large scale systems, which will be taken up later.

*Corresponding author. *Email addresses:* lakshmik@fit.edu (V.Lakshmikantham), jvdevi@gmail.com (J.Devi)

2. Uniqueness and Continuous Dependence

Let E be a real Banach Space with the norm $|\cdot|$. Let $0 < q < 1$ and $p = 1 - q$. We let $C_p([t_0, t_0 + a], E) = [u : C([t_0, t_0 + a], E) \text{ and } (t - t_0)^{1-q}u(t) \in C([t_0, t_0 + a], E)]$. Let us consider the initial value problem (IVP) for fractional differential equations in E given by

$$D^q x = f(t, x), \quad x(t)(t - t_0)^{1-q}|_{t=t_0} = x^0, \quad (2.1)$$

where $f \in C[R_0, E]$ with $R_0 = [(t, x) : t_0 \leq t \leq t_0 + a \text{ and } B(x_0, b)]$, $D^q x$ is the fractional derivative of x of order $0 < q < 1$ and $x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$. Since f is assumed continuous, the IVP (2.1) is equivalent to the following fractional Volterra integral

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s)) ds, \quad t_0 \leq t \leq t_0 + a; \quad (2.2)$$

that is, every solution of (2.2) is a solution of (2.1) and vice versa. Here and in what follows Γ is the Gamma function.

Remark 2.1: When we change the initial condition in time from zero to $t_0 \geq 0$, we need to change the same in (2.2). In some earlier papers [9]- [11], we did use $t_0 = 0$ for convenience. However, in dealing with, for example, the continuous dependence of solutions with respect to initial values (t_0, x_0) , as well as discussing the qualitative theory including Lyapunov stability theory, we need to employ nonzero initial time $t_0 > 0$ instead of only $t_0 = 0$. The difference in such a change appears only when we have to utilize the Gamma and Beta functions, in which case, the limits of the transformed integral is required to be zero to one. A suitable transformation does the trick, that is, setting $s = t_0 + (t - t_0)\sigma$.

We need the following known results [10], [11] before we proceed further.

Lemma 2.1: Let $m : R_+ \rightarrow R$ be locally Hölder continuous such that for any $t_1 \in [t_0, \infty)$, one has

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \text{ or } m(t) \geq 0 \text{ for } t_0 \leq t \leq t_1.$$

Then it follows that $D^q m(t_1) \geq 0$ or $D^q m(t_1) \leq 0$ respectively.

Lemma 2.2: Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0, t_0 + T]$, for each $\epsilon > 0$ where $D^q x_\epsilon(t) = f(t, x_\epsilon(t))$, $x_\epsilon^0 = x_\epsilon(t)(t - t_0)^{1-q}|_{t=t_0}$ and $|f(t, x_\epsilon(t))| \leq M$ for $t_0 \leq t \leq t_0 + T$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $t_0 \leq t \leq t_0 + T$.

Lemma 2.3: Assume that $g \in C[\Omega, R]$ where Ω is an open (t, u) -set in R^2 and $(t_0, u_0) \in \Omega$. Suppose that $[t_0, t_0 + a]$ is the largest interval of existence of the maximal solution $r(t)$ of $D^q u = g(t, u)$, $u(t)(t - t_0)^{1-q}|_{t=t_0} = u^0$. Let $[t_0, t_1]$ be a compact interval of $[t_0, t_0 + a]$. Then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, \epsilon)$ of $D^q u = g(t, u) + \epsilon$, $u(t)(t - t_0)^{1-q} + \epsilon|_{t=t_0} = u^0 + \epsilon$ exists on $[t_0, t_1]$ and $\lim_{\epsilon \rightarrow \infty} r(t, \epsilon) = r(t)$ uniformly on $[t_0, t_1]$.

Lemma 2.4: Assume that $m : [t_0, t_0 + a] \rightarrow R_+$ be locally Hölder continuous, $g \in C([t_0, t_0 + a] \times R_+, R_+)$ and for $t_0 \leq t \leq t_0 + a$,

$$D^q m(t) \leq g(t, m(t)), \quad m^0 = m(t)(t - t_0)^{1-q}|_{t=t_0}.$$

Let $r(t)$ be the maximal solution of

$$D^q u = g(t, u), \quad u(t)(t - t_0)^{1-q}|_{t=t_0} = u^0 \geq 0,$$

existing on $[t_0, t_0 + a]$. Then we have

$$m(t) \leq r(t), \quad t_0 \leq t \leq t_0 + a.$$

We are now in a position to prove the following general existence and uniqueness result.

Theorem 2.1: Assume that

- (a) $f \in C(R_0, E)$ and $|f(t, x)| \leq M_0$ on R_0 ;
- (b) $g \in C([t_0, t_0 + a] \times [0, 2b], R_+)$, $g(t, u) \leq M_1$ on $[t_0, t_0 + a] \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, u)$ is nondecreasing in u for each t and $u(t) \equiv 0$ is the only solution of

$$D^q u = g(t, u), \quad u(t)[t - t_0]^{1-q}|_{t=t_0} = 0, \tag{2.3}$$

on $[t_0, t_0 + a]$;

- (c) $|f(t, x) - f(t, y)| \leq g(t, |x - y|)$ on R_0 .

Then, the successive approximation defined by

$$x_{n+1}(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x_n(s)) ds, \quad n = 0, 1, 2, \dots \tag{2.4}$$

on $[t_0, t_0 + \alpha]$, where $\alpha = \min \left(a, \left[\frac{b\Gamma(q+1)}{M} \right]^{\frac{1}{q}} \right)$, $M = \max(M_0, M_1)$, are continuous and converge uniformly to the unique solution $x(t)$ of the IVP (2.1) on $[t_0, t_0 + \alpha]$.

Proof: For $t_0 \leq t_1 \leq t_2 \leq t_0 + \alpha$, we find

$$\begin{aligned} |x_1(t_1) - x^0(t_1) - x_1(t_2) + x^0(t_2)| &\leq \frac{M}{\Gamma(q)} \left| \int_{t_0}^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right| \\ &\leq \frac{M}{\Gamma(q+1)} [(t_1 - t_0)^q - (t_2 - t_0)^q + 2(t_2 - t_1)^q] \\ &\leq \frac{2M}{\Gamma(q+1)} (t_2 - t_1)^q < \epsilon, \end{aligned}$$

provided $|t_2 - t_1| < \delta$, where $\delta = \left[\frac{\epsilon\Gamma(q+1)}{2M} \right]^{\frac{1}{q}}$, proving that $x_1(t)$ is continuous on $[t_0, t_0 + \alpha]$. Similarly,

$$|x_1(t) - x^0(t)| \leq \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} |f(s, x_0)| ds \leq \frac{M(t - t_0)^q}{\Gamma(q+1)} \leq \frac{M\alpha^q}{\Gamma(q+1)} \leq b.$$

Hence it is easily seen by induction that the successive approximations are continuous and satisfy $|x_n(t) - x_0| \leq b$, $n = 0, 1, 2, 3, \dots$

We shall next define the successive approximations for the IVP (2.3) as follows:

$$\begin{aligned} u_0(t) &= \frac{M(t - t_0)^q}{\Gamma(q+1)} \\ u_{n+1} &= \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} g(s, u_n(s)) ds, \quad t_0 \leq t \leq t_0 + \alpha. \end{aligned} \tag{2.5}$$

Since $g(t, u)$ is assumed to be nondecreasing in u for each t , an easy induction shows that the successive approximations (2.5) are well defined and satisfy

$$0 \leq u_{n+1}(t) \leq u_n(t), \quad t_0 \leq t \leq t_0 + \alpha.$$

Moreover, $|D^q u_n(t)| = g(t, u_{n-1}(t)) \leq M$ and therefore, we can conclude by Ascoli-Arzelà theorem and the monotonicity of the sequence $\{u_n(t)\}$ that $\lim_{n \rightarrow \infty} u_n(t) = u(t)$ uniformly on $[t_0, t_0 + \alpha]$. It is also clear that $u(t)$ satisfies the IVP (2.3) and hence by (b) $u(t) \equiv 0$ on $[t_0, t_0 + \alpha]$. To get the equicontinuity of the sequence $\{u_n(t)\}$, one can use Lemma 2.2.

Now from the earlier estimate

$$|x_1(t) - x^0(t)| \leq \frac{M(t - t_0)^q}{\Gamma(q + 1)} = u_0(t).$$

Assume that $|x_k(t) - x_{k-1}(t)| \leq u_{k-1}(t)$ for some given k . Since

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &= \frac{1}{\Gamma(q)} \left| \int_{t_0}^t (t-s)^{q-1} f(s, x_k(s)) ds - \int_{t_0}^t (t-s)^{q-1} f(s, x_{k-1}(s)) ds \right| \\ &\leq \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} |f(s, x_k(s)) - f(s, x_{k-1}(s))| ds, \end{aligned}$$

using condition (c) and the monotone character of $g(t, u)$, we get

$$|x_{k+1}(t) - x_k(t)| \leq \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} g(s, |x_k(s) - x_{k-1}(s)|) ds = u_k(t).$$

Thus by induction, the inequality

$$|x_{n+1}(t) - x_n(t)| \leq u_n(t), \quad t_0 \leq t \leq t_0 + \alpha,$$

holds for all n . Also,

$$\begin{aligned} |D^q x_{n+1}(t) - D^q x_n(t)| &\leq |f(t, x_n(t)) - f(t, x_{n-1}(t))| \\ &\leq g(t, |x_n(t) - x_{n-1}(t)|) \leq g(t, u_n(t)). \end{aligned}$$

Let $n \leq m$. Then we can easily obtain

$$\begin{aligned} D^{+q} |x_n(t) - x_m(t)| &\leq |D^q x_n(t) - D^q x_m(t)| \\ &\leq g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) + g(t, |x_n(t) - x_m(t)|). \end{aligned}$$

Since $u_{n+1}(t) \leq u_n(t)$ for all n , it follows that

$$D^{+q} |x_n(t) - x_m(t)| \leq g(t, |x_n(t) - x_m(t)|) + 2g(t, u_{n-1}(t)),$$

where D^{+q} denotes the corresponding Dini derivative to D^+ . An application of comparison result Lemma 2.4 gives

$$|x_n(t) - x_m(t)| \leq r_n(t), \quad t_0 \leq t \leq t_0 + \alpha,$$

where $r_n(t)$ is the maximal solution of the IVP

$$D^q v = g(t, v) + 2g(t, u_{n-1}(t)), \quad v(t)(t - t_0)^{1-q}|_{t=t_0} = 0,$$

for each n . Since as $n \rightarrow \infty$, $2g(t, u_{n-1}(t)) \rightarrow 0$ uniformly on $[t_0, t_0 + \alpha]$, it follows by Lemma 2.3 that $r_n(t) \rightarrow 0$ uniformly on $[t_0, t_0 + \alpha]$. This implies that $\{x_n(t)\}$ converges uniformly to $x(t)$ and it is now easy to show that $x(t)$ is a solution of IVP (2.1).

To show that this solution $x(t)$ is unique, let $y(t)$ be another solution of the IVP (2.1) on $[t_0, t_0 + \alpha]$. Define $m(t) = |x(t) - y(t)|$ and note that $m(t_0) = 0$. Then $D^+ m(t) \leq |D^+ x(t) - D^+ y(t)| \leq |f(t, x(t)) - f(t, y(t))| \leq g(t, m(t))$, using condition (c). Again applying the comparison result Lemma 2.4, we have

$$m(t) \leq r(t), \quad t_0 \leq t \leq t_0 + \alpha,$$

where $r(t)$ is the maximal solution of IVP (2.3). By assumption (b), $r(t) \equiv 0$ and this proves that $x(t) = y(t)$ on $[t_0, t_0 + \alpha]$. Hence the proof is complete.

Corollary 2.1: The function $g(t, u) = Lu, L > 0$ is admissible in Theorem 2.1.

Let us note first that when the initial time is changed the corresponding fractional differential equation becomes different because the notion of fractional derivative is nonlocal. We are therefore content with proving the continuous dependence of solutions $x(t, t_0, x_0)$ of IVP (2.1) with respect to x_0 only.

Theorem 2.2: Let $f \in C(R_+ \times E, E)$ and for $(t, x) = R_+ \times E$,

$$|f(t, x) - f(t, y)| \leq g(t, |x - y|) \tag{2.6}$$

where $g \in C(R_+^2, R_+)$. Assume that $u(t) \equiv 0$ is the unique solution of the fractional differential equation

$$D^q u = g(t, u) \tag{2.7}$$

with $u(t)(t - t_0)^{1-q}|_{t=t_0} = 0$. Then, if the solutions $u(t, t_0, u^0)$ where $u^0 = u(t)(t - t_0)^{1-q}|_{t=t_0}$ of (2.7) are continuous with respect to the initial condition u^0 , the solutions $x(t, t_0, x^0)$ of (2.1) are unique and continuous relative to x^0 .

Proof: Since uniqueness follows from Theorem 2.1, we have to consider continuity part only. To the end, let $x(t, t_0, x^0), y(t, t_0, y^0)$ be the two solutions of (2.1) through $(t_0, x^0), (t_0, y^0)$ respectively. Defining $m(t) = |x(t, t_0, x^0) - y(t, t_0, y^0)|$, condition (2.6) implies the inequality

$$D^+ m(t) \leq g(t, m(t)),$$

and by Lemma 2.4, we get $m(t) \leq r(t, t_0, |x^0 - y^0|), t \geq t_0$, where $r(t, t_0, u^0)$ is the maximal solution of (2.7) such that $u^0 = |x^0 - y^0|$. Since the solutions $u(t, t_0, u^0)$ are assumed to be continuous relative to u^0 , it follows that $\lim_{x^0 \rightarrow y^0} r(t, t_0, |x^0 - y^0|) = r(t, t_0, 0) \equiv 0$ by hypothesis. It then follows that $\lim_{x^0 \rightarrow y^0} x(t, t_0, x^0) = y(t, t_0, y^0)$, proving the continuity with respect to x^0 . The proof is complete.

3. Flow Invariance and Inequalities in Cones

The definition of the fractional derivative of an arbitrary order $0 < q < 1$ of a function $x \in C([t_0, \infty), E)$ is given by [14]

$$D^q x(t) = \lim_{h \rightarrow 0} h^{-q} \sum_{\eta=0}^n (-1)^\eta \binom{q}{\eta} \times (t - \eta h) = \lim_{h \rightarrow 0} x^{(q)} h(t), \quad (3.1)$$

where

$$x_h^{(q)}(t) = h^{-q} \sum_{\eta=0}^n (-1)^\eta \binom{q}{\eta} \times (t - \eta h). \quad (3.2)$$

This implies, on expanding

$$\begin{aligned} x_h^{(q)}(t) &= \frac{1}{h^q} \left[x(t) - qx(t-h) + \frac{q(q-1)}{2!} x(t-2h) - \frac{q(q-1)(q-2)}{3!} x(t-3h) + \dots \right] \\ &= \frac{1}{h^q} [x(t) - s(t, h, q)]. \end{aligned} \quad (3.3)$$

Let us consider the IVP for fractional differential equation

$$D^q x = f(t, x), \quad x(t)(t-t_0)^{1-q}|_{t=t_0} = x^0 \in F, \quad (3.4)$$

where $f \in C([t_0, \infty) \times E, E)$ and $F \subset E$ is a closed set. We define

$$\lim_{h \rightarrow \infty} \frac{1}{h^q} d[x - h^q f(t, x), F] = 0, \quad (3.5)$$

where $d(x, F) = \inf_{y \in F} |x - y|$. The set F is said to be flow invariant relative to f , if every solution $x(t)$ of IVP (3.4) on $[t_0, \infty)$ is such that $x(t) \in F$ for $t_0 \leq t < \infty$. A set $A \subset E$ is called a distance set, if to each $x \in E$ there corresponds a point $y \in A$ such that $d(x, A) = |x - y|$. A function $g \in ([t_0, \infty) \times R_+, R_+)$ is said to be a uniqueness function, if the following holds: if $m \in C([t_0, \infty), R_+)$ is such that $m(t)(t-t_0)^{1-q}|_{t=t_0} \leq 0$ and $D^q m(t) \leq g(t, m(t))$, wherever $m(t) > 0$, then $m(t) \leq 0$ for $t_0 \leq t < \infty$.

We are now in a position to prove the following result on flow invariance of F .

Theorem 3.1: Let $F \subset E$ be a closed and distance set. Assume further that

- (i) $\lim_{h \rightarrow 0} \frac{1}{h^q} d[x - h^q f(t, x), F] = 0, \quad t \in [t_0, \infty), \quad x \in \varphi F$
- (ii) $|f(t, x) - f(t, y)| \leq g(t, |x - y|), \quad x \in E - F, \quad y \in \varphi F$, where $g(t, u)$ is a uniqueness function.

Then F is flow invariant with respect to f .

Proof: Let $x(t)$ be a solution of (3.4). Suppose that $x(t) \in F$ for $t_0 \leq t \leq t_0 + a$, where $t_0 + a < \infty$ is maximal, that is, $x(t)$ leaves the set F at $t = t_0 + a$ for the first time. Let $t_1 \in (t_0 + a, \infty)$ and $x(t_1) \notin F$ and let $y_0 \in \varphi F$ be such that $d(x(t_1), F) = |x(t_1) - y_0|$. Set

for $t \in [t_0, \infty)$ $m(t) = d[x(t), F]$ and $v(t) = |x(t) - y_0|$. For sufficiently small $h > 0$, letting $x = x(t_1)$, we have

$$\begin{aligned} d[s(t_1, h, q), F] &\geq d[x - h^q f(t_1, x), F] - \epsilon(h^q) \\ &= d[x - y_0 - h^q(f(t_1, x) - f(t_1, y_0)), F] \\ &\quad - d[x - y_0 - h^q(f(t_1, x) - f(t_1, y_0)), F] \\ &\quad + d[x - h^q f(t_1, x), F] - \epsilon(h^q) \\ &\geq d[x - y_0 - h^q(f(t_1, x) - f(t_1, y_0)), F] - d[y_0 - h^q f(t_1, y_0), F] - \epsilon(h^q) \\ &\geq -h^q|f(t_1, x) - f(t_1, y_0)| - \epsilon(h^q) + d[x, F]. \end{aligned} \tag{3.6}$$

Since $m(t_1) = v(t_1) > 0$ and $m(t_1) = d[x, F] = |x(t_1) - y_0|$, we find

$$|x(t_1) - y_0| - d[s(t_1, h, q), F] \leq h^q|f(t_1, x) - f(t_1, y_0)| - \epsilon(h^q),$$

which yields the inequality

$$D^q m(t_1) \leq g(t_1, m(t_1)).$$

This gives, in view of the facts that g is the uniqueness function and $m(t)(t - t_0)^{1-q}|_{t=t_0} = 0$, the relation $m(t) \leq 0, t_0 \leq t < \infty$. But, $d[x(t_1), F] = m(t_1) > 0$, which is a contradiction. Hence the set F is flow invariant relative to $f(t, x)$ and the proof is complete.

We shall next develop the theory fractional differential inequalities. To do this, we need the concept of a cone which induces a partial order in E . A proper subset k of E is said to be a cone if $\lambda k \subset k, \lambda \geq 0, k + k \subset k, k = \bar{k}$ and $k \cap \{-k\} = 0$, where 0 denotes the null element of E and \bar{k} is the closure of k . Let k^0 denote the interior of k and assume that k^0 is nonempty. The cone k induces the order relations in E defined by

$$x \leq y \text{ iff } y - x \in k \text{ and } x < y \text{ iff } y - k \in k^0.$$

Let k^* be the set of all continuous linear functionals c on E such that $cx \geq 0$ for all $x \in k$ and let k_0^* be the set such that $cx > 0$ for all $x \in k^0$. A function $f : E \rightarrow E$ is said to be quasimonotone nondecreasing if $x \leq y$ and $cx = cy$ for some $c \in k_0^*$, then $cf(x) \leq cf(y)$.

Theorem 3.2: Let k be a cone with nonempty interior. Assume that

- (a) $u, v \in C[R_+, E]$ such that $D^q v, D^q u$ exist, $f \in C[R_+ \times E, E]$ and $f(t, x)$ is quasimonotone nondecreasing in x for each $t \in R_+$;
- (b) $D^q u(t) - f(t, u(t)) < D^q v(t) - f(t, v(t)), t \in [t_0, \infty)$.

Then $v^0 < w^0$ implies that $u(t) < v(t), t_0 \leq t < \infty$.

Proof: Suppose that the assertion of the Theorem is false. Then there exists a $t_1 > t_0$ such that

$$v(t_1) - u(t_1) \in \varphi k \text{ and } v(t) - u(t) \in k^0, \quad t \in [t_0, t_1).$$

By Mazur's Lemma, there exists a $c \in k_0^*$ with $cv(t_1) - cu(t_1) = 0$. Setting $m(t) = c(v(t) - u(t))$, we see that $m(t) > 0$ for $t_0 \leq t \leq t_1$ and $m(t_1) = 0$. Consequently, by Lemma 2.1, we

get $D^q m(t_1) \leq 0$. At $t = t_1$, we have $u(t_1) \leq v(t_1)$ and $c(u(t_1)) = c(v(t_1))$. Hence using quasimonotone property of f and (b), it follows that

$$D^q m(t_1) = c(D^q v(t_1) - D^q u(t_1)) > c(f(t_1, v(t_1)) - f(t_1, u(t_1))) \geq 0.$$

This contradiction proves the Theorem.

Remark 3.1: Observe that theorem 3.1 is true when $F = k$. Although k^0 is not assumed to have nonempty interior, Theorem 3.1 requires that k must be a distance set. This, however, a weaker assumption because the cones in L^p -spaces are distance sets whose interior is empty. We note also that every closed convex set in a reflexive Banach space is a distance set.

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